TCM315 Fall 2022: Introduction to Open Quantum Systems

Lecture 18: Quantum trajectories driven by white noise

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CONTENTS

1. INTRODUCTION

Unraveling of the Lindblad-Gorini-Kossakowski-Sudarshan completely positive equation by Wiener process driven stochastic differential equations is an example of **quantum state diffusion**. A nice physics style introduction to quantum state diffusion is [10]. Chapter 4 in particular is a source for the present notes. A mathematically more advanced presentation is the monograph [1].

2. UNRAVELING BY WIENER NOISE

We take as starting point the Lindblad–Gorini–Kossakowski–Sudarshan equation

$$\dot{\boldsymbol{\rho}}_t = -\imath\,[\mathbb{H}\,,\boldsymbol{\rho}_t] + \sum_{a=1}^{\mathscr{A}} \mathscr{a}_{a,t}\left(\mathbb{A}_a\boldsymbol{\rho}_t\mathbb{A}_a^{\dagger} - \frac{\mathbb{A}_a^{\dagger}\mathbb{A}_a\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\mathbb{A}_a^{\dagger}\mathbb{A}_a}{2}\right) \tag{1}$$

where for each $a = 1, \dots, \mathscr{A}$ and time t complete positivity requires

$$\mathscr{a}_{a,t} \geq 0$$

Our aim is to construct an Itô stochastic differential equation satisfied by the realizations of the state vector process $\{\boldsymbol{\psi}_t\}_{t\geq 0}$ such that

$$\boldsymbol{\rho}_t = \mathrm{E}\,\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger} \tag{2}$$

2.1. Complex Wiener noise

To our goal we posit that the state vector satisfies an **Itô stochastic differential equation** of the form

$$\mathrm{d}\boldsymbol{\psi}_t = \boldsymbol{f}_t\mathrm{d}t + \sum_{a=1}^{\mathscr{A}} \boldsymbol{g}_{a,t}\mathrm{d}w_{a,t} \tag{3}$$

$\{w_{a,t}\}_{t\geq 0}$ with $a = 1, \dots, \mathscr{A}$ are a collection of a complex-valued Wiener processes. We determine the vector-valued drift $\{\boldsymbol{f}_t\}_{t\geq 0}$ and diffusion coefficients $\{\boldsymbol{g}_{a,t}\}_{t\geq 0}$ in (3) by requiring that

i (2) holds true,

ii pathwise probability conservation:

$$\|\boldsymbol{\psi}_t\|^2 = 1$$

Satisfying condition **i** entails the evaluation of

$$\mathrm{d}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t^\dagger) = (\mathrm{d}\boldsymbol{\psi}_t)\boldsymbol{\psi}_t^\dagger + \boldsymbol{\psi}_t \mathrm{d}\boldsymbol{\psi}_t^\dagger + (\mathrm{d}\boldsymbol{\psi}_t)\mathrm{d}\boldsymbol{\psi}_t^\dagger \tag{4}$$

which yields

$$\mathrm{E}\,\mathrm{d}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t^\dagger) = \mathrm{E}\left(\boldsymbol{f}_t\boldsymbol{\psi}_t^\dagger + \boldsymbol{\psi}_t\boldsymbol{f}_t^\dagger + \sum_{a=1}^{\mathscr{A}} \boldsymbol{g}_{a,t}\boldsymbol{g}_{a,t}^\dagger\right)\mathrm{d}t \tag{5}$$

Similarly **ii** imposes

$$0 = \mathrm{d}(\boldsymbol{\psi}_t^\dagger\boldsymbol{\psi}_t) = \left(\boldsymbol{\psi}_t^\dagger\boldsymbol{f}_t + \boldsymbol{f}_t^\dagger\boldsymbol{\psi}_t + \sum_{a=1}^{\mathscr{A}} \|\boldsymbol{g}_{a,t}\|^2\right)\mathrm{d}t + \sum_{a=1}^{\mathscr{A}}\left(\boldsymbol{\psi}_t^\dagger\boldsymbol{g}_{a,t}\mathrm{d}w_{a,t} + \mathrm{d}\bar{w}_{a,t}\boldsymbol{g}_{a,t}^\dagger\boldsymbol{\psi}_t\right) \tag{6}$$

