INVARIANT SUBSPACE PROBLEM: SUBNORMAL OPERATORS

ABSTRACT. Subnormal operators have nontrivial invariant subspaces. Why? We will take a look at Thomson [6].

1. INTRODUCTION

It will be assumed that Hilbert spaces are complex and subspaces are closed. Hilbert spaces will be denoted by $\mathcal{H}$ and $\mathcal{K}$. Notation $\mathcal{H} \leq \mathcal{K}$ will mean that $\mathcal{H}$ is a subspace of $\mathcal{K}$. The set of all bounded linear operators $\mathcal{H} \to \mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$.

An **invariant subspace** of $A \in \mathcal{B}(\mathcal{H})$ is a subspace $\mathcal{M} \leq \mathcal{H}$ such that $A\mathcal{M} \subseteq \mathcal{M}$. It is called a **nontrivial invariant subspace (n.i.s.)** of A if also $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq \mathcal{H}$. The **adjoint** of an operator $A \in \mathcal{B}(\mathcal{H})$ is the unique operator $A^* \in \mathcal{B}(\mathcal{H})$ satisfying $(Ax \,|\, y) = (x \,|\, A^*y)$ for all $x, y \in \mathcal{H}$.

Definition 1.1. Operator $N \in \mathcal{B}(\mathcal{H})$ is called **normal** if $N^*N = NN^*$. Operator $S \in \mathcal{B}(\mathcal{H})$ is called **subnormal** if there exists a Hilbert space $\mathcal{K}$ and a normal operator $N \in \mathcal{B}(\mathcal{K})$ such that $\mathcal{H} \leq \mathcal{K}$ and $S = N|\mathcal{H}$. Operator N is called a **normal extension** of S.

Obviously all normal operators are subnormal.

Examples 1.2. We assume that $\mathcal{H} = \ell^2$.

(1) Shift operator $S \in \mathcal{B}(\mathcal{H})$ given by $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ is subnormal but not normal.
(2) Operator S^* given by $S^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$ is not subnormal.

Our goal is to examine James Thomson's short proof [6] of the following theorem originally proved by Scott Brown [3]:

Theorem (Brown [3]). *If* $\dim(\mathcal{H}) \geq 2$ *and* $S \in \mathcal{B}(\mathcal{H})$ *is subnormal, then* S *has a nontrivial invariant subspace.*

2. BACKGROUND

The set of all polynomial functions $\mathbb{C} \to \mathbb{C}$ will be denoted by P.

Definition 2.1. Operator $A \in \mathcal{B}(\mathcal{H})$ is called **cyclic** if for some $x \in \mathcal{H}$ we get

$$\overline{\{p(A)x : p \in P\}} = \mathcal{H}.$$

Vector x is called a **cyclic vector** of A.

Theorem 2.2. *If $A \in \mathcal{B}(\mathcal{H})$ is not cyclic, then A has a n.i.s.*

Proof. We can fix $x \in \mathcal{H} \setminus \{0\}$ and define $\mathcal{M} = \overline{\{p(A)x : p \in P\}}$. Since x is not a cyclic vector of A, the set $\mathcal{M}$ is a nontrivial invariant subspace of A. $\square$

Definition 2.3. Operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are called **unitarily equivalent** if there exists an isometric isomorphism $U : \mathcal{H} \to \mathcal{K}$ such that $A = U^{-1}BU$. This will be denoted by $A \simeq B$.

Remark 2.4. Clearly $\simeq$ is reflexive, symmetric, and transitive. Also, if $\mathcal{M}$ is a nontrivial invariant subspace of A and $A \simeq B$ with isomorphism U, then $U\mathcal{M}$ is a nontrivial invariant subspace of B.

