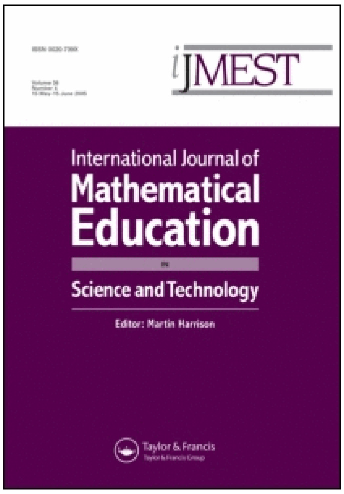

International Journal of Mathematical Education in Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713736815

Ideas and results in teaching beginning maths students

Juha Oikkonen [a]
[a] Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

Online Publication Date: 01 January 2009

To cite this Article Oikkonen, Juha(2009)'Ideas and results in teaching beginning maths students',International Journal of Mathematical Education in Science and Technology,40:1,127 — 138

To link to this Article: DOI: 10.1080/00207390802582961

URL: http://dx.doi.org/10.1080/00207390802582961

International Journal of Mathematical Education in Science and Technology, Vol. 40, No. 1, 15 January 2009, 127–138

Taylor & Francis
Taylor & Francis Group

Ideas and results in teaching beginning maths students

Juha Oikkonen*

Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

(*Received 12 October 2008*)

At the University of Helsinki, special effort has been put to help beginning maths students to get started with their studies. This involves new ways of teaching a lecture course and new ways of giving peer support to the students. The results have been promising: the number of students passing certain important courses has been at least doubled compared to the earlier prevailing level. The present article describes the work done in connection to a lecture course and the thinking that lies behind it. A paper on peer support is planned.

Keywords: teaching analysis; teaching definitions; beginning students; new approaches to lectures; increase in pass rate; two sides of mathematics

1. Background

Recruitment of maths students is not so far a big problem for departments of mathematics in Finland. This is true especially with respect to the department of mathematics and statistics in Helsinki. Besides receiving enough students (about 200 per year), the department receives also rather good students. In the Finnish matriculation exam one can do two alternative versions in maths, a 'shorter' and 'longer' version. Students can pass these exams with six alternative grades. Most of the students the department receives have passed the exam in the 'longer' maths with one of the three highest grades.

There are many polytechnics in the country and they have problems in recruiting students with satisfactory mathematical background. This is related at least partially to the fact that there are places to study at universities and polytechnics for more students with a background in mathematics than the schools manage to educate.

The above remarks mean that the starting point for teaching mathematics at the department of mathematics and statistics in Helsinki is moderately good. But the cultural gap between school maths and university mathematics remains in Helsinki as well as elsewhere. The gap means a difference in the appearance of mathematics. School maths for many students and teachers is rather algorithmic: one has to learn certain rules and then to use them in a rather standardized way. In university, mathematics is a science where exact definitions, abstract thinking and rigorous arguments become more important. The gap has also many social aspects like entering the world of university mathematicians.

The department of mathematics and statistics in Helsinki has suffered the loss of students during the first year. Partially, this is related to the gap between school maths and university mathematics, but one has to observe that there are also many students who,

*Email: oikkonen@mappi.helsinki.fi

ISSN 0020–739X print/ISSN 1464–5211 online

DOI: 10.1080/00207390802582961
http://www.informaworld.com

even on entering the department, are considering going to study in some other faculty or university.

The real rate of loss of students during the first year is hard to compute since there are always students who enter the university as maths majors but who in reality do not even begin their studies in mathematics. (Some years ago about 1/3rd of beginners belonged to this group.)

On the other hand, those that go on with their studies after the first (two) year(s) seem to have a good probability of eventually receiving a master's degree in mathematics.

The difficulties related to the first year have been especially severe in connection to a first-year course in analysis. For these reasons, special effort has been put into supporting first-year students in general and in improving the course in analysis, especially. An overview of what has been done in Helsinki is given in Oikkonen [1]. The present article goes deeper in the developments in the course in analysis. Some aspects discussed here are related to Oikkonen [2], too. Ideas in general support will be presented by Terhi Hautala in a later paper. Before analysing in detail what has been done with the analysis courses, we shall describe the results briefly.

