# TCM315 Fall 2022: Introduction to Open Quantum Systems

## Lecture 16: Unraveling of the completely positive master equation by Poisson processes

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**CONTENTS**



## 1. INTRODUCTION

The scope of these notes is to present a phenomenological derivation of the Lindblad–Gorini–Kossakowski–Sudarshan equation pinpointing the physical intuition behind it. In the same spirit we derive the stochastic Schrödinger equation following [1]. A clear presentation can also be found in § 4.2 of [3].

## 2. ANOTHER LOOK AT THE LINDBLAD–GORINI–KOSSAKOWSKI–SUDARSHAN EQUATION

The simplest way to conceptualize an open quantum system is to think of a **bipartite system**, consisting of the system of interest, referred from now on simply as the system, in interaction with an environment. Stinespring dilation theorem (Krauss representation) states that we can construct any completely positive trace preserving (CPTP) map by composing three three fundamental operations:

- tensor product at an initial time between the state operator of the system and that of the environment,
- a joint unitary evolution,
- partial trace with respect to the environment degrees of freedom.

A tensor product form of the initial value of the state operator

$$\boldsymbol{\rho}_0 = \boldsymbol{\rho}_0^{(S)} \otimes \boldsymbol{\rho}_0^{(E)}$$

is a further **necessary condition** in order to insure that the evolution law of the system as specified by the partial trace is independent of the initial conditions or, in other words, is an universal dynamical map.

These considerations suggest an intuitive picture of the way an environment with a number of degrees of freedom much larger than those of the system affects the dynamics of this latter. Namely, the interaction with the environment is effectively equivalent to a generalized measurement of the state of the system at any instant of time. The **monitoring** of the system by the environment **measures** the state of the system **without recording** the outcomes thus leaving the system state operator in a linear combination of the possible outcomes. More explicitly, if the state of the system at time $t$ is specified by a state operator $\boldsymbol{\rho}_t^{(S)}$ the measurement of $\alpha$ yields

$$\boldsymbol{\rho}_t^{(S,\alpha)} = \frac{\mathbb{M}_\alpha \boldsymbol{\rho}_t^{(S)} \mathbb{M}_\alpha^\dagger}{\operatorname{Tr}\left(\mathbb{M}_\alpha \, \boldsymbol{\rho}_t^{(S)} \, \mathbb{M}_\alpha^\dagger\right)}$$

where

- $\mathbb{M}_\alpha$ is the measurement operator associated with the observation of $\alpha$
- $\mathbb{M}_\alpha^\dagger \mathbb{M}_\alpha$ is the effect or probability operator associated to $\alpha$:

$$\sum_\alpha \mathbb{M}_\alpha^\dagger \mathbb{M}_\alpha = \mathbb{1}_{\mathcal{H}_S} \tag{1}$$

- $\wp_\alpha = \operatorname{Tr}\left\{\mathbb{M}_\alpha \, \boldsymbol{\rho}^{(S)} \, \mathbb{M}_\alpha^\dagger\right\}$ is the probability of observing $\alpha$ if the state of the system is $\boldsymbol{\rho}^{(S)}$ **before** the measurement.

If the measurement time $\tau$ is infinitesimal it is possible to *continuously monitor the system*. Furthermore, **not recording** the monitoring outcomes means that the **after** any generalized measurement the state of the system is the average of the possible outcomes:

$$\boldsymbol{\rho}_{t+\mathrm{d}t}^{(S)} = \sum_\alpha \wp_\alpha \, \boldsymbol{\rho}_t^{(S,\alpha)} = \sum_\alpha \mathbb{M}_\alpha \boldsymbol{\rho}_t^{(S)} \mathbb{M}_\alpha^\dagger \tag{2}$$

Classical master equations governing the evolution of a probability distribution take the form

$$\mathrm{p}(i, t + \mathrm{d}t) = \sum_{j \,\in\, \mathbb{I}} \mathrm{T}(i|j) \, \mathrm{p}(j, t)$$

where the transition probabilities $\mathrm{T}(i|j)$ for $i, j$ ranging in some set $\mathbb{I} \subseteq \mathbb{N}$ satisfy

$$\sum_{i \,\in\, \mathbb{I}} \mathrm{T}(i|j) = 1$$

If we contrast the classical master equation with (2) we see that $\wp_\alpha$ plays a role analogous to that of the transition rate.