Definition 2.5. Suppose that $t \in [1, \infty)$ and that $\mu : \mathrm{Bor}(\mathbb{C}) \to [0, \infty)$ is a compactly supported measure. We denote by $P^t(\mu)$ the closure of P in $L^t(\mu)$. We also define an operator $S_\mu : P^2(\mu) \to P^2(\mu)$ by $S_\mu f(z) = z\, f(z)$. [1]

Theorem 2.6 (Bram [1])**.** *If $S \in \mathcal{B}(\mathcal{H})$ is cyclic and subnormal, there exists a compactly supported measure $\mu : \mathrm{Bor}(\mathbb{C}) \to [0, \infty)$ such that $S \simeq S_\mu$.*

Proof. Fix a cyclic vector $x \in \mathcal{H}$ and a normal extension $N \in \mathcal{B}(\mathcal{K})$ of S. By spectral theorem there exists a spectral measure $E : \mathrm{Bor}(\mathbb{C}) \to \mathcal{B}(\mathcal{K})$ such that $N = \int z \, dE(z)$. We will now define

$$\mu(B) = (E(B)x \,|\, x)$$

for $B \in \mathrm{Bor}(\mathbb{C})$. For every polynomial $p \in P$ we now have

$$(p(S)x \,|\, p(S)x) = (p(N)^* p(N)x \,|\, x) = \left(\left(\int |p|^2 \, dE \right) x \,\middle|\, x \right) = \int |p|^2 \, d\mu.$$

Because P is dense in $P^2(\mu)$ and $\{p(S)x : p \in P\}$ is dense in $\mathcal{H}$, there exists an isometric isomorphism $U : P^2(\mu) \to \mathcal{H}$ such that $Up = p(S)x$ for all $p \in P$. If we set $q(z) = zp(z)$, we also see that

$$US_\mu p = Uq = q(S)x = Sp(S)x = SUp.$$

Therefore $US_\mu = SU$, which implies $S_\mu = U^{-1}SU$. Thus $S_\mu \simeq S$. $\square$

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[1]Here $\mathrm{Bor}(\mathbb{C})$ denotes the collection of all Borel subsets of $\mathbb{C}$. It should be noted that $P^t(\mu)$ is actually the closure of the equivalence classes of polynomials with respect to μ. That is, $P^t(\mu) = \overline{\{[p]_\mu : p \in P\}}^{L^t(\mu)}$, where $[p]_\mu$ is the set of all Borel functions $f : \mathbb{C} \to \mathbb{C}$ such that f and p are same μ almost everywhere. It can happen that $p \neq q$ but $[p]_\mu = [q]_\mu$. Beware, we are careless about this distinction!

3. Brown's Theorem

Lebesgue area measure on $\mathbb{C}$ is denoted by m_2. The space of all compactly supported functions $f : \mathbb{C} \to \mathbb{C}$ with continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is denoted by $C_c^1(\mathbb{C})$. Cauchy-Riemann operator $\overline{\partial}$ is defined by

$$\overline{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Lemma 3.1 (Cauchy). *If $f \in C_c^1(\mathbb{C})$ and $\lambda \in \mathbb{C}$, then*

$$f(\lambda) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\overline{\partial} f(z)}{z-\lambda}\, dm_2(z).$$

Proof. See Rudin [5], Lemma 20.3. [Hint: $\overline{\partial} = \frac{1}{2}e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)$.] □

We will now fix a compactly supported measure $\mu : \mathrm{Bor}(\mathbb{C}) \to [0,\infty)$.

Lemma 3.2. *For every $g \in L^{3/2}(\mu) \setminus \{0\}$ there is $\lambda \in \mathbb{C}$ such that $\mu(\{\lambda\}) = 0$,*

$$\int \left|\frac{g(z)}{z-\lambda}\right|^{3/2} d\mu(z) < \infty,$$

and

$$\int \frac{g(z)}{z-\lambda}\, d\mu(z) \neq 0.$$

Proof. ([4], [2]) We will temporarily set $a/0 = 0$ for all $a \in \mathbb{C}$. Fix $R > 0$ such that $\mathrm{supp}(\mu) \subseteq \mathbb{D}(0,R)$. For every $r > 0$ we have, by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{D}(0,r)} \int \left|\frac{g(z)}{z-\lambda}\right|^{3/2} d\mu(z)\, dm_2(\lambda) &= \int |g(z)|^{3/2} \int_{\mathbb{D}(0,r)} |z-\lambda|^{-3/2}\, dm_2(\lambda)\, d\mu(z) \\ &\leq \int_{\mathbb{D}(0,r+R)} |\lambda|^{-3/2}\, dm_2(\lambda) \int |g|^{3/2}\, d\mu \\ &= 4\pi(r+R)^{1/2}\, \|g\|_{3/2}^{3/2} < \infty.\end{aligned}$$

