Delay-differential equations; Local stability analysis.

Consider the equation
\[ \dot{n} = f(n, n_T) \]
where \( n_T(t) = u(t-T), T > 0. \)
Suppose \( u(t) = \bar{u} \) (constant) is an equilibrium of (1):
\[ 0 = f(\bar{u}, \bar{u}) \]

Linearization (= 1st order Taylor approx) about the equilibrium gives:
\[ \dot{\bar{n}} = a u + b u_T \]
where \( u = n - \bar{u} \) and \( u_T = n_T - \bar{u}, \) and
\[ a = \partial_1 f(\bar{u}, \bar{u}) \] and \[ b = \partial_2 f(\bar{u}, \bar{u}) \]
Substitute \( u(t) = e^{2t} \) to get the characteristic equation:
\[ \lambda = a + b e^{-2T} \]
where \( \lambda \) is an eigenvalue.
Write \( z = \mu + iw \) and separate the real and imaginary parts of the characteristic equation:

\[
\begin{align*}
\mu T &= aT + bT e^{-\mu T} \cos \omega T \\
\omega T &= -bT e^{-\mu T} \sin \omega T
\end{align*}
\]

**Real eigenvalues**

Substitute \( \omega T = 0 \) into (3) and rewrite as follows:

\[
aT = \mu T - bT e^{-\mu T}
\]

**Case: \( bT < 0 \)**

- \( aT \)
- \( aT - \mu T \)
- \( \mu T \)
- \( \log(-bT) \)

\[
1 + \log(-bT) \quad \mu T
\]

**Case: \( bT \geq 0 \)**

- \( aT \)
- \( aT - \mu T \)
- \( \mu T \)

\[
\Rightarrow \text{There exist no real eigenvalues if and only if } bT < 0 \text{ and } aT < 1 + \log(-bT)
\]
Graphically, in the \((aT, bT)\)-plane:

- Real eigenvalues exist.
- All eigenvalues have a non-zero imaginary part.

\[ aT = 1 + \log(-bT) \]

**Positive (real) eigenvalues**

1. Case: \(bT < 0 \& \log(-bT) < 0\)

Positive eigenvalue exists if and only if

\[ aT > -bT \]
case: $bT < 0 \quad \& \quad \log(-bT) > 0$

$$\begin{align*}
\text{Pos. eq. val. exists if and only if} \\
aT > 1 + \log(-bT)
\end{align*}$$

case: $bT > 0$

$$\begin{align*}
\text{Pos. eq. val. exists if and only if} \\
aT > -bT
\end{align*}$$
Graphically, in the \((aT, bT)\) plane,

- **Case III**
  - \(bT = -aT\)
  - Positive real eigenvalue exists.
  - Negative eigenvalue exists.
  - No real eigenvalues exist.

- \(aT = 1 + \log(-bT)\)

Note that if positive (real) eigenvalues exist, then the equilibrium is unstable.

**Complex eigenvalues**

Crossing the imaginary axis at \(pT = 0\):

\[
0 = aT + bT \cos \omega T
\]

\[
\omega T = -bT \sin \omega T
\]
Solve 4 for \((aT, bT)\):
\[
\begin{align*}
aT &= \omega T \cot \omega T \\
bT &= -\omega T / \sin \omega T.
\end{align*}
\]
Parameterized curves in the \((aT, bT)\)-plane.

For any \((aT, bT)\) on one of the solid curves, at least one eigenvalue has a zero real part.

Which of these curves correspond to the dominant eigenvalue?
Write the char. eqn. (page 3) as

\( q_a \) \hspace{1cm} \mu T = aT - \omega T \cot \alpha \omega T \hspace{1cm} \text{and}

\( q_b \)

\[
\mu T = \begin{cases} 
\log bT + \log \left( -\frac{\tan \omega T}{\omega T} \right) & \text{for } bT > 0 \text{ and } \frac{(2k-1)\pi}{2} < |\omega T| < 2k\pi \\
\log bT + \log \left( \frac{\tan \omega T}{\omega T} \right) & \text{for } bT < 0 \text{ and } 2k\pi < |\omega T| < (2k+1)\pi
\end{cases}
\]

for \( k = 0, 1, 2, \ldots \).

The graph of \( q_a \):

\[\text{To see where the eigenvalues are, this graph has to be intersected with the graph of } q_b\;\text{.}\]
Graph of $y_A$ intersected with graph of $y_B$, case $bT<0$:

The dominant eigenvalues (i.e., with the largest real part) are found in the strip $|\omega T|<\pi$.
Complete linear stability bifurcation plot:

\[ bT = aT \]

stable, (exponential convergence)

unstable (exponential divergence)

stable (oscillatory convergence)

unstable (oscillatory divergence)

\[ aT = 1 + \log|bT| \]

\[ \omega T = -\omega T \text{ cotan} \omega T \]

\[ \omega T = -\omega T / \sin \omega T \] (for \( |\omega T| < \pi \))

\[ e^\omega \quad -1 \quad -\pi/2 \]

\[ \nu \rightarrow \]

exp. convergence

exp. divergence

osc. convergence

osc. divergence
Example (prev. lecture, page 5):

\[ N = \frac{2\beta F(T)}{\gamma} (S_0 - N) N_T - \delta N =: g(N, N_T) \]

\( \lambda \): reproduction rate of territory owners

\( \beta \): territory colonization rate

\( S \): death rate territory owner

\( \gamma \): "" free individual

\( F(T) \): survival probability from birth to maturation

\( S_0 \): territory density

Equilibria:

\[ N = 0 \quad \text{and} \quad \bar{N} = S_0 = \frac{\partial g}{2\beta F(T)} > 0 \]

\( a := \partial g(\bar{N}, \bar{N}) = -\frac{2\beta F(\bar{T})}{\gamma} \quad S_0 < 0 \)

\( b := \partial^2 g(\bar{N}, \bar{N}) = \delta > 0 \)

Conclusions:

\( S \) is stable whenever \( a \) is positive.