2. Results

Both of the courses Analyses I and II are meant for first-year maths students, but they are also taken by several students of other subjects too (like physics, chemistry and computer science). They are 'analysis' courses and not 'calculus' courses in the sense that rather much effort is put into learning exact definitions and (making) proofs. The main themes of Analysis I are: the real number system and especially use of inequalities, absolute value, limit of a sequence of reals, limit of a function (of one variable), continuity, derivative and its uses and some transcendental functions. The main themes of Analysis II are: Riemann integral and its definition, improper integrals, series, uniform convergence of sequences of functions, power series, Taylor polynomials and Taylor series. Analysis I is lectured during the fall term and Analysis II during the spring term.

During the period of the table below, the number of students entering the department annually has remained rather stable. So the change cannot be explained by a change in the number of students coming to the department. (According to the information that the author has from other departments of mathematics in Finland, no similar changes have taken place elsewhere.)

The author began teaching the first-year analysis coursed in fall 2001 and the approach discussed in this article seems to have led to a clear increase in the number of students passing the analysis courses.

The following table (Table 1) shows numbers of maths majors passing the analysis courses. Since rather many students of other subject take these, the total numbers of students passing are higher. For example, that has been since 2001 over 200 per year for Analysis I. This indicates that teaching the course means coping with a rather large group.

One reason for annual differences is that the number of (nominally) beginning maths majors changes annually, too.

The loss of students between Analysis I and Analysis II needs consideration in the future. The proportion has anyway not been weakening while the number of students passing the analysis courses has increased. The loss may be related to the fact that some students aiming to apply for some other university or faculty in the spring perhaps take Analysis I but not Analysis II. But it is known that part of the content of Analysis II is

Table 1. Numbers of maths students passing Analyses I and II.

Acad year	Analysis I	Analysis II	A II/A I
1997–1998	70	58	0.829
1998–1999	88	61	0.693
1999–2000	82	48	0.585
2000–2001	75	50	0.667
2001–2002	139	120	0.863
2002–2003	122	125	1.025
2003–2004	162	117	0.722
2004–2005	150	115	0.767
2005–2006	131	102	0.779
2006–2007	165	142	0.860
2007–2008	134	112	0.836

Table 2. Numbers passing three Helsinki courses.

Acad year	Measure	Probability	Teacher	Sum
2000–2001	19	13	21	53
2001–2002	21	17	10	48
2002–2003	31	15	33	79
2003–2004	33	20	55	108
2004–2005	61	14	42	117
2005–2006	41	14	63	118
2006–2007	32	18	57	107

Note: The academic year 2007–2008 is omitted since it is not comparable to the others. The spring course in probability is exceptionally going on during fall 2008 and the course in teacher training was exceptionally not given a second time in the summer.

considered rather hard by the students. So there may be need for more development in teaching and studying Analysis II.

When we look at beginning successfully the master-level studies we also have good news. Roughly speaking, most of the increased number of students passing the first-year analysis courses seems to be able to go on successfully with their studies.

Three courses can be considered as entry to master-level studies in maths during the period 2000–2006. These are Measure and Integration, (higher) Probability Theory and a certain course in teacher training. In Table 2, we give some numbers of maths majors passing these courses.

It is interesting to compare the 'sum' column in the above table to the number of students passing Analysis II. It is hard to tell the exact time between making these two steps in one's studies and one also has to notice that some students take more than one of these master-level courses. But it seems that rather few maths majors passing Analysis II after 2002 fail to pass the master-level course.

Thus, it should be possible for the department to increase essentially the annual number of students receiving a master's degree. (Between 2004 and 2007, the number has been: 43, 55, 48 and 62.)

We shall now begin to look at what has been done.

3. About passing a course

Traditionally, a university course is organized like this: first some teaching is done. Then an exam is used to decide which students happen to pass.

But those students who do not pass the course fall behind from the correct timing of their studies: certain (number of) courses are supposed to be studied each semester and if a student has to come back to some of the earlier courses, it is hard to avoid the lengthening of the time they need to get a degree. This is a personal failure for the plans of the students. But this means also problems for the department. Each department is expected to educate masters in 5 years. If the average time needed for a master's degree is much longer, then the department has failed in its task. Since the universities have to negotiate their funding with the state authorities, this may cause problems.

