TCM315 Fall 2022: Introduction to Open Quantum Systems

Lecture 13: Axiomatic derivation of the completely positive quantum master equation

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CONTENTS

1. INTRODUCTION

The axiomatic derivation of the Lindblad–Gorini–Kossakowski–Sudarshan (LGKS) master equation can be found in chapter 4 of [9] and in § 3.2.2 of [1]. Together with the article [7], these references are the main sources for the present notes.

2. THE LINDBLAD–GORINI–KOSSAKOWSKI–SUDARSHAN QUANTUM MASTER EQUATION

Lindblad [4] and Gorini Kossakowski and Sudarshan [3] separately discovered the form of the master equation that any semi-group (i.e. family of infinitely divisible) of completely positive dynamical maps must satisfy. The result is a necessary and sufficient condition: given the equation the semi-group (i.e. the fundamental solution) is completely positive.

2.1. Semigroup of completely positive maps

We consider a two parameter family of linear maps

$$\mathbf{\Phi}_{ts} \colon \mathbb{R} \times \mathbb{R} \times \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H}) \tag{1}$$

This means that for any $t, s \in \mathbb{R}$, $\mathbf{\Phi}_{ts}$ is a **linear map** of the space $\mathcal{B}(\mathcal{H})$ into itself. $\mathcal{B}(\mathcal{H})$ is the space of bounded operators acting on an Hilbert space $\mathcal{H}$. In reference to Heisenberg's picture evolution of operators the family is also referred to a **linear development**. If the Hilbert space is finite dimensional, operators are bounded by default and $\mathcal{B}(\mathcal{H}) \equiv \mathcal{M}_d(\mathbb{C})$ the space of matrices $d \times d$ matrices with complex entries.

We suppose that

$$\text{for any operator } \mathbb{A} \in \mathcal{B}(\mathcal{H})$$

the development map $\mathbf{\Phi}_{ts}$ enjoys the following properties:

i $\mathbf{\Phi}_{ts}(\mathbb{A})$ is completely positive for any $t \geq s$

ii $\mathbf{\Phi}_{ts}(\mathbb{A})$ is strongly continuous: i.e. it is continuous with respect to the norm induced by the inner product.

iii Semi-group property: for any $t \geq s$ and $u \geq 0$ $\mathbf{\Phi}$ enjoys the infinite divisibility property

$$\mathbf{\Phi}_{t+us}(\mathbb{A}) = \mathbf{\Phi}_{t+ut}(\mathbf{\Phi}_{ts}(\mathbb{A})) \qquad \forall s \leq u \leq t \tag{2}$$

A dynamical map is completely positive if we construct it by tracing out the environment under the assumption that the state operator specifying the initial datum is the tensor product of the state operators of the system and the environment

$$\mathbf{\Phi}_{t\,s}(\boldsymbol{\rho}_s) = \mathrm{Tr}_{\mathcal{H}_E}\left(\mathbb{U}_{t\,s}\boldsymbol{\rho}_s \otimes \boldsymbol{\rho}^{(E)}\mathbb{U}_{t\,s}\right)$$

It must be emphasized that unitary evolution generates entanglement and therefore does not preserve the tensor product form. This means that hypotheses i and ii guarantee that an evolution starting from time s is completely positive at any later time. It **does not** however guarantee that the completely positive evolution can be sub-divided in iterations of completely positive elements. Infinite divisibility i.e. iii

$$\frac{\mathrm{d}\mathbf{\Phi}_{ts}}{\mathrm{d}t}(\mathbb{A}) = \lim_{\varepsilon\searrow 0}\frac{\mathbf{\Phi}_{t+\varepsilon s}(\mathbb{A}) - \mathbf{\Phi}_{ts}(\mathbb{A})}{\varepsilon} = \lim_{\varepsilon\searrow 0}\frac{\mathbf{\Phi}_{t+\varepsilon t}(\mathbf{\Phi}_{ts}(\mathbb{A})) - \mathbf{\Phi}_{ts}(\mathbb{A})}{\varepsilon} = \mathbb{L}_t(\mathbf{\Phi}_{ts}(\mathbb{A})) \tag{3}$$

which define the **generator** $\mathbb{L}_t$ of the linear map $\mathbf{\Phi}$.