The simplest non-trivial measurement admits only two outcomes. Let us define the "null outcome" by associating to it the operation

$$\mathbb{M}_0 = \mathbb{1}_{\mathcal{H}_S} - \left(\imath \, \mathbb{H} + \frac{\mathbb{R}}{2}\right) \mathrm{d}t + o(\mathrm{d}t) \tag{3}$$

with $\mathbb{H}^\dagger = \mathbb{H}$ and $\mathbb{R}^\dagger = \mathbb{R}$ *positive definite*:

$$\mathbb{R} = \mathbb{A}^\dagger \, \mathbb{A} \tag{4}$$

for some linear operator $\mathbb{A}$ acting on the Hilbert space of the system.

**Remark.** *We can always decompose a generic element of $\mathcal{M}_d(\mathbb{C})$ as a linear combination of self-adjoint operators weighed by a purely imaginary coefficient. Essential physics phenomenology (dissipation towards the environment) is embodied by the assumption that $\mathbb{R}$ be positive definite.*

* *

The effect corresponding to the null operation is

$$\mathbb{M}_0^\dagger \mathbb{M}_0 = \left[\mathbb{1}_{\mathcal{H}_S} - \left(-\imath \, \mathbb{H} + \frac{\mathbb{R}}{2}\right) \mathrm{d}t\right] \left[\mathbb{1}_{\mathcal{H}_S} - \left(\imath \, \mathbb{H} + \frac{\mathbb{R}}{2}\right) \mathrm{d}t\right] = \mathbb{1}_{\mathcal{H}_S} - \mathbb{R}\mathrm{d}t + o(\mathrm{d}t) \tag{5}$$

The operator $\mathbb{R}$ models dissipation due to interaction with the environment. We use hypothesis that the environment contains a number of degrees of freedom much larger than those of the system to neglect effects on the environment of the interaction with the system. In order to satisfy (1) for non vanishing $\mathbb{R}$, we need the measurement process to have at least a second outcome. Using the representation (4) we define the "jump" outcome as

$$\mathbb{M}_1 = \mathbb{A}\sqrt{\mathrm{d}t} \tag{6}$$

Within $O(\mathrm{d}t)$ accuracy, this choice enforces effects' completeness relation

$$\mathbb{M}_0^\dagger \mathbb{M}_0 + \mathbb{M}_1^\dagger \mathbb{M}_1 = \mathbb{1}_{\mathcal{H}_S} \tag{7}$$

The evolution of the system under the monitoring of

$$\begin{aligned}\boldsymbol{\rho}_{t+\mathrm{d}t}^{(S)} &= \left[\mathbb{1}_{\mathcal{H}_S} - \left(\imath\,\mathbb{H} + \frac{\mathbb{R}}{2}\right)\mathrm{d}t\right]\boldsymbol{\rho}_t^{(S)}\left[\mathbb{1}_{\mathcal{H}_S} - \left(-\imath\,\mathbb{H} + \frac{\mathbb{R}}{2}\right)\mathrm{d}t\right] + \mathbb{A}\boldsymbol{\rho}_t^{(S)}\mathbb{A}^\dagger\,\mathrm{d}t \\ &= \boldsymbol{\rho}_t^{(S)} - \left(\imath\,[\mathbb{H}\,,\boldsymbol{\rho}_t^{(S)}] + \frac{\mathbb{A}^\dagger\,\mathbb{A}\,\boldsymbol{\rho}_t^{(S)} + \boldsymbol{\rho}_t^{(S)}\,\mathbb{A}^\dagger\,\mathbb{A}}{2} - \mathbb{A}\,\boldsymbol{\rho}_t^{(S)}\,\mathbb{A}^\dagger\right)\mathrm{d}t \equiv \boldsymbol{\rho}_t^{(S)} + \mathbb{L}\boldsymbol{\rho}_t^{(S)}\,\mathrm{d}t\end{aligned} \tag{8}$$

We heuristically derived the Lindblad–Gorini–Kossakowski–Sudarshan master equation.