In Europe, the students are nowadays supposed to study 3 years to get the degree of a candidate and 2 more years to get the master's degree. It is a task of a department (the university) to make this possible. If a notable amount of students do not pass a compulsory first-year course then it will become difficult for a department to fulfill its task.

So can there be an arrangement in which 'everybody' can pass a course in mathematics? Beginning 1998, there has been in use in the department of mathematics and statistics a version of small group teaching, which will be called here study group teaching. There, students combine project work with various forms of exams and in principle everybody who has completed all the work required will pass. So it is possible to think about teaching university mathematics so that 'everybody' will pass.

This study group teaching will be described better below.

We shall first go back to exams and later discuss what has actually happened to teaching the analysis courses.

4. Rethinking exams

In the analysis courses, the students are encouraged to do additional work to compensate the outcome of the exams.

The role of the analysis courses in the curriculum is that in them the students learn conceptual foundations for further studies (and for students becoming maths teachers, understanding what lies behind the elements of calculus taught at school). It seems that difficulties in learning the courses lies more often in the conceptual aspects than in lacking the skill to do the calculations. From the point of view of a typical beginning student the analysis courses contain plenty of hard maths. But from the point of view of the lecturer there are actually quite a few ways of thinking that appear again and again.

These observations have led to the following kind of additional work. First, a theorem together with its proof is chosen from the course material. It is done so that the student looks for a candidate and then discusses the choice with the author. For example, limits of sequences are important for the first exam and a good candidate might choose the theorem stating that a sequence can have only one limit or the theorem saying that the limit of the sum of two convergent sequences is the sum of their limits. These are very elementary but in their proofs the definition of a limit is used in a way that is hard for a student who has difficulties. And the way in which the proofs are written in the course material (or, in most textbooks) does not make things easier.

The task of the student is to make sense of these rather few lines of mathematical writing using pen and paper. The task is done, when everything in the proof has become completely clear to the student. It is up to the student to decide when this is the case.

The student is encouraged to discuss the task with fellow students and help can be requested from the teacher. While returning the documented work to the teacher, they often discussed whether something still remains unclear. Indeed, emphasis in the additional work is in learning, not in examining.

Since the details of the analysis course are very similar in the way of thinking underlying them, the author believes that this kind of studying has very strong transfer effect within the course: when you understand one detail well, it will be rather easy to understand other details, too.

In using this kind of additional work in connection to the analysis courses it has been rather striking to observe that actually fewer students than originally expected have asked for additional work. Especially, this arrangement means very little extra work for the teacher.

The above seems to suggest that even more important than the additional work actually done, has been the option that one can always compensate the outcome of exams. There seems to have been less stress related to the exams.

Also another minor change has been done in connection to the exams. In a course with 200–300 students it is very probable that always some students are not able to take part in the exams. For example, it has turned out that there are always cases where students have exams in some other department simultaneously with exams in analysis. To overcome this difficulty, alternative ways to do the exams are given. (This may sound complicated but in practice it has caused very little extra work for the lecturer.)

5. Experiences behind teaching analysis

Finally, we go to the actual ideas in teaching the analysis course. The ideas came originally from maths days and were later developed in the study group teaching mentioned earlier in this article.

During the late 1990s the author organized several 'maths days' at schools. There he became impressed by what the children were able to do. Also, there was a rather striking similarity between his discussions concerning maths problems with children and discussions with colleague researchers of mathematics. Typical to these discussions is that mathematical questions appear in them in a meaningful form.

School children and students at high schools and universities are in a certain sense between children taking part in maths days and researchers of mathematics. So it seemed natural to ask whether teaching maths at schools and in universities could be similar to what happened in the discussions mentioned above?

The first attempt to give a positive answer to this was the above mentioned study group teaching. It was begun during autumn 1998. In it a group of 15–25 students work together with a teacher on the theme of one of the usual courses studied by first- or second-year students. A characteristic feature in study group teaching is doing things together with a familiar group where everybody is expected to support each other in their studies.

In study group teaching, short lectures by the teacher are combined with research discussions led by the teacher and presentations of projects by the students. The students do projects in smaller groups already during their first semester.