	$\mathrm{d}t$	$\mathrm{d}w_{a,t}$	$\mathrm{d}\bar{w}_{a,t}$	$\mathrm{d}w_{b,t}$	$\mathrm{d}\bar{w}_{b,t}$
$\mathrm{d}t$	0	0	0	0	0
$\mathrm{d}w_{a,t}$	0	0	$\mathrm{d}t$	0	0
$\mathrm{d}\bar{w}_{a,t}$	0	$\mathrm{d}t$	0	0	0
$\mathrm{d}w_{b,t}$	0	0	0	0	$\mathrm{d}t$
$\mathrm{d}\bar{w}_{b,t}$	0	0	0	$\mathrm{d}t$	0

TABLE I: Differential table for $\mathrm{d}w_{a,t}$, $\mathrm{d}w_{b,t}$ and corresponding complex conjugate Wiener processes $a \neq b$. Increments at time t are independent from state of the system and of Wiener processes at time t. Note the normalization of the complex noise: $(\mathrm{Re}\,\mathrm{d}w_{a,t})^2 + (\mathrm{Im}\,\mathrm{d}w_{a,t})^2 = \mathrm{d}t$

We readily infer from (6) the conditions

$$\boldsymbol{f}_t = -\imath\,\mathbb{H}\boldsymbol{\psi}_t - \frac{1}{2}\left(\boldsymbol{\psi}_t\sum_{a=1}^{\mathscr{A}}\|\boldsymbol{g}_{a,t}\|^2 + \boldsymbol{g}_{0,t}\right) \tag{7}$$

and

$$\boldsymbol{\psi}_t^\dagger\boldsymbol{g}_{a,t} = 0 \qquad \textbf{for } a = 0, \dots, \mathscr{A} \tag{8}$$

From the geometric point of view, the set of conditions (8) require the state vector to be orthogonal with respect to the inner product in $\mathcal{H}$ to any stochastic increment.

$$\mathrm{E}\,\mathrm{d}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t^\dagger) = \mathrm{E}\left(-\imath\,\mathbb{H}\boldsymbol{\psi}_t\boldsymbol{\psi}_t^\dagger + \imath\,\boldsymbol{\psi}_t\boldsymbol{\psi}_t^\dagger\mathbb{H} - \frac{1}{2}\left(\sum_{a=1}^{\mathscr{A}}\boldsymbol{\psi}_t\,\|\boldsymbol{g}_{a,t}\|^2 + \boldsymbol{g}_{0,t}\right)\boldsymbol{\psi}_t^\dagger - \frac{1}{2}\boldsymbol{\psi}_t\left(\sum_{a=1}^{\mathscr{A}}\boldsymbol{\psi}_t^\dagger\,\|\boldsymbol{g}_{a,t}\|^2 + \boldsymbol{g}_{0,t}^\dagger\right) + \sum_{a=1}^{\mathscr{A}}\boldsymbol{g}_{a,t}\boldsymbol{g}_{a,t}^\dagger\right)\mathrm{d}t \tag{9}$$

In order to recover (1), we fulfill (8) by setting

$$\boldsymbol{g}_{0,t} = \sum_{a=1}^{\mathscr{A}} \alpha_{a,t}\left(\mathbb{A}_a^\dagger\mathbb{A}_a - 2\,(\boldsymbol{\psi}_t^\dagger\mathbb{A}_a^\dagger\boldsymbol{\psi}_t)\mathbb{A}_a - (\|\mathbb{A}_a\boldsymbol{\psi}_t\|^2 - 2\,|\boldsymbol{\psi}_t^\dagger\mathbb{A}_a\boldsymbol{\psi}_t|^2)\mathbb{1}\right)\boldsymbol{\psi}_t$$

$$\boldsymbol{g}_{a,t} = \sqrt{\alpha_{a,t}}\left(\mathbb{A}_a - \boldsymbol{\psi}_t^\dagger\mathbb{A}_a\boldsymbol{\psi}_t\mathbb{1}\right)\boldsymbol{\psi}_t \qquad a = 1, \dots, \mathscr{A}$$