### 2.1. Case of multiple outcomes

We can immediately extend the foregoing argument to the case of multiple measurement outcomes. We just set

$$\mathbb{M}_0 = \mathbb{1}_{\mathcal{H}_S} - \left(\imath\,\mathbb{H} + \frac{1}{2}\sum_{\alpha=1}^{\mathscr{M}} \mathbb{A}_\alpha^\dagger \mathbb{A}_\alpha\right)\mathrm{d}t$$

and define

$$\mathbb{M}_\alpha = \mathbb{A}_\alpha\sqrt{\mathrm{d}t} \qquad \alpha = 1, \dots, \mathscr{M}$$

We verify that

$$\mathbb{M}_0^\dagger \mathbb{M}_0 + \sum_{\alpha=1}^{\mathscr{M}} \mathbb{M}_\alpha^\dagger \mathbb{M}_\alpha = \left(\mathbb{1}_{\mathcal{H}_S} + \left(-\imath\,\mathbb{H} + \frac{1}{2}\sum_{\alpha=1}^{\mathscr{M}} \mathbb{A}_\alpha^\dagger \mathbb{A}_\alpha\right)\mathrm{d}t\right)\left(\mathbb{1}_{\mathcal{H}_S} - \left(\imath\,\mathbb{H} + \frac{1}{2}\sum_{\alpha=1}^{\mathscr{M}} \mathbb{A}_\alpha^\dagger \mathbb{A}_\alpha\right)\mathrm{d}t\right) + \sum_{\alpha=1}^{\mathscr{M}} \mathbb{A}_\alpha^\dagger \mathbb{A}_\alpha \mathrm{d}t$$

yields

$$\mathbb{M}_0^\dagger \mathbb{M}_0 + \sum_{\alpha=1}^{\mathscr{M}} \mathbb{M}_\alpha^\dagger \mathbb{M}_\alpha = \mathbb{1}_{\mathcal{H}_S} + O(\mathrm{d}t)^2$$

### 2.2. Summary: comparison with classical master equations

| Classical master equation | Non selective measurement |
|---|---|
| A set $\{\mathfrak{s}_\alpha\}_{\alpha=1}^{d}$ of states. | A set $\{\mathbb{M}_\alpha\}_{\alpha=1}^{\mathcal{M}}$ of operators corresponding to distinct **measurement outcomes**. |
| Transition probability $$\mathrm{T}(i\|j) = \mathrm{Prob}(\mathfrak{s}_j \to \mathfrak{s}_i)$$ | Probability of observing the measurement $\alpha$ outcome if the state of the system is $\boldsymbol{\rho}^{(S)}$ **before** the measurement: $$\wp_\alpha = \mathrm{Tr}\left(\mathbb{M}_\alpha\,\boldsymbol{\rho}^{(S)}\,\mathbb{M}_\alpha^\dagger\right)$$ |
| Probability conservation: $$\sum_{i\,\in\,\mathbb{I}} \mathrm{T}(i\|j) = 1$$ | Probability conservation: $$\sum_{\alpha=1}^{\mathcal{M}} \mathbb{M}_\alpha^\dagger \mathbb{M}_\alpha = \mathbb{1}_{\mathcal{H}_S}$$ |
| **Before** the transition: $$\mathrm{p}(i,t) = \mathrm{Prob}(\xi_t = \mathfrak{s}_i)$$ probability to find the system in the state $\mathfrak{s}_i$ at time $t$ | **Selective** measurement: $$\boldsymbol{\rho}_t^{(S,\alpha)} = \frac{\mathbb{M}_\alpha \boldsymbol{\rho}_t^{(S)} \mathbb{M}_\alpha^\dagger}{\mathrm{Tr}\left(\mathbb{M}_\alpha\,\boldsymbol{\rho}_t^{(S)}\,\mathbb{M}_\alpha^\dagger\right)}$$ |
| **After** the transition $$\mathrm{p}(i,t+\mathrm{d}t) = \sum_{j\,=\,1}^{d} \mathrm{T}(i\|j)\,\mathrm{p}(j,t)$$ | **Non-selective** measurement: $$\boldsymbol{\rho}_t^{(S)\prime} = \sum_{\alpha} \wp_\alpha\,\boldsymbol{\rho}_t^{(S,\alpha)} := \boldsymbol{\rho}_{t+\mathrm{d}t}^{(S)}$$ |