In a typical course in mathematics the course material is approached going from the beginning to the end. The beginning of the course material often contains concepts and lemmas meant only to support the main content that will come later. It is often hard to motivate students for this part of the course.

In study groups, the mathematical content is sometimes approached starting from the most interesting ideas and then details are worked around these special points of interest. In this kind of approach many of the lemmas get special interest: they are what is needed to carry out the ideas developed in the research discussions.

For example, the author on one occasion began discussing why a continuous function obtains the value zero between places where it gets negative and positive values. The starting point was that the students knew this fact from school. So the group was working for a justification of this assertion. Then the discussion went to recalling how approximations for the zeros were computed in school by dividing the interval repeatedly so that the function get a negative value at one end point and a positive one at the other. And then it was natural to start to discuss whether this way of computing the zero could be transformed into a proof of the existence of one. After a while the group interviewed by the teacher had an argument with essentially one detail missing: the definition of continuity. This means that the research discussion in a way simultaneously led to an argument and to a motivated definition.

Since its beginning in 1998 the study groups have been taught by many teachers and their ways of teaching have been rather varied. Therefore, it is interesting to note that almost always the students have found them successful. During the last few years study group teaching has become an essential part in the teacher training given by the department.

6. Developing the analysis course

It was impressive to know from the study groups how fruitful it can be to base teaching on an active role of the students. In 2000, the author began an attempt to share as much as possible of the good experiences of study group teaching to all beginning students of the department of mathematics and statistics: from about 20 students to over 200 students in a big lecture hall.

The first step was to hire new kinds of teachers for some of the exercise classes in 2000. Younger people that were more capable of approaching beginning students were used. To help these in their work a seminar called ‘seminar of mathematical thinking’ was organized. (Here ideas of Miroslav Lovric from McMaster University were borrowed. The author learned about these during an AMS Scandinavian colloquium in Odense, Denmark.) But it turned out that intervention in the exercise classes was not enough to change the results of the course.

In 2001, the author began to lecture first-year analysis with an aim to give a new meaning to the lectures. One aim was to stimulate as much discussion as is possible with 100–300 students.

Another aim was to reveal to the students how mathematicians think and what lies behind the definitions, examples and proofs of the course material. The course material was then regarded as kind of a handbook. During the lectures most of the time was given to the central concepts and ideas. The analysis courses became really a year of the ‘epsilon–delta’ method.

There were two aims: to try to give better understanding of the central concepts of the course to an average student passing the course; and to radically increase the number of students passing the course. The total number of students that passed both courses more than doubled compared to the prevailing level. Later this ‘phenomenon of a large number of students passing courses’ has gone through other courses as well and several teachers have played a role in the developments.

These aims grew from the fact that it was found out at the department that the outcome of the analysis course taught in a traditional way was strongly unsatisfactory in both respects. As a matter of fact, it was felt that the situation was much the same in many other universities around the world.

7. Lecturing the analysis course

The author encourages (and sometimes requires) the students to take an active part in the lecture theatre: to ask anything, to suggest what shall be done next. The students are, for example, asked whether they wish to have a theorem next or an example; if an example to say what kind of an example. Such examples are of course created *ex tempore*. It is also tried to extend this discussion outside the lectures and for example to encourage the students to ask for hints for getting started with the exercise problems.

Often when discussing with students about why they do not ask more questions during lectures, they say that they do not understand enough to ask a good question. To this the author remarks that a most positive comment during a lecture is to say 'I do not understand anything'. Indeed, after such a comment the lecturer has complete permission to stop to explain.

The author does not try to go through all the course material during the lectures. Instead, the lectures concentrate on the central concepts and their exact definitions together with those ways of thinking that lie behind these concepts. This is done by use of theorems and examples as interesting case studies used to open the thinking behind the analysis course. In this way, everything in the two analysis courses is used as a tool to learn the 'epsilon–delta' method of contemporary analysis.

Let us take a closer look at some aspects of the everyday life of the course. First a flavour of the local level of the course.

In introducing, say, the definition for a limit of a sequence, the author starts from the idea 'x_n gets close to a when we make n big' that is more or less (vaguely) known to the students already from school. Then the author draws pictures (as probably most teachers of a similar course do) about situations where a is a limit and where a is not a limit of the x_n. Then this discussion is translated in the form of the definition in terms of 'epsilons'.

