Orlicz spaces as a generalization of L^p spaces

Sauli Lindberg

When $1 \leq p < \infty$, we can define $\Phi : [0, \infty) \to [0, \infty)$ by $\Phi(t) = t^p$ and present the L^p norm of a function $f \in L^p(\mathbb{R}^n)$ in the form

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} = \Phi^{-1} \left(\int_{\mathbb{R}^n} \Phi(|f(x)|)\, dx \right).$$

In this talk we define Orlicz spaces by replacing Φ by a more general function than $\Phi(t) = t^p$. We indicate why we can't generalize L^p norms by simply defining

$$\|f\|_{L^\Phi} \equiv \Phi^{-1} \left(\int_{\mathbb{R}^n} \Phi(|f(x)|)\, dx \right)$$

for suitable Φ. We show why it is, instead, natural to use the rather odd-looking Luxemburg functional given by

$$\|f\|_{L^\Phi} \equiv \inf \left\{ A > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{A} \right) dx \leq 1 \right\}. \tag{1}$$

When Φ is convex and increasing, $\Phi(0) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$, the set

$$\left\{ f : \mathbb{R}^n \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{A} \right) dx \leq 1 \text{ for some } A > 0 \right\}$$

becomes a Banach space when endowed with the norm given by (1).

As a special case we treat $\Phi(t) = t \log^+ t$, where $\log^+ t = \max\{0, \log t\}$. We show that when $f : \mathbb{R}^n \to \mathbb{R}$ is measurable and supported on a compact set $K \subset \mathbb{R}^n$ and Mf is the Hardy-Littlewood maximal function of f, the uniform estimate

$$\|Mf\|_{L^1(K)} \leq C_K \|f\|_{L^\Phi(K)}$$

holds.

The talk will be given in English unless all the participants of the seminar are Finnish speaking.