Inverse Problems with Sparsity Constraints

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Outline

1. Sparsity concepts
2. Inverse Problems with Sparsity Constraints
3. Research directions
4. Parameter identification for PDEs
The usual setting

\[ A : X \rightarrow Y, \quad g \in \text{range}(A) \]

- solutions \( \{ f \in X \mid Af = g \} \)
- noise \( \| g^\varepsilon - g \|_Y \leq \varepsilon \)
- approximate solutions \( \{ f \in X \mid \| Af - g^\varepsilon \| \leq \varepsilon \} \)
Set of possible solutions

\[ \{ f \mid \|Af - g^\varepsilon\| \leq \varepsilon \} \]

\[ \|Af_1 - g^\varepsilon\| \leq \varepsilon \]

\[ \|Af_2 - g^\varepsilon\| \leq \varepsilon \]

\[ \|f_1\| < \|f_2\| \]

Which one is the preferred approximation?
Sparse structures

critical structures $\Delta$ sparse representations

- High precision surfaces
- Linear guideways
- Quality control
- LC-MS spectra
- Aero engines
- Turning processes
- Quality control
- Test design
Sparse Decomposition

Definition

$f$ is called sparse with respect to a basis/frame/dictionary $\{\varphi_i\}$, if

$$f = \sum_{i \in I} f_i \varphi_i, \quad |I| < \infty,$$

i.e. there exists a finitely supported decomposition of $f$.

Examples of bases:
- pixel basis
- Fourier basis
- object basis
- wavelets
- frames, dictionaries
Sparsity in Signal/Image Processing

\[ A = I \]

- sparsity with respect to basis \( \{ \varphi_i \}_{i \in \mathbb{Z}} \), i.e. \( f = \sum f_i \varphi_i \)
- variational approach/ Tikhonov functional

\[
\min_f \| f - g^\varepsilon \|^2 + \alpha \| f \|_{\ell^p}^p
\]

- shrinkage operator \( f^\varepsilon_\alpha = S^p_\alpha (\{ g^\varepsilon_i \}) = \sum S^p_\alpha (\langle g^\varepsilon, \varphi_i \rangle) \varphi_i \), e.g.

\[
S^1_\alpha (u_i) := \begin{cases} 
  u_i - \alpha, & \text{for } u_i > \alpha, \\
  0, & \text{for } -\alpha \leq u_i \leq \alpha, \\
  u_i + \alpha, & \text{for } u_i < \alpha.
\end{cases}
\]

D. Donoho. De-noising by soft thresholding. 1995. and many others
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Sparsity and Inverse Problems

\[ \min_{f} \| A f - g^\varepsilon \|^2 + \alpha \| f \|_{\ell^p} \]

- Regularization Properties
  \[ f_{\alpha}^\varepsilon \rightarrow f^{\dagger}, \quad \varepsilon \rightarrow 0 \]

- Iterated Soft Shrinkage
  \[ f(k+1) = S_{\alpha}^p [ f(k) - A^* (A f(k) - g^\varepsilon) ] \]

Bregman distance

\[ D_{jp}(x, y) := \frac{1}{p} \|y\|^p - \frac{1}{p} \|x\|^p - \langle j_p(x), y - x \rangle \]

- measures the gap between the functional \( \frac{1}{p} \|x\|^p \) and its linearization
- in \( \ell^p, L^p, W^p_k, B^s_{p,q} \) with \( 1 < p, q < \infty \) can be bounded from above resp. below by some power of \( \|x - y\| \)
- better suited for analysis than the norm
Duality mapping

- may be multivalued ($j_p$ selection of $J_p$)
- $\langle j_p(x), x \rangle = \|x\|^p$ and $\|j_p(x)\| = \|x\|^{p-1}$
- in $\ell^p, L^p, W^p_k, B^s_{p,q}$ with $1 < p, q < \infty$

\[
J_p = \partial\left(\frac{1}{p} \| \cdot \|^p\right)
\]

\[
J_p^*(J_p(x)) = x \quad \text{and} \quad J_p(J_p^*(x^*)) = x^*
\]

- needed for Bregman distances, Source Conditions, Minimization schemes
Major research directions

- Generalizations
  - Non-linear operators
  - $X, Y$ Banach spaces
  - Iteration methods for sparse approximations

- Some open problems
  - Efficient algorithms for minimizing Tikhonov functionals
  - Source conditions
  - Applications
Other sparse/local reconstruction schemes

- BV-regularization, $A : BV(\Omega) \rightarrow L^2(\Omega)$,

$$\min \|Af - g^\varepsilon\| + \alpha \sup \int_\Omega f \div p \, dx$$


- Sampling methods/ factorization schemes → A. Kirsch

- Mollifier methods → A.K. Louis
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Non-linear operators

\[ \min_f \| A(f) - g^\varepsilon \|_2^2 + \alpha \| f \|_{\ell^p} \]

\[ f^{(k+1)} = S^p_{\alpha} \left[ f^{(k)} - \left[ A'(f^{(k)}) \right]^* \left( A(f^{(k)}) - g^\varepsilon \right) \right] \]


Grasmair, Haltmeier, Scherzer. *Sparse Regularization with \( \ell^q \) Penalty Term.* 2008.