Thus $\int \left|\frac{g(z)}{z-\lambda}\right|^{3/2} d\mu(z) < \infty$ for m_2 almost every $\lambda \in \mathbb{C}$. Because $\mu(\{\lambda\}) \neq 0$ holds only for countably many $\lambda \in \mathbb{C}$, we can abandon our temporary adjustment and conclude that for m_2 almost every $\lambda \in \mathbb{C}$ we have $\mu(\{\lambda\}) = 0$ and $\int \left|\frac{g(z)}{z-\lambda}\right|^{3/2} d\mu(z) < \infty$.

We will now assume that $\int \frac{g(z)}{z-\lambda}\, d\mu(z) = 0$ for m_2 almost every $\lambda \in \mathbb{C}$. It suffices to show that this leads to a contradiction. Suppose that $f \in C_c^1(\mathbb{C})$. Previous lemma shows that $f(z) = \frac{1}{\pi}\int \frac{\overline{\partial} f(\lambda)}{z-\lambda}\, dm_2(\lambda)$ for all $z \in \mathbb{C}$. Because

$$\int\int \left|\frac{\overline{\partial} f(\lambda)}{z-\lambda}\, g(z)\right| dm_2(\lambda)\, d\mu(z) < \infty,$$

we have, by using Fubini's theorem, that

$$\begin{aligned}\int fg\,d\mu &= \int\left(\frac{1}{\pi}\int\frac{\bar\partial f(\lambda)}{z-\lambda}\,dm_2(\lambda)\right)g(z)\,d\mu(z)\\ &= \frac{1}{\pi}\int\bar\partial f(\lambda)\left(\int\frac{g(z)}{z-\lambda}\,d\mu(z)\right)dm_2(\lambda)\\ &= 0.\end{aligned}$$

Since $C_c^1(\mathbb{C})$ is dense in $L^3(\mu)$, we have $g = 0$. [2] This is a contradiction. □

The proofs of the following theorems are from J. E. Thomson's article [6].

Theorem 3.3. *If* $P^2(\mu) \neq L^2(\mu)$*, we can find a point* $\lambda \in \mathbb{C}$ *and vectors* $x \in P^2(\mu)$ *and* $y \in L^2(\mu)$ *such that*

$$p(\lambda) = (px\,|\,y)$$

for all $p \in P$ *and* $\mu(\{\lambda\}) = 0$.

Proof. ([6]) By assumption there exists $g \in L^2(\mu) \setminus \{0\}$ such that $\overline{g} \perp P^2(\mu)$. Because $g \in L^{3/2}(\mu) \setminus \{0\}$, we can choose λ as in Lemma 3.2. By scaling g we can assume that

$$\int\frac{g(z)}{z-\lambda}\,d\mu(z) = 1.$$

Let $\phi \in (P^3(\mu))^*$ be the functional given by

$$\phi(f) = \int f(z)\frac{g(z)}{z-\lambda}\,d\mu(z).$$

Suppose that $p \in P$. If we set $f(z) = p(z) - p(\lambda)$, we can find $q \in P$ such that $f(z) = (z-\lambda)q(z)$. Because $\phi(1) = 1$ and $\overline{g} \perp P^2(\mu)$, we have

$$\phi(p) - p(\lambda) = \phi(f) = \int(z-\lambda)q(z)\frac{g(z)}{z-\lambda}\,d\mu(z) = (q\,|\,\overline{g}) = 0.$$

Thus $p(\lambda) = \phi(p)$ for all $p \in P$.

By Hahn-Banach theorem there is $h \in L^{3/2}(\mu)$ such that $\|h\|_{3/2} = \|\phi\|$ and $\phi(f) = \int fh\,d\mu$ for all $f \in P^3(\mu)$. We will now factorize h into $x\overline{y}$.