## 3. UNRAVELING OF THE CLASSICAL MASTER EQUATION: THE CASE OF THE TELEGRAPH PROCESS

As an example of unraveling of a classical master equation we consider the **telegraph noise** [2]. The telegraph noise is a classical random two-level system. Namely, a telegraph noise process $\xi_t$ has just two values, $\xi_t = 0$ and $\xi_t = 1$, and is subject to two random jump processes. The first kind of jump flips the state from $\xi_t = 0$ to $\xi_t = 1$, with rate $r_1$. The second kind of jump flips the state back from $\xi_t = 1$ to $\xi_t = 0$ this time with the jump rate $r_0$. The process is described by the Itô stochastic differential equation

$$\mathrm{d}\xi_t = (1 - \xi_t)\,\mathrm{d}\nu_t^{(1)} - \xi_t\,\mathrm{d}\nu_t^{(0)} \tag{9}$$

with, by hypothesis, admissible initial data

$$\xi_0 = \{0, 1\}$$

In (9) we surmise that $\mathrm{d}\nu_t^{(i)}$, $i = 0, 1$ are **mutually independent** stochastic processes satisfying the following hypotheses

- time decorrelation: $\mathrm{d}\nu_t^{(i)}$, $i = 0, 1$ are independent of the values at previous times
- at fixed $t$

$$
\begin{aligned}
&(\mathrm{d}\nu_t^{(i)})^2 = \mathrm{d}\nu_t^{(i)} \qquad \Rightarrow \qquad \mathrm{d}\nu_t^{(i)} = \{0, 1\} \\
&\mathrm{E}\left(\mathrm{d}\nu_t^{(i)} | \xi_t\right) = r_i \,\mathrm{d}t
\end{aligned}
$$

The conditional expectation means that $\xi_t$ at time $t$ is **independent** of the realization of $\mathrm{d}\nu_t^{(i)}$.

| | $\mathrm{d}t$ | $\mathrm{d}\nu_t^{(0)}$ | $\mathrm{d}\nu_t^{(1)}$ |
|---|---|---|---|
| $\mathrm{d}t$ | 0 | 0 | 0 |
| $\mathrm{d}\nu_t^{(0)}$ | 0 | $\mathrm{d}\nu_t^{(0)}$ | 0 |
| $\mathrm{d}\nu_t^{(1)}$ | 0 | 0 | $\mathrm{d}\nu_t^{(1)}$ |

TABLE I: With probability one the products of the two increments $\mathrm{d}\nu_t^{(0)}$, $\mathrm{d}\nu_t^{(1)}$ vanishes. This means that they cannot take unit value at the same time.

Under our assumptions we see that

$$\mathrm{d}\xi_t\big|_{\xi_t=0} = \mathrm{d}\nu_t^{(1)}$$

In othr words, if the system occupies the lower level it can only jump to the upper in consequence of the increment of the Poisson process. Similarly, if the process occupies the upper level it can only decay to the lower:

$$\mathrm{d}\xi_t\big|_{\xi_t=1} = -\mathrm{d}\nu_t^{(0)}$$

**Proposition.** *Let us denote by*

$$\mathrm{p}_t(0) = \mathrm{P}\left(\xi_t = 0\right) \qquad \& \qquad \mathrm{p}_t(1) = \mathrm{P}\left(\xi_t = 1\right)$$

*the* ***(relative) populations*** *of the telegraph noise at time t i.e. the probabilities at time t to find the process in one of the two admissible states. Then these probabilities obey the master equation*

$$
\begin{aligned}
\partial_t \mathrm{p}_t(0) = -r_1\, \mathrm{p}_t(0) + r_0\, \mathrm{p}_t(1) \\
\partial_t \mathrm{p}_t(1) = r_1\, \mathrm{p}_t(0) - r_0\, \mathrm{p}_t(1)
\end{aligned}
$$

*Proof.*
By definition the identities

$$\partial_t\, \mathrm{E}\, f(\xi_t) \,\equiv\, \lim_{\varepsilon \searrow 0}\, \mathrm{E}\, \frac{f(\xi_{t+\varepsilon}) - f(\xi_t)}{\varepsilon}$$

and

$$\partial_t\, \mathrm{E}\, f(\xi_t) \equiv \partial_t \sum_{x=0,1} f(x)\mathrm{p}(x, t)$$

must hold true. We notice that for any test function $f$

$$\mathrm{d}f(\xi_t) = f(\xi_{t+\varepsilon}) - f(\xi_t) = f(\xi_t + \mathrm{d}\xi_t) - f(\xi_t) = f(\xi_t + (1 - \xi_t)\, \mathrm{d}\nu_t^{(1)} - \xi_t\, \mathrm{d}\nu_t^{(0)}) - f(\xi_t)$$