When going to the first examples or theorems, the author expresses the fact that one is not meant to be surprised by the results – rather the situation is like that when one has bought a new computer program: one plays with it like with a new toy and tests it to see whether it works as expected.

Imagine then that after some examples it is time for the first theorems about limits. The author does not go through them one by one but rather collects them in a list. It is told to the students that the list is meant to be used as a collection of theoretical exercises aimed to help in getting hold of the definition.

One of the first theorems tells that if $\lim x_n = a$ and $\lim x_n = b$, then $a = b$. To work with this, we begin by discussing what sense can be in this kind of the theorem: why can there be anything to prove. Then the lecture goes on to making observations underlying the proof as follows:

Assume $a < b$. The author draws a vertical line – part of the y-axis – and marks a and b there. Then 'short vertical lines' are drawn near a and b. (The reader is suggested to do this.) It is remarked that x_n is in each place for large n. Then this impossibility is turned into finding an obvious candidate for epsilon that can be used to draw a contradiction.

The students are helped also on a global level. This means organization of material and choice of examples and problems.

For example, exact definitions of limits are hard for at least two reasons: the 'epsilon-thinking' is hard and in using the definition one is using inequalities in a way that looks strange for one coming from school. For this reason, the course is begun with a careful look at examples and examples of the way in which inequalities are used in connection to the definition of a limit.

An example of this is the following: show that if $2 < x < 2 + 5^{-100}$, then $4 < x^2 < 4 + 5^{-99}$. The solution is based on the following two observations: $x^2 - 4 = (x + 2)(x - 2)$ and that $x + 2 < 5$. A related problem might be: find such an $h > 0$ that $x^2 < 4 + 7^{-999}$ whenever $2 < x < x + h$. In a sense, working with limit is here begun during the first week of the course.

Such examples are important since in school inequalities are problems given by the teacher that have to be solved. But in the analysis course inequalities are something quite different. They are a way of speaking about the size of various expressions. Moreover, one can rather freely change the expressions while making estimates. It is a major challenge for a beginning student to make such a change in thinking. The students are of course told why such examples are considered.

Another example of changing the order in which material is considered, is that immediately after defining the notion of a limit of a function continuity and derivatives are defined. This makes available more natural examples of limits. Moreover, several properties of continuous functions or derivatives will become examples of the notion and properties of limits. For example, continuity and differentiability of the sum of two (continuous or differentiable, respectively) functions will be consequences of the interplay of the limit of a sum of functions. (Proper 'theory of continuous functions' like existence of maxima and minima and 'theory of derivatives' like the mean value theorem will be considered later in their own parts of the course.)

An interesting example for such a lemma that is usually considered already in the part of limits of functions is the following:

Assume that $f'(a) > 0$. Then there is $h > 0$ such that for x,

(i) if $a - h < x < a$, then $f(x) < f(a)$;
(ii) if $a < x < a + h$, then $f(a) < f(x)$.

(There is a similar assertion for $f'(x) < 0$.)

The proof of this fact is very easy as long as one considers this as an exercise about the concept of limit (of the difference quotient). If one begins to think in terms of tangents or other ideas related to actual differential calculus, then one easily gets stuck. This result has such an important role in working towards Rolle's theorem that the above lemma has got the name 'the derivative lemma' in the course.

This discussion is related to a more general observation. It seems to the author that for improvements in teaching mathematics, rethinking of the actual mathematics to be studied is needed besides pedagogical thinking.

While lecturing some theorem or example, most time is given in the analysis courses for revealing the thinking that lies behind 'proofs' or 'solutions'. There is not much of this in textbooks or the course material in use in the course. To emphasize this, work with a proof or a solution begins with a stage called 'observations'. Notice that in connection to the examples given above the related observations were mentioned. In the 'observations' part the task is approached with pictures, calculations, inequalities (the course is really about inequalities), etc. When enough of these observations have been gathered, the proper proof or solution will be just a version of some of the facts already on the blackboard edited to fulfill the formal requirements.