We then verify that

$$\|\boldsymbol{g}_{a,t}\|^2 = \alpha_{a,t}\,\boldsymbol{\psi}_t^\dagger\left(\mathbb{A}_a^\dagger - (\boldsymbol{\psi}_t^\dagger\mathbb{A}_a\boldsymbol{\psi}_t)\mathbb{1}\right)\left(\mathbb{A}_a - (\boldsymbol{\psi}_t^\dagger\mathbb{A}_a\boldsymbol{\psi}_t)\mathbb{1}\right)\boldsymbol{\psi}_t$$
$$= \alpha_{a,t}\left(\|\mathbb{A}_a\boldsymbol{\psi}_t\|^2 - |\boldsymbol{\psi}_t^\dagger\mathbb{A}_a\boldsymbol{\psi}_t|^2\right)$$

and

$$\boldsymbol{g}_{a,t}\boldsymbol{g}_{a,t}^{\dagger} = \alpha_{a,t}\left(\mathbb{A}_a - (\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a\boldsymbol{\psi}_t)\mathbb{1}\right)\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger}\left(\mathbb{A}_a^{\dagger} - (\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a^{\dagger}\boldsymbol{\psi}_t)\mathbb{1}\right)$$
$$= \alpha_{a,t}\left(\mathbb{A}_a\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a^{\dagger} + |\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a\boldsymbol{\psi}_t|^2\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger} - (\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a^{\dagger}\boldsymbol{\psi}_t)\mathbb{A}_a\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger} - (\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a\boldsymbol{\psi}_t)\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a^{\dagger}\right)$$

bring about the necessary cancellation to recover

$$\mathrm{d}\,\mathrm{E}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger}) =$$
$$\left(-\imath\left[\mathbb{H}\,,\mathrm{E}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger})\right] - \frac{1}{2}\sum_{a=1}^{\mathscr{A}}\alpha_{a,t}\left(\mathbb{A}_a^{\dagger}\mathbb{A}_a\,\mathrm{E}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger}) - \frac{1}{2}\,\mathrm{E}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t)\mathbb{A}_a^{\dagger}\mathbb{A}_a + \mathbb{A}_a\,\mathrm{E}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger})\mathbb{A}_a^{\dagger}\right)\right)\mathrm{d}t$$

2.1.1. Stochastic Schrödinger equation driven by a Wiener process

We conclude that the Itô stochastic differential equation

$$\mathrm{d}\boldsymbol{\psi}_t =$$
$$-\left(\imath\,\mathbb{H} + \frac{1}{2}\sum_{a=1}^{\mathscr{A}}\alpha_{a,t}\left(\mathbb{A}_a^{\dagger}\mathbb{A}_a - 2\,(\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a^{\dagger}\boldsymbol{\psi}_t)\,\mathbb{A}_a + |\boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a\boldsymbol{\psi}_t|^2\mathbb{1}\right)\right)\boldsymbol{\psi}_t\,\mathrm{d}t + \sum_{a=1}^{\mathscr{A}}\mathrm{d}w_{a,t}\sqrt{\alpha_{a,t}}\left(\mathbb{A}_a - \boldsymbol{\psi}_t^{\dagger}\mathbb{A}_a\boldsymbol{\psi}_t\mathbb{1}\right)\boldsymbol{\psi}_t \quad (10)$$

describes **an** unraveling in terms of **continuous random paths** of (1).

Remark. *The unraveling above is **not unique**. Averaging over the realizations of the solutions of stochastic differential equation (10) allows us to construct via (2) a the **unique solution** of (1) for any given initial data. The construction itself, however, is non unique as we can find other stochastic differential equations generating random state vectors with the same property.*

* *

2.2. Comparison with unraveling by Poisson noise

We can adopt the same strategy to derive an unraveling of (1) by a stochastic differential equation driven by increments of the Poisson noise. In this latter case we surmise

$$\mathrm{d}\boldsymbol{\psi}_t = \boldsymbol{f}_t\mathrm{d}t + \sum_{a=1}^{\mathscr{A}}\boldsymbol{g}_{a,t}\mathrm{d}\nu_{a,t}$$

with

$$\mathrm{d}\nu_{a,t}\mathrm{d}\nu_{b,t} = \delta_{ab}\mathrm{d}\nu_{a,t}$$
$$\mathrm{E}\left(\mathrm{d}\nu_{a,t}\middle|\boldsymbol{\psi}_t\right) = r_a(\boldsymbol{\psi}_t)\,\mathrm{d}t$$