As the two increments $\mathrm{d}\nu_t^{(0)}$, $\mathrm{d}\nu_t^{(1)}$ cannot take unit value at the same time, we then find

$$\mathrm{d}f(\xi_t) = \left((1 - \mathrm{d}\nu_t^{(0)} - \mathrm{d}\nu_t^{(1)}) f(\xi_t) + \mathrm{d}\nu_t^{(0)} f(0) + \mathrm{d}\nu_t^{(1)} f(1)\right) - f(\xi_t)$$

and therefore

$$\mathrm{d}f(\xi_t) = \mathrm{d}\nu_t^{(0)}\, (f(0) - f(\xi_t)) + \mathrm{d}\nu_t^{(1)}\, (f(1) - f(\xi_t))$$

Itô prescription allows us to evaluate the expected value of the increment as

$$\lim_{\varepsilon \searrow 0} \frac{\mathrm{E}\, \mathrm{d}f(\xi_t)}{\varepsilon} = r_0\, (f(0) - \mathrm{E}\, f(\xi_t)) + r_1\, (f(1) - \mathrm{E}\, f(\xi_t)) = r_0\, p_t(1)\, (f(0) - f(1)) + r_1\, p_t(0)\, (f(1) - f(0))$$

As $f$ is arbitrary the identity

$$0 = \partial_t \sum_{x=0,1} f(x)\mathrm{p}(x,t) - \Big( r_0\, p_t(1)\,(f(0) - f(1)) + r_0\, p_t(0)\,(f(1) - f(0)) \Big)$$

must hold for any choice of $f(0)$, $f(1)$. More explicitly, if we enforce the condition that the coefficients of $f(0)$, $f(1)$ in

$$0 = \big(\partial_t \mathrm{p}_t(0) + r_1\,\mathrm{p}_t(0) - r_0\,\mathrm{p}_t(1)\big) f(0) + \big(\partial_t \mathrm{p}_t(1) - r_1\,\mathrm{p}_t(0) + r_0\,\mathrm{p}_t(1)\big) f(1)$$

should vanish, we recover the claim. □

**Remark.** *If $r_1 = r_2$ it is immediate to verify that (9) is statistically equivalent to the stochastic differential equation*

$$\mathrm{d}\xi_t = (1 - 2\,\xi_t)\mathrm{d}\nu_t$$

*driven by just one jump process*

$$(\mathrm{d}\nu_t)^2 = \mathrm{d}\nu_t$$
$$\mathrm{E}\,(\mathrm{d}\nu_t|\xi_t) = r\mathrm{d}t$$

* *

## 4. UNRAVELING OF THE QUANTUM MASTER EQUATION

We now wish to rephrase the generalized measurement process of section 2 in terms of **state vectors**

- "Null result" with probability

$$\wp_0 = \mathrm{Tr}\left(\mathbb{M}_0\, \boldsymbol{\psi}_t \boldsymbol{\psi}_t^\dagger\, \mathbb{M}_0^\dagger\right) = 1 - \mathrm{Tr}\left(\mathbb{R}\boldsymbol{\psi}_t \boldsymbol{\psi}_t^\dagger\right)\mathrm{d}t \tag{10}$$

- "Detection" with (infinitesimally small) probability

$$\wp_1 = \mathrm{Tr}\left(\mathbb{M}_1\, \boldsymbol{\psi}_t \boldsymbol{\psi}_t^\dagger\, \mathbb{M}_1^\dagger\right) = \mathrm{Tr}\left(\mathbb{R}\, \boldsymbol{\psi}_t \boldsymbol{\psi}_t^\dagger\right)\mathrm{d}t \tag{11}$$

  When a detection occurs, the system undergoes a *quantum jump* induced by $\mathbb{M}_1$. The rate

$$\mathrm{Tr}\left(\mathbb{R}\boldsymbol{\psi}_t \boldsymbol{\psi}_t^\dagger\right) = \|\mathbb{A}\boldsymbol{\psi}_t\|^2$$

  depends on the value of the state vector at time $t$.