For example, the proof of the continuity of x^2 at $x = 2$ would be approached as follows: at the beginning of the 'observations', it would be recalled that the problem is actually about making $| x^2 - 4 |$ as small as required. Then one observes that $| x^2 - 4 | = |x + 2||x–2|$. Next it would be observed that if $| x - 2 | < 1$, then we could go on with $|x + 2||x–2| = (x + 2) | x - 2 | \leq 5 | x - 2 |$. The final step in the 'observations' would be to notice that for $\varepsilon > 0$, $5 \mid x - 2 \mid < \varepsilon$ if and only if $\mid x - 2 \mid < \varepsilon/5$. After such observations, it is a straightforward task to write the actual proof.

The way of lecturing described above leads to the following problem to the students and challenge to the lecturer: if not everything in the course material is covered in the lectures, then how to know or tell what is actually required in the exams? The solution to this problem that is used in the author's analysis course is two-fold. To the first, it is important to choose all the exercise problems carefully so that they help in learning what is considered central in the lecturer's interpretation of the course. Then one can say that those skills that have actually been exercised will suffice in the exams. To the second, the author has prepared special training material consisting 20–30 problems that are formulated according to the content of the exercises. And finally, the most central material for the exams is sketched during the last lectures before the exams.

Some of the students naturally and quickly find the good points in this kind of teaching, but some others have negative feelings at first. Later, the students have got so much experience of traditional teaching that the majority seems to be strongly in favour of the approach described here. The lecture theatre seems to be almost as full at the end of the course as it is during the first weeks. Here is a spontaneous anonymous comment by a student taking part in the course.

'I am really positively surprised by how Analysis I is lectured. It is very good that you try to help the student really to understand by use of concrete examples and comparisons. There is a magical atmosphere during the lectures when all the audience is concentrated to follow the teaching. In my earlier maths courses (maths is a minor subject to me) proofs and exact definitions have felt mostly like useless additional weight, but now I have finally begun to understand their meaning.'

8. What mathematics is – for the author

The approach to teaching mathematics described above is closely related to the question about what mathematics is. In the writings of several researchers there are strong resemblances to this approach. Such authors include Dubinsky, Sfard and Tall, etc. (See for instance [3–5].)

But the aim in this part of the article is to describe those actual ideas that come out of the teaching process described here and which continuously inspire developing it further. Therefore, we do not go into a discussion of the interesting writings of these or other authors. Rather we give a crude description of the pedagogical (and philosophical) thinking underlying the teaching by the author. This part of the article is closely related to Oikkonen [2].

When we work with mathematics, it seems to have two sides.

We use intuitive ideas and meet heuristic descriptions. These belong to the human side of mathematics. This side of mathematics can be thought of also as the subjective and social (or psychological) side of mathematics.

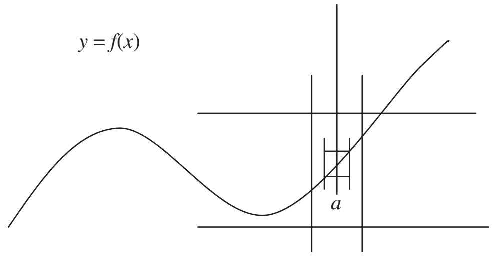

Figure 1. The human side of continuity.

We use rules and algorithms to handle mathematical tasks and we work with mathematical formulas. These belong to the formal side of mathematics. The formal side can be thought also as the objective side of mathematics since formulas and rules usually appear in written mathematical text, which is objective in the sense that it is independent of any persons.

Of course, the physical reality is also something objective related to mathematics and it is often thought that mathematics rose originally from attempts to cope with it. But in this article we are mainly interested in mathematics within itself and hence the world around us will be neglected.

Next, we try to see where we are led if we take the above simple observation seriously. (Interest in this kind of considerations comes from the history of the author: he is originally a research logician now involved in looking for new ideas for teaching mathematics.)

Let us consider an example taken from the realm of teaching analysis: the notion of continuity.

First, the human side of continuity. The idea is simple: a function $f(x)$ is continuous at a point a, if $f(x)$ is near $f(a)$ when x is near a.

More exactly, if we draw two horizontal lines, one above and one below $y = f(a)$, then there is are two such vertical lines, one left and one right of $x = a$, that the graph of $f(x)$ does not 'cut' the 'floor' or 'roof' of the rectangle formed by the four lines. And we can make such rectangles as low as we like. Two such rectangles are shown in the following picture (Figure 1).