we obtain by retaining second variation terms as in (4) the conditions

$$\mathrm{E}\,\mathrm{d}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger}) = \mathrm{E}\left(\boldsymbol{f}_t\boldsymbol{\psi}_t^{\dagger} + \boldsymbol{\psi}_t\boldsymbol{f}_t^{\dagger} + \sum_{a=1}^{\mathscr{A}}r_a(\boldsymbol{\psi}_t)\left(\boldsymbol{g}_{a,t}\boldsymbol{\psi}_t^{\dagger} + \boldsymbol{\psi}_t\boldsymbol{g}_{a,t}^{\dagger} + \boldsymbol{g}_{a,t}\boldsymbol{g}_{a,t}^{\dagger}\right)\right)\mathrm{d}t \quad (11)$$

and

$$0 = \mathrm{d}(\boldsymbol{\psi}_t^{\dagger}\boldsymbol{\psi}_t) = \mathrm{d}t\left(\boldsymbol{f}_t^{\dagger}\boldsymbol{\psi}_t + \boldsymbol{\psi}_t^{\dagger}\boldsymbol{f}_t\right) + \sum_{a=1}^{\mathscr{A}}\mathrm{d}\nu_{a,t}(\boldsymbol{g}_{a,t}^{\dagger}\boldsymbol{\psi}_t + \boldsymbol{\psi}_t^{\dagger}\boldsymbol{g}_{a,t} + \boldsymbol{g}_{a,t}^{\dagger}\boldsymbol{g}_{a,t}) \quad (12)$$

From (12) we get the conditions

$$\boldsymbol{f}_t = -\imath\,\mathbb{H}\,\boldsymbol{\psi}_t + \boldsymbol{v}_{0,t} \quad (13a)$$
$$\boldsymbol{\psi}_t^{\dagger}\boldsymbol{v}_{0,t} = 0 \quad (13b)$$

and

$$\boldsymbol{g}_{a,t} = \boldsymbol{v}_{a,t} - \boldsymbol{\psi}_t \tag{14a}$$

$$\|\boldsymbol{v}_{a,t}\|^2 = 1 \qquad a = 1, \dots, \mathscr{A} \tag{14b}$$

We fulfill (13b) by setting

$$\boldsymbol{v}_{0,t} = -\frac{1}{2}\sum_{a=1}^{\mathscr{A}} \alpha_{a,t}\left(\mathbb{A}_a^\dagger \mathbb{A}_a - \|\mathbb{A}_a \boldsymbol{\psi}_t\|^2 \mathbb{1}\right)\boldsymbol{\psi}_t$$

(14b) by setting

$$\boldsymbol{v}_{a,t} = \frac{\mathbb{A}_a \boldsymbol{\psi}_t}{\|\mathbb{A}_a \boldsymbol{\psi}_t\|}$$

Finally we have the freedom to choose the rates of the Poisson increments

$$r_a(\boldsymbol{\psi}_t) = \alpha_{a,t}\, \|\mathbb{A}_a \boldsymbol{\psi}_t\|^2$$

These choices recover (1) and yield

$$\mathrm{d}\boldsymbol{\psi}_t = -\mathrm{d}t\left(\imath\,\mathbb{H} + \frac{1}{2}\sum_{a=1}^{\mathscr{A}} \alpha_{a,t}\left(\mathbb{A}_a^\dagger \mathbb{A}_a - \|\mathbb{A}_a \boldsymbol{\psi}_t\|^2\, \mathbb{1}\right)\right)\boldsymbol{\psi}_t + \sum_{a=1}^{\mathscr{A}} \mathrm{d}\nu_{a,t}\left(\frac{\mathbb{A}_a}{\|\mathbb{A}_a \boldsymbol{\psi}_t\|} - \mathbb{1}\right)\boldsymbol{\psi}_t \tag{15}$$

3. WIENER QUANTUM TRAJECTORIES AS LIMIT OF POISSON JUMPS

We now surmise that the collection of Lindblad (collapse) operators $\{\mathbb{A}_a\}_{a=1}^{\mathscr{A}}$ are amenable to the form

$$\mathbb{A}_a = \frac{1}{\varepsilon}\mathbb{1} + \mathbb{B}_a \tag{16a}$$

$$\mathbb{B}_a^\dagger = \mathbb{B}_a \tag{16b}$$

where

$$\varepsilon \ll 1$$

is a non-dimensional control parameter. Note that (16b) naturally holds true if we identify the $\mathbb{B}_a$'s with projectors on the eigenstates of a dynamical variable.