In order to model this situation we suppose that

1. we model the counting of these events by a Poisson process and denote by $\nu_t$ the number of detected events at time $t$.

2. the system at time $t$ is in a pure state $\boldsymbol{\psi}_t$

The increment of the counting process satisfies

$$(\mathrm{d}\nu_t)^2 = \mathrm{d}\nu_t \qquad \text{the increment is either } 0 \text{ or } 1 \tag{12a}$$
$$\mathrm{E}\left(\mathrm{d}\nu_t|\boldsymbol{\psi}_t\right) = \wp_1 \equiv \|\mathbb{A}\boldsymbol{\psi}_t\|^2\mathrm{d}t \tag{12b}$$

The relation (12b) means that the expected value of the Poisson increment *conditioned* upon the fact that the state of the system takes the value $\psi(t)$ equals the detection probability. Then

$$\boldsymbol{\psi}_{t+\mathrm{d}t} = \boldsymbol{\psi}_t + (1 - \mathrm{d}\nu_t)\left(\frac{\mathbb{M}_0}{\|\mathbb{M}_0\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\boldsymbol{\psi}_t + \mathrm{d}\nu_t\left(\frac{\mathbb{M}_1}{\|\mathbb{M}_1\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\boldsymbol{\psi}_t$$

We know that

$$\frac{\mathbb{M}_1\boldsymbol{\psi}_t}{\|\mathbb{M}_1\boldsymbol{\psi}_t\|} \equiv \frac{\mathbb{M}_1\boldsymbol{\psi}_t}{\sqrt{\left\langle\, \boldsymbol{\psi}_t \,,\, \mathbb{M}_1^{\dagger}\mathbb{M}_1\boldsymbol{\psi}_t \,\right\rangle}} = \frac{\mathbb{A}\,\boldsymbol{\psi}_t}{\|\mathbb{A}\boldsymbol{\psi}_t\|} \tag{13}$$

and

$$\frac{\mathbb{M}_0\boldsymbol{\psi}_t}{\|\mathbb{M}_0\boldsymbol{\psi}_t\|} = \frac{\mathbb{1}_{\mathcal{H}_S} - \left(\imath\mathbb{H} + \mathbb{A}^{\dagger}\,\mathbb{A}/2\right)\mathrm{d}t}{\sqrt{1 - \|\mathbb{A}\boldsymbol{\psi}_t\|^2\mathrm{d}t}}\boldsymbol{\psi}_t = \left(\mathbb{1}_{\mathcal{H}_S} - \left(\imath\mathbb{H} + \frac{\mathbb{A}^{\dagger}\,\mathbb{A}}{2} - \frac{\|\mathbb{A}\boldsymbol{\psi}_t\|^2}{2}\right)\mathrm{d}t\right)\boldsymbol{\psi}_t \tag{14}$$

Gleaning these results we get into

$$\mathrm{d}\boldsymbol{\psi}_t = -(1-\mathrm{d}\nu_t)\,\mathrm{d}t\left(\imath\mathbb{H} + \frac{\mathbb{A}^{\dagger}\,\mathbb{A}}{2} - \frac{\|\mathbb{A}\boldsymbol{\psi}_t\|^2}{2}\right)\boldsymbol{\psi}_t + \mathrm{d}\nu_t\left(\frac{\mathbb{A}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\boldsymbol{\psi}_t \tag{15}$$

We note that by (12)

$$\mathrm{E}\left((\mathrm{d}\nu_t)^n\,\mathrm{d}t\right) = \mathrm{d}t\,\mathrm{E}\left(\mathrm{d}\nu_t\right) = o(\mathrm{d}t^2) \tag{16}$$

hence, in a probabilistic sense which can be made precise (15), is equivalent to

$$\mathrm{d}\boldsymbol{\psi}_t = -\mathrm{d}t\left(\imath\mathbb{H} + \frac{\mathbb{A}^{\dagger}\,\mathbb{A}}{2} - \frac{\|\mathbb{A}\boldsymbol{\psi}_t\|^2}{2}\right)\boldsymbol{\psi}_t + \mathrm{d}\nu_t\left(\frac{\mathbb{A}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\boldsymbol{\psi}_t \tag{17}$$