Non-linear operators with projections on \( \ell_1 \) ball

\[ \min_{f \in B_R} \| A(f) - g^\varepsilon \|_2^2 \]

\[ f^{(k+1)} = P_{B_R} \left[ f^{(k)} - \frac{\beta^k}{r} \left[ A'(f^{(k+1)}) \right]^* \left( A(f^{(k)}) - g^\varepsilon \right) \right] \]

Teschke, Borries. *Accelerated Projected Steepest Descent Methods,* 2010
Tikhonov-functionals $T_\alpha$ in Banach spaces

$$T_\alpha(x) = \frac{1}{r} \| Ax - y^\delta \|_Y^r + \alpha \frac{1}{p} \| x \|_X^p$$

$X, Y$ Banach spaces

$x_{n+1} = j_p^*(j_p(x_n) - \mu_n \partial T_\alpha(x_n))$

$x_{n+1}(x) = x_n - \mu_n j_p^*(\partial T_\alpha(x_n))$

- update in the dual space or in the primal space
- Tikhonov or generalized Landweber-iteration ($\alpha = 0$)


Bonesky et al. 2008, Kazimierski 2009
Efficient minimization Algorithms

\[ \min_f \| Af - g^\varepsilon \|^2 + \alpha \| f \|^{\ell_p} \]

- **active set methods**

- **iterative methods**

- **gradient descent**

- **forward-backward splitting**


- **FISTA** Beck-Teboulle (2009), SpaRSA Wright-Nowak-Figueiredo (2009)
Source Conditions

**Assumptions for \(\ell^1\) penalty**

- \(K\) has finite basis injectivity property
- minimum-\(\| \cdot \|_{\ell^1}\)-norm-solution \(f^\dagger\) is finitely supported
- source condition

\[
\text{range } A^* \cap \text{Sign}(f^\dagger) \neq \emptyset.
\]


**Extended concepts**

- approximate source conditions Hofmann, Düvelmeyer, Krumbiegel 2006
- variational source conditions Hofmann, Kaltenbacher, Pöschl, Scherzer 2007, Hein
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Impedance tomography (EIT)


\[ f^{n+1} = S_\alpha \left( f^n - [A'(f^n)]^*(A(f^n) - g^\delta) \right) \]

determine \( \sigma \) from EIT measurements

\[
\begin{align*}
-\text{div}(\sigma \nabla u) &= 0 \text{ in } \Omega \\
\sigma \frac{\partial u}{\partial n} &= j \text{ on } \partial \Omega
\end{align*}
\]

forward solver: \( u = F^\sigma(j) \)

measurements (restr. of \( u \) on \( \partial \Omega \)) \( g = \gamma_0 u \)
**Sparsity reconstruction**  Bangti Jin, T. Khan, P. Maass

- assumption: known background $\sigma_0$
- then calculating $\sigma^\dagger = \sigma_0 + \delta\sigma^\dagger$
- $\delta\sigma^\dagger$: localized inclusion


\[
\min_{\sigma} \max_j \left\| \gamma_0 F^{\sigma}(j) - \gamma_0 F^{\sigma^\dagger}(j) \right\|
\]

**reconstruction algorithm**

\[
\delta\sigma^{n+1} = S_\alpha \left( \delta\sigma^n - \left[ \frac{\partial}{\partial \sigma} \gamma_0 F^{\sigma^n}(j) \right]^* (\gamma_0 F^{\sigma^n}(j) - \gamma_0 F^{\sigma^\dagger}(j)) \right)
\]
Analytic prerequisites

\[
\frac{d}{d\sigma} F^\sigma(j) : L_\infty(\Omega) \to H^1(\Omega)
\]


adjoint not defined

\[
\frac{d}{d\sigma} F^\sigma(j) : L_q(\Omega) \to H^1(\Omega)
\]

algorithm

- step size selection: Barzilai-Borwein rule
- Sobolev smoothing

Ref: Jin B, Khan T, Maass P. Sparse reconstruction in electrical impedance tomography, preprint.
Results (continuum model)

- single inclusion (1% noise)
- complex model (5% noise)
Results (../../../bilder/complete electrode model)

32 electrodes, 2% noise

- **real data, 3D**: Kaipio et al (2010),
- **finite element analysis**: Chen-Zou (1999), Jin-Zou (2009)