Plugging (16) into (15) yields

$$\begin{aligned}\mathrm{d}\boldsymbol{\psi}_t &= -\mathrm{d}t\, \imath\, \mathbb{H}\, \boldsymbol{\psi}_t \\ &+ \sum_{a=1}^{\mathscr{A}}\left(-\frac{\mathrm{d}t}{2}\alpha_{a,t}\left(\frac{\mathbb{1}}{\varepsilon^2} + \frac{2}{\varepsilon}\mathbb{B}_a + \mathbb{B}_a^2 - \frac{\|(\mathbb{1} + \varepsilon\,\mathbb{B}_a)\,\boldsymbol{\psi}_t\|^2}{\varepsilon^2}\mathbb{1}\right) + \mathrm{d}\nu_{a,t}\left(\frac{\mathbb{1} + \varepsilon\,\mathbb{B}_a}{\|(\mathbb{1} + \varepsilon\,\mathbb{B}_a)\,\boldsymbol{\psi}_t\|} - \mathbb{1}\right)\right)\boldsymbol{\psi}_t\end{aligned} \tag{17}$$

Our goal is now to recover from a formal expansion in powers of ε a Wiener noise driven stochastic Schrödinger equation.

3.1. Martingale decomposition of Poisson increments

Wiener noise enjoys the **martingale property**

$$\mathrm{E}\left(\mathrm{d}w_t \,|\, w_t\right) = \mathrm{E}\left(\mathrm{d}w_t\right) = 0$$

In order to approximate a Wiener process by high frequence jump process it is preliminary expedient to “de-trend” the jump process. In stochastic analysis this operation is called Doob-Meyer decomposition. The decomposition consists in writing increments of Poisson processes as

$$\mathrm{d}\nu_{a,t} = r_a(\boldsymbol{\psi}_t)\,\mathrm{d}t + \mathrm{d}\mu_{a,t}$$

where on the right hand side there appear

- the **compensator** (predictable increment) characterized by the property

$$\mathrm{E}\left(\mathrm{d}\nu_{a,t}\,|\,\boldsymbol{\psi}_t\right)=r_a(\boldsymbol{\psi}_t)\,\mathrm{d}t$$

- the martingale component with conserved vanishing expectation value

$$\mathrm{E}\left(\mathrm{d}\mu_{a,t}\,|\,\boldsymbol{\psi}_t\right)=0$$

General results in stochastic analysis guarantee the uniqueness with probability one of the decomposition (see e.g. [8] for more information conveyed in a practitioner attuned presentation).

Physically the compesator describes the (conditional) mean instantaneous drift induced by the presence of jumps.

We derive the statistical properties of the martingale components observing that

$$\begin{aligned}&\delta_{ab}\,\mathrm{d}\nu_{a,t}=\mathrm{d}\nu_{a,t}\mathrm{d}\nu_{b,t}=\\&\big(r_a(\boldsymbol{\psi}_t)\,\mathrm{d}t+\mathrm{d}\mu_{a,t}\big)\big(r_b(\boldsymbol{\psi}_t)\,\mathrm{d}t+\mathrm{d}\mu_{b,t}\big)=\mathrm{d}\mu_{a,t}\,\mathrm{d}\mu_{b,t}\end{aligned}$$

We conclude that

1. Increments of intependent process simultaneously occur with vanishing probability.
2. powers of increments of the martingale component have a **non-vanishing** conditional expectation fully specified by the **compensator**.

	$\mathrm{d}t$	$\mathrm{d}\mu_{a,t}$	$\mathrm{d}\mu_{b,t}$
$\mathrm{d}t$	0	0	0
$\mathrm{d}\mu_{a,t}$	0	$\mathrm{d}\nu_{a,t}$	0
$\mathrm{d}\mu_{b,t}$	0	0	$\mathrm{d}\nu_{b,t}$

TABLE II: Differential table for the martingale components of two independent Poisson processes $\nu_{a,t}$, and $\nu_{b,t}$ ($a\neq b$). Powers of increments of the martingale components **coincide** with increments of the Poisson process itself.