Because $L^3(\mu)$ is reflexive, Banach-Alaoglu theorem implies that the closed unit ball of $P^3(\mu)$ is weakly compact. Since ϕ is weakly continuous, there exists $x \in P^3(\mu) \subseteq P^2(\mu)$ such that $\|x\|_3 = 1$ and $\phi(x) = \|\phi\|$. Using Hölder's inequality we now have

$$\|h\|_{3/2} = \|\phi\| = \phi(x) = \int xh\,d\mu \leq \int |x|\,|h|\,d\mu \leq \|x\|_3\,\|h\|_{3/2} = \|h\|_{3/2}.$$

Because of equality in Hölder's inequality, there exists $\alpha > 0$ such that [3]

$$|x|^3 = \alpha\,|h|^{3/2}.$$

[2] By using mollifiers we can show that $C_c^1(\mathbb{C})$ is dense in $C_c(\mathbb{C})$ with $\|\cdot\|_\infty$-norm. Now combine this to the fact that $C_c(\mathbb{C})$ is dense in $L^3(\mathbb{C})$. (See Rudin [5], Thm. 3.14.)

[3] This holds for equivalence classes. We will choose representative functions for h and x, so that this will hold exactly.

Therefore $|x|^2 = \alpha^{2/3}\,|h|$. Let $y : \mathbb{C} \to \mathbb{C}$ be such that

$$y(z) = \begin{cases} \overline{h(z)/x(z)} & \text{if } x(z) \neq 0 \\ 0 & \text{if } x(z) = 0. \end{cases}$$

We now have $y \in L^2(\mu)$ because $h \in L^{3/2}(\mu) \subseteq L^1(\mu)$ and

$$\int |y|^2\, d\mu = \int \alpha^{-2/3}|h|\, d\mu < \infty.$$

Moreover $h = x\overline{y}$. Summing all up, we have proved that

$$p(\lambda) = \phi(p) = \int ph\, d\mu = \int px\overline{y}\, d\mu = (px \,|\, y)$$

for every $p \in P$. □

Theorem 3.4 (Brown [3]). *If* $\dim(\mathcal{H}) \geq 2$ *and* $S \in \mathcal{B}(\mathcal{H})$ *is subnormal, then* S *has a nontrivial invariant subspace.*

Proof. ([6]) By Theorem 2.2. we can assume that S is cyclic. Bram's theorem now gives us a measure μ such that $S \simeq S_\mu$, so it suffices to consider S_μ.

$\boxed{P^2(\mu) = L^2(\mu)}$ Because $\dim(\mathcal{H}) \geq 2$, we can choose $B \in \mathrm{Bor}(\mathbb{C})$ such that $\mu(B) > 0$ and $\mu(\mathbb{C} \setminus B) > 0$. Thus S_μ has a nontrivial invariant subspace

$$\mathcal{M} = \{\chi_B f : f \in L^2(\mu)\}.$$

$\boxed{P^2(\mu) \neq L^2(\mu)}$ Choose λ, x, and y as in Theorem 3.3. and define

$$\mathcal{M} = \overline{\{px : p \in P \text{ and } p(\lambda) = 0\}}.$$

Fix $p \in P$ such that $p(\lambda) = 0$. For polynomial $q(z) = zp(z)$ we also have $q(\lambda) = 0$, so $S_\mu px = qx \in \mathcal{M}$. Thus $\mathcal{M}$ is an invariant subspace of S_μ. Because $(x \,|\, y) = 1$ and $(px \,|\, y) = 0$ whenever $p(\lambda) = 0$, we have $x \notin \mathcal{M}$. Therefore $\mathcal{M} \neq P^2(\mu)$. Because $x \neq 0$ and $\mu(\{\lambda\}) = 0$, we have $px \neq 0$ for the particular polynomial $p(z) = z - \lambda$. Hence $\mathcal{M} \neq \{0\}$. □

The constructions made in the last two proofs are related to the so called *bounded point evaluation problem*. For further reading see Thomson [7] and Conway [4], and search for J. E. Brennan's contributions.

References

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