### 4.1. Relation with the Lindblad–Gorini–Kossakowski–Sudarshan operator

In order to shed light on the physical meaning of (17) we associate with $\boldsymbol{\psi}_t$ the pure state operator

$$\mathbb{P}_t = \boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger} \tag{18}$$

As in the case of $\boldsymbol{\psi}_t$, $\mathbb{P}_t$ is a functional of the Poisson process up to time $t$ but independent of $\mathrm{d}\nu_t$. From (17) we can compute the stochastic differential of $\mathbb{P}_t$.

$$\mathrm{d}\mathbb{P}_t = \boldsymbol{\psi}_{t+\mathrm{d}t}\boldsymbol{\psi}_{t+\mathrm{d}t}^{\dagger} - \boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger}$$

Simple algebra allows us to write

$$\mathrm{d}\mathbb{P}_t = \mathrm{d}\boldsymbol{\psi}_t\boldsymbol{\psi}_t^{\dagger} + \boldsymbol{\psi}_t\mathrm{d}\boldsymbol{\psi}_t^{\dagger} + \mathrm{d}\boldsymbol{\psi}_t\,\mathrm{d}\boldsymbol{\psi}_t^{\dagger}$$

The first two addends on the right hand side yield the usual Leibniz differential formula. We retain the third term because $\mathrm{d}\nu_t$ is not an ordinary differential. We need instead to take into account that products of differentials obey the Poisson increment differential table. In the table, we wrote zero whenever the product is of order higher than $\mathrm{d}t$. Taking into account these product rules the *stochastic differential* of (18)

| | $\mathrm{d}t$ | $\mathrm{d}\nu_t$ |
|---|---|---|
| $\mathrm{d}t$ | 0 | 0 |
| $\mathrm{d}\nu_t$ | 0 | $\mathrm{d}\nu_t$ |

$$\begin{aligned}\mathrm{d}\mathbb{P}_t =\ & \mathrm{d}\nu_t\left(\frac{\mathbb{A}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\mathbb{P}_t - \mathrm{d}t\left(\imath\mathbb{H} + \frac{\mathbb{A}^{\dagger}\,\mathbb{A}}{2} - \frac{\|\mathbb{A}\boldsymbol{\psi}_t\|^2}{2}\right)\mathbb{P}_t \\ & + \mathbb{P}_t\mathrm{d}\nu_t\left(\frac{\mathbb{A}^{\dagger}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right) - \mathbb{P}_t\mathrm{d}t\left(-\imath\mathbb{H} + \frac{\mathbb{A}^{\dagger}\,\mathbb{A}}{2} - \frac{\|\mathbb{A}\boldsymbol{\psi}_t\|^2}{2}\right) + \mathrm{d}\nu_t\left(\frac{\mathbb{A}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\mathbb{P}_t\left(\frac{\mathbb{A}^{\dagger}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\end{aligned} \tag{19}$$

Upon reordering contributions we obtain

$$\begin{aligned}\mathrm{d}\mathbb{P}_t =\ & -\left(\imath[\mathbb{H}\,,\mathbb{P}_t] + \frac{\mathbb{A}^{\dagger}\,\mathbb{A}\,\mathbb{P}_t + \mathbb{P}_t\,\mathbb{A}^{\dagger}\,\mathbb{A} - 2\,\|\mathbb{A}\boldsymbol{\psi}_t\|^2\,\mathbb{P}_t}{2}\right)\mathrm{d}t \\ & + \mathrm{d}\nu_t\left(\left(\frac{\mathbb{A}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\mathbb{P}_t + \mathbb{P}_t\left(\frac{\mathbb{A}^{\dagger}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\right) + \mathrm{d}\nu_t\left(\frac{\mathbb{A}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\mathbb{P}_t\left(\frac{\mathbb{A}^{\dagger}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\end{aligned} \tag{20}$$