3.2. Martingale decomposition of the stochastic Schrödinger equation

To shorten the notation we define

$$b_{a,t}=\boldsymbol{\psi}_t^{\dagger}\mathbb{B}_a\boldsymbol{\psi}_t$$

We use the martingale decomposition of Poisson increments with

$$r_a(\boldsymbol{\psi}_t)=\frac{\alpha_{a,t}}{\varepsilon^2}\,\left\|\left(\mathbb{1}+\varepsilon\,\mathbb{B}\right)\boldsymbol{\psi}_t\right\|^2$$

to couch (17) into the form

$$\begin{aligned}\mathrm{d}\boldsymbol{\psi}_t=&-\mathrm{d}t\,\imath\,\mathbb{H}\,\boldsymbol{\psi}_t-\frac{\mathrm{d}t}{2}\sum_{a=1}^{\mathscr{A}}\alpha_{a,t}\left(2\frac{\mathbb{B}_a-b_{a,t}\mathbb{1}}{\varepsilon}+\mathbb{B}_a^2-\left\|\mathbb{B}_a\boldsymbol{\psi}_t\right\|^2\mathbb{1}\right)\boldsymbol{\psi}_t\\&+\sum_{a=1}^{\mathscr{A}}\left(\mathrm{d}t\,\alpha_{a,t}\,\frac{1+2\,\varepsilon\,b_{a,t}+\varepsilon^2\,\left\|\mathbb{B}_a\boldsymbol{\psi}_t\right\|^2}{\varepsilon^2}+\mathrm{d}\mu_{a,t}\right)\left(\frac{\mathbb{1}+\varepsilon\,\mathbb{B}_a}{\sqrt{1+2\,\varepsilon\,b_{a,t}+\varepsilon^2\,\left\|\mathbb{B}_a\boldsymbol{\psi}_t\right\|^2}}-\mathbb{1}\right)\boldsymbol{\psi}_t\end{aligned}$$

We now recall the formula

$$\frac{1+\varepsilon x_1}{\sqrt{1+\varepsilon\,2\,x_2+\varepsilon^2\,x_3}}=1+(x_1-x_2)\,\varepsilon+\frac{3x_2^2-2\,x_1\,x_2-x_3}{2}\varepsilon^2+O(\varepsilon^3)\tag{18}$$

and write

$$\begin{aligned}\mathrm{d}\boldsymbol{\psi}_t=&-\mathrm{d}t\,\imath\,\mathbb{H}\,\boldsymbol{\psi}_t-\frac{\mathrm{d}t}{2}\sum_{a=1}^{\mathscr{A}}\alpha_{a,t}\left(2\frac{\mathbb{B}_a-b_{a,t}}{\varepsilon}+\mathbb{B}_a^2-\left\|\mathbb{B}_a\boldsymbol{\psi}_t\right\|^2\mathbb{1}\right)\boldsymbol{\psi}_t\\&+\sum_{a=1}^{\mathscr{A}}\left(\mathrm{d}t\,\alpha_{a,t}\,\frac{1+2\,\varepsilon\,b_{a,t}+\varepsilon^2\,\left\|\mathbb{B}_a\boldsymbol{\psi}_t\right\|^2}{\varepsilon}+\varepsilon\,\mathrm{d}\mu_{a,t}\right)\left(\mathbb{B}_a-b_{a,t}\mathbb{1}-\varepsilon\frac{2\,\mathbb{B}_a b_{a,t}-3\,b_{a,t}^2\mathbb{1}+\left\|\mathbb{B}_a\boldsymbol{\psi}_t\right\|^2\mathbb{1}}{2}+O(\varepsilon^2)\right)\boldsymbol{\psi}_t\end{aligned}$$

Straightforward algebra allows us to reorder the expansion in ε of the drift:

$$\begin{aligned}\frac{1+2\,\varepsilon\, b_{a,t}}{\varepsilon}\left(\mathbb{B}_a - b_{a,t}\mathbb{1} - \varepsilon\frac{2\,\mathbb{B}_a b_{a,t} - 3\, b_{a,t}^2\mathbb{1} + \|\mathbb{B}_a \boldsymbol{\psi}_t\|^2\,\mathbb{1}}{2}\right) + O(\varepsilon)\\ = \frac{\mathbb{B}_a - b_{a,t}}{\varepsilon} + \mathbb{B}_a\, b_{a,t} - \frac{b_{a,t}^2 + \|\mathbb{B}_a \boldsymbol{\psi}_t\|^2}{2}\mathbb{1} + O(\varepsilon)\end{aligned}$$