There is nothing special in this. We are used to seeing pictures more or less like this in textbooks and during lectures.

Then the formal side of continuity: f is continuous at a, if (and only if) for every $\varepsilon > 0$ there is such a $\delta > 0$ that $|f(x) - f(a)| < \varepsilon$ for all x satisfying $|x - a| < \delta$.

This is certainly formal and exact. But there is nothing new here, either.

Since the formal definition is not very long, it should not be so difficult to memorize. But does mere memorizing suffice? And there is an even more important question.

We have now actually two meanings for continuity – the human (intuitive) and the formal (exact) definition. Which is the right one? The same question can also be put somewhat differently: what is the real the notion of continuity? These questions will be discussed in a more general form below.

9. Which side is right?

Mathematics has two sides – or, to put it differently: there are two viewing points for observing mathematics: human and formal. So it is natural to ask which of these is the right one. In the case of the notion of continuity this means: what is the concept of continuity, really. Is it a collection of ideas and mental images like the visual one described above. Or is it the formal definition?

The author has found it necessary and inspiring to think is that continuity is neither of these! Mathematics is not any one of its two sides.

In a sense mathematics is a human endeavour – something that people learn and do. So it seems tempting to say that mathematics is in reality more or less what we called the human side above. But how to understand then the role of formulas, exact proofs, or formalized definitions (like that of continuity).

For many mathematicians (including high school teachers) the attribute 'mathematical' seems to mean more or less the same as 'formal(ized)'. So it seems that these people would like to say that mathematics is really what we called the formal side above. What we called the human side would then only be our discussion about mathematics. But would it then be enough to correctly memorize the formal definition of continuity in order to say that one has learned continuity?

The way of seeing mathematics that has grown from the experiences of the author in teaching and research is that mathematics is an interplay between what has been called its human and formal sides above. This seems to lead to fruitful developments in teaching.

For instance, to master continuity, one has to be able to relate one's own mental images to the concept and its definition and to be able to use these mental images as a tool to handle questions about continuity in terms of the formal definition. Simultaneously, one has to be able to test one's mental images against the formal definition. In a sense, our mental images are private and can be shared only to a limited degree. They are highly subjective.

At the same time the formal definition or formal proofs are something public which is open to everyone. In this sense, the formal side of mathematics is objective, independent of any person.

This is related to ideas for better teaching. The author thinks that the interplay between the human and formal sides of mathematics is the place for discovery and understanding in learning or 'doing' mathematics. To teach or learn mathematics is not about some given set of 'mental images' or about the definitions etc. It is all about teaching and learning the interplay between the two sides of mathematics. In this sense teaching mathematics is teaching mathematical thinking to the author.

Building an active interplay between the two sides of mathematics has much to give in improving mathematical teaching and learning on all levels from elementary school to university.

In Oikkonen [2], the author has sketched how the above way of looking at mathematics can be carried further to the level of philosophy of mathematics. There the challenge is to put together metamathematics closely related to a study of the logical foundations of mathematics with the so-called new philosophy of mathematics closely related to modern learning theory.

References

[1] J. Oikkonen, *Good experiences in teaching beginning maths students in Helsinki*, ICMI-Bulletin 62 (2008), pp. 74–80.

[2] J. Oikkonen, *Mathematics between its two faces*, in *Matematiikan Opetuksen Tutkimuspäivät Oulussa 25. – 26. 11*, J. Lasse, K. Tapio, and K. Kari, eds., University of Oulu, Oulu, Finland, 2004, pp. 23–30, ISBN 951-42-7886-0 (http://herkules.oulu.fi/isbn9514278879/).

[3] E. Dubinsky and M.A. McDonald, *APOS: A constructivist theory of learning in undergraduate mathematics and education research*, in *The Teaching and Learning of Mathematics at University Level: An ICMI Study*, D. Holton, ed., Kluwer Academic Publishers, Dordrecht, The Netherland, 2001, pp. 273–280.

[4] A. Sfard, *On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin*, Educ. Stud. Math. 22(1) (1991), pp. 1–36.

[5] D. Tall, *A theory of mathematical growth through embodiment*, *symbolism and proof*, International Colloquium on Mathematical Learning from Early Childhood to Adulthood, Belgium, 2005.