12. In the following epidemiological model $S$ denotes a healthy individual (a so-called susceptible) and $I$ an infected individual, and the processes on the individual level are modeled by the following reaction network of monomolecular and bimolecular reaction:

$$\begin{align*}
S + I & \xrightarrow{\alpha} 2I \quad \text{(transmission)} \\
I & \xrightarrow{\beta} S \quad \text{(recovery)} \\
S & \xrightarrow{\lambda} 2S \quad \text{(reproduction)} \\
S & \xrightarrow{\mu} \dagger \quad \text{(death)} \\
I & \xrightarrow{\nu} \dagger \quad \text{(death)}
\end{align*}$$

with $\nu > \mu$. (a) Give the corresponding differential equations for the population densities $S$ and $I$. (b) Perform a complete phase-plane analysis. (c) Use local stability analysis where the phase-plane analysis is inconclusive.

13. The following model is similar to the previous but has an extra $i$-state $R$ of resistant individuals who after some time become susceptible again:

$$\begin{align*}
S + I & \xrightarrow{\alpha} 2I \quad \text{(transmission)} \\
I & \xrightarrow{\beta} R \quad \text{(recovery)} \\
R & \xrightarrow{\gamma} S \quad \text{(loss of immunity)} \\
S & \xrightarrow{\lambda} 2S \quad \text{(reproduction)} \\
R & \xrightarrow{\lambda} R + S \quad \text{(reproduction)} \\
S & \xrightarrow{\mu} \dagger \quad \text{(death)} \\
R & \xrightarrow{\mu} \dagger \quad \text{(death)} \\
I & \xrightarrow{\nu} \dagger \quad \text{(death)}
\end{align*}$$
with \( \nu > \mu \). (a) Give the corresponding differential equations for the population densities of S, I and R. (b) Show that if transmission, recovery and loss of immunity are fast processes, the system can be written as a fast-slow system where total population density of S, I and R together is a slow variable and S and I are fast variables. (c) Give a phase-plane analysis and (if necessary) a local stability analysis of the 2-dimensional fast system for S and I. (d) Use Bendixon’s theorem to show that there are no limit cycles in the fast system. (e) Analyze the slow dynamics of the total population size.

14. Numerically solve the orbits of the 3-dimensional system in exercise 13. For different magnitudes of the parameters \( \alpha \), \( \beta \) and \( \gamma \) project the orbits in the \((S, I)\)-plane of the fast dynamics and also plot the total population density as a function of time.