We thus obtain

$$\mathrm{d}\boldsymbol{\psi}_t = -\left(\imath\,\mathbb{H} + \frac{1}{2}\sum_{a=1}^{\mathscr{A}} \alpha_{a,t}\left(\mathbb{B}_a - (\boldsymbol{\psi}_t^{\dagger}\mathbb{B}_a\boldsymbol{\psi}_t)\mathbb{1}\right)^2\right)\boldsymbol{\psi}_t\,\mathrm{d}t + \sum_{a=1}^{\mathscr{A}} \varepsilon\,\mathrm{d}\mu_{a,t}\left(\mathbb{B}_a - \boldsymbol{\psi}_t^{\dagger}\mathbb{B}_a\boldsymbol{\psi}_t\mathbb{1}\right)\boldsymbol{\psi}_t$$

We now want to model the stochastic differential as a **real Wiener process**. This is possible because the ε prefactor of the stochastic differential insures that only the second moment of the noise contributes

$$\begin{aligned}\varepsilon^2\,\mathrm{E}\,(\mathrm{d}\mu_{a,t}\mathrm{d}\mu_{b,t}) &= \varepsilon^2\delta_{ab}\,\mathrm{E}\,(\mathrm{d}\nu_{a,t})\\ &= \alpha_{a,t}\,\delta_{ab}\,\|(\mathbb{1}+\varepsilon\,\mathbb{B})\,\boldsymbol{\psi}_t\|^2\end{aligned}$$

whereas

$$\varepsilon^n\,\mathrm{E}\left(\mathrm{d}\mu_{a,t}^n\right) = \varepsilon^n\,\mathrm{E}\left(\mathrm{d}\mu_{a,t}^{n-2}\mathrm{d}\nu_{a,t}\right) = \varepsilon^n\,\mathrm{E}\,(\mathrm{d}\nu_{a,t}) = O(\varepsilon^{n-2})$$

We therefore set

$$\varkappa\,\mathrm{d}w_{a,t} = \varepsilon\,\mathrm{d}\mu_{a,t}$$

and fix the prefactor $\varkappa$ by requiring

$$\varkappa^2\,\mathrm{E}\,\mathrm{d}w_{a,t}^2 = \alpha_{a,t}\,\mathrm{d}t$$

We obtain in this way the quantum state diffusion equation driven by **a real Wiener process**

	$\mathrm{d}t$	$\mathrm{d}w_{a,t}$
$\mathrm{d}t$	0	0
$\mathrm{d}w_{a,t}$	0	$\mathrm{d}t$

Differential table for a real Wiener process

$$\mathrm{d}\boldsymbol{\psi}_t = -\left(\imath\,\mathbb{H} + \frac{1}{2}\sum_{a=1}^{\mathscr{A}} \alpha_{a,t}\left(\mathbb{B}_a - (\boldsymbol{\psi}_t^{\dagger}\mathbb{B}_a\boldsymbol{\psi}_t)\mathbb{1}\right)^2\right)\boldsymbol{\psi}_t\,\mathrm{d}t + \sum_{a=1}^{\mathscr{A}} \sqrt{\alpha_{a,t}}\,\mathrm{d}w_{a,t}\left(\mathbb{B}_a - \boldsymbol{\psi}_t^{\dagger}\mathbb{B}_a\boldsymbol{\psi}_t\mathbb{1}\right)\boldsymbol{\psi}_t \tag{19}$$

Remark. *We obtain an unraveling driven by a* ***real Wiener process*** *because we do not take into account the possibility of relations that distinct Lindblad (collapse) operators may satisfy. Examples leading to complex Wiener processes as scaling limits of jump processes are discussed in § 6.4 and § 6.5 of [6].*

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4. CONTEXTUALITY OF QUANTUM TRAJECTORIES

> My own conclusion is that today there is no interpretation of quantum mechanics that does not have serious flaws ... In my view, we ought to take seriously the possibility of finding some more satisfactory other theory, to which quantum mechanics is only a good approximation
>
> *"Lectures on Quantum Mechanics" [11] pag. 102*
> *Steven Weinberg*

The possibility to associate to the same Lindblad–Gorini–Kossakowski–Sudarshan equation multiple description in terms of quantum trajectories poses (once again) the problem of interpretation. As pointed out in [12], there are, at least, three distinct way to interpret this result.