**Proposition.** *The expected value*

$$\boldsymbol{\rho}_t^{(S)} = \mathrm{E}\,\mathbb{P}_t \tag{21}$$

*over the realizations of the Poisson process satisfies the Lindblad–Gorini–Kossakowski–Sudarshan equation*

$$\mathrm{d}\boldsymbol{\rho}_t^{(S)} = -\left(\imath[\mathbb{H}\,,\boldsymbol{\rho}_t^{(S)}] + \frac{\mathbb{A}^\dagger\,\mathbb{A}\,\boldsymbol{\rho}_t^{(S)} + \boldsymbol{\rho}_t^{(S)}\,\mathbb{A}^\dagger\,\mathbb{A}}{2} - \mathbb{A}\,\boldsymbol{\rho}_t^{(S)}\,\mathbb{A}^\dagger\right)\mathrm{d}t \tag{22}$$

*Proof.*
Owing to (12) this equation is again *linear* in the Poisson process increment. Furthermore the Poisson process has **independent increments**. We can avail us of this fact to derive a *closed* equation for the expected value:

$$\boldsymbol{\rho}_t^{(S)} \equiv \mathrm{E}\,\mathbb{P}_t \tag{23}$$

Namely, for any functional $\mathcal{F}$ depending upon the Poisson process up to time $t$ the chain of identities

$$\mathrm{E}\left(\mathrm{d}\nu_t\mathcal{F}\right) = \mathrm{E}\left(\mathrm{E}\left(\mathrm{d}\nu_t|\boldsymbol{\psi}_t\right)\mathcal{F}\right) = \mathrm{E}\left(\|\mathbb{A}\boldsymbol{\psi}_t\|^2\mathcal{F}\right)\,\mathrm{d}t \tag{24}$$

holds true. The first identity is the **the law of iterated expectation** (or tower rule), a property (conditional) expectation values enjoy by definition. The second identity follows from (12b). For the expectation value of (20), the identities (24) translate into

$$\begin{aligned}\mathrm{d}\boldsymbol{\rho}_t^{(S)} =& -\left(\imath[\mathbb{H}\,,\boldsymbol{\rho}_t^{(S)}] + \frac{\mathbb{A}^\dagger\,\mathbb{A}\,\boldsymbol{\rho}_t^{(S)} + \boldsymbol{\rho}_t^{(S)}\,\mathbb{A}^\dagger\,\mathbb{A}}{2} - \mathrm{E}\left(\|\mathbb{A}\boldsymbol{\psi}_t\|^2\,\mathbb{P}_t\right)\right)\mathrm{d}t \\ &+ \mathrm{d}t\,\mathrm{E}\left(\|\mathbb{A}\boldsymbol{\psi}_t\|^2\left(\left(\frac{\mathbb{A}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\mathbb{P}_t + \mathbb{P}_t\left(\frac{\mathbb{A}^\dagger}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\right)\right) \\ &+ \mathrm{d}t\,\mathrm{E}\left(\|\mathbb{A}\boldsymbol{\psi}_t\|^2\left(\frac{\mathbb{A}}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\mathbb{P}_t\left(\frac{\mathbb{A}^\dagger}{\|\mathbb{A}\boldsymbol{\psi}_t\|} - \mathbb{1}_{\mathcal{H}_S}\right)\right)\end{aligned} \tag{25}$$

Straightforward algebra brings about the cancellation of the terms non-linearly depending upon the stochastic state vector of the system. We are therefore left with the *linear* equation:

$$\mathrm{d}\boldsymbol{\rho}_t^{(S)} = -\left(\imath[\mathbb{H}\,,\boldsymbol{\rho}_t^{(S)}] + \frac{\mathbb{A}^\dagger\,\mathbb{A}\,\boldsymbol{\rho}_t^{(S)} + \boldsymbol{\rho}_t^{(S)}\,\mathbb{A}^\dagger\,\mathbb{A}}{2} - \mathbb{A}\,\boldsymbol{\rho}_t^{(S)}\,\mathbb{A}^\dagger\right)\mathrm{d}t \tag{26}$$

This is the Lindblad–Gorini–Kossakowski–Sudarshan equation satisfied by the state operator $\boldsymbol{\rho}_t$. Furthermore since $\boldsymbol{\rho}_{t_\iota}^{(S)} = \boldsymbol{\rho}_{t_\iota}$ we have

$$\boldsymbol{\rho}_t^{(S)} = \mathrm{E}\,\mathbb{P}_t \tag{27}$$

at any further time. □

* * *

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