I.1 Quantum trajectories are merely **mathematical tool** to compute the solution of the Lindblad–Gorini–Kossakowski–Sudarshan. This is the motivation put forward in [7] one of the pioneering works on the stochastic Schrödinger equation with Poisson noise. The merit of quantum trajectories consists in bringing down computer memory requirements for a system with N states from $O(N^2)$ to $O(N)$ in the evaluation of ensemble averages. Although this motivation for quantum trajectory is certainly valid, the interpretation is too conservative.

I.2 Quantum trajectories are **subjectively real**: their existence and features are determined only **once** a particular measurement scheme has been chosen. In this sense quantum trajectory can be regarded [12] are "*real in so far as the system is a subject of observation*".

I.3 Quantum trajectories are **real**: they are an element of a still missing theory of quantum state reduction. Spontaneous collapse models are physical theory aiming at encompassing in a single dynamical principle without the need of approximations or limit procedures the two distinct dynamical postulates (undisturbed unitary evolution and measurement) assumed in the postulates. We refer to [2, 3] for thorough presentation.

The validity of the point of view I.**2** is corroborated by many convincing experimental evidences see e.g. [9], whereas I.**3** is a possibility which deseves to be further explored.

4.1. Do quantum trajectory contain more information than the quantum master equation?

A further distinction with the classical case is that the solution of the Lindblad–Gorini–Kossakowski–Sudarshan equation coincides with the expectation value of outer product of state vectors (2). At variance with the Fokker-Planck equation the Lindblad–Gorini–Kossakowski–Sudarshan equation is not enough to reconstruct the full statistics generated by the realizations of the solutions of a stochastic Schrödinger equation. The question naturally arises whether such statistics contains extra physical information. From the theoretical point of view [4, 5] showed that , a "Liouville master equation" governing the probability distribution of a system+environment state vector "*uniquely defines in the weak coupling limit a stochastic process on projective Hilbert space of the reduced system*". This result and experimental evidences such as [9] allow us to uphold that at least at weak coupling solving a stochastic Schrödinger equation is not only more computationally efficient but also more informative about the physics emerging from measurements in a definite experimental setup.

* * *

[1] A. Barchielli and M. Gregoratti. *Quantum Trajectories and Measurements in Continuous Time*, volume 782 of *Lecture Notes in Physics*. Springer Berlin Heidelberg, 2009.
[2] A. Bassi and G. Ghirardi. Dynamical reduction models. *Physics Reports*, 379(5-6):257–426, Jun 2003.
[3] A. Bassi, K. Lochan, S. Satin, T. P. Singh, and H. Ulbricht. Models of wave-function collapse, underlying theories, and experimental tests. *Reviews of Modern Physics*, 85(2):471–527, apr 2013, arXiv:1204.4325.
[4] H.-P. Breuer and F. Petruccione. Reduced System Dynamics as a Stochastic Process in Hilbert Space. *Physical Review Letters*, 74(19):3788–3791, May 1995.
[5] H.-P. Breuer and F. Petruccione. Stochastic dynamics of quantum jumps. *Physical Review E*, 52(1):428, July 1995.
[6] H.-P. Breuer and F. Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, reprint edition, 2002.
[7] J. Dalibard, Y. Castin, and K. Mølmer. Wave-function approach to dissipative processes in quantum optics. *Physical Review Letters*, 68(5):580–583, February 1992.
[8] F. C. Klebaner. *Introduction to stochastic calculus with applications*. Imperial College Press, 2 edition, 2005.
[9] Z. K. Minev, S. O. Mundhada, S. Shankar, P. Reinhold, R. Gutiérrez-Jáuregui, R. J. Schoelkopf, M. Mirrahimi, H. J. Carmichael, and M. H. Devoret. To catch and reverse a quantum jump mid-flight. *Nature*, pages 1–14, June 2019, arXiv:1803.00545.
[10] I. C. Percival. *Quantum State Diffusion*. Cambridge University Press, 2003.
[11] S. Weinberg. *Lectures on Quantum Mechanics*. Cambridge University Press, 2015.
[12] H. M. Wiseman. Quantum trajectories and quantum measurement theory. *Quantum and Semiclassical Optics: Journal of the European Optical Society Part B*, 8(1):205–222, Feb 1996.