Definition. *A map linear map $\mathbf{\Phi}_{ts}$ satisfying **i**, **ii**, **iii** is called a **completely positive semi-group**.*

Remark. *The flow or fundamental solution of a differential equation is the map that applied to an initial datum produces a particular solution of the equation. A completely positive flow ensures that any particular solution is completely positive. This fact does not exclude that a **particular solution** of a time local master equations obtained only assuming hypotheses ii and iii be completely positive. In other words, the Lindblad-Gorini-Kossakowski-Sudarshan master equation is not the most general quantum master equation.*

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2.2. Complete positivity and semi-group property

The fundamental solution or flow $\mathbf{\Phi}_{t\,s}$of an differential ordinary differential equation enjoys the group property

$$\mathbf{\Phi}_{t\,s}(\boldsymbol{x}) = \mathbf{\Phi}_{t\,u}(\mathbf{\Phi}_{u\,s}(\boldsymbol{x}))$$

irrespective of $\mathbb{A}$ and of the ordering of s, u, and t. This is true for unitary flows generated by the Schrödinger equation.

$$\mathbb{U}_{t\,u}\mathbb{U}_{u\,s} = \mathbb{U}_{t\,s}$$

The lift to $\mathcal{M}_d$ give rise to self-adjoint and trace preserving completely positive unitary dynamical maps

$$\mathbf{\Phi}_{t\,s}(\boldsymbol{\rho}) = \mathbb{U}_{t\,s}\,\boldsymbol{\rho}\,\mathbb{U}_{t\,s}^{\dagger} \tag{4}$$

which themselves enjoy the flow property. Thus the inverse of an element of the family is also an element of the family:

$$\mathbf{\Phi}_{t\,s}^{-1}(\boldsymbol{\rho}) = \mathbf{\Phi}_{s\,t}(\boldsymbol{\rho}) = \mathbb{U}_{s\,t}\,\boldsymbol{\rho}\,\mathbb{U}_{s\,t}^{\dagger} = \mathbb{U}_{t\,s}^{\dagger}\,\boldsymbol{\rho}\,\mathbb{U}_{t\,s} \qquad \forall t, s$$

and by Choi theorem is completely positive. The question naturally arises whether the inverse operation preserves in general complete positivity.

Proposition. *A self-adjoint and trace preserving completely positive map in $\mathcal{M}_d(\mathcal{H})$ admits a self-adjoint and trace preserving completely positive **inverse** if and only if it is also **unitary***

Proof.
We have already seen that if a completely positive map is also unitary then it admits a completely positive inverse. Hence the non-trivial content of the proposition is the **only if** or necessary condition. We follow §3.2.2 of cite [8]. We observe that given the completely positive map

$$\boldsymbol{\Phi}(\boldsymbol{\rho}) = \sum_{i=1}^{\mathscr{W}} \mathbb{W}_i \boldsymbol{\rho} \mathbb{W}_i^\dagger$$

then $\boldsymbol{\Psi}$ specifies its the inverse

$$\boldsymbol{\Psi}(\boldsymbol{\rho}) = \sum_{i=1}^{\mathscr{V}} \mathbb{V}_i \boldsymbol{\rho} \mathbb{V}_i^\dagger$$

if and only if for any state operator

$$\boldsymbol{\rho} = \boldsymbol{\Psi}(\boldsymbol{\Phi}(\boldsymbol{\rho})) = \sum_{i=1}^{\mathscr{V}} \sum_{i=1}^{\mathscr{W}} \mathbb{V}_i \mathbb{W}_j \boldsymbol{\rho} \mathbb{W}_j^\dagger \mathbb{V}_i^\dagger$$

This means that for every $\boldsymbol{v} \in \mathcal{H}$

$$0 \leq \boldsymbol{v}^\dagger \boldsymbol{\rho} \boldsymbol{v} = \sum_{i=1}^{\mathscr{V}} \sum_{i=1}^{\mathscr{W}} \boldsymbol{v}^\dagger \mathbb{V}_i \mathbb{W}_j \boldsymbol{\rho} \mathbb{W}_j^\dagger \mathbb{V}_i^\dagger \boldsymbol{v} \tag{5}$$

and in particular

$$\boldsymbol{v}^\dagger \mathbb{V}_i \mathbb{W}_j \boldsymbol{\rho} \mathbb{W}_j^\dagger \mathbb{V}_i^\dagger \boldsymbol{v} = \left(\mathbb{W}_j^\dagger \mathbb{V}_i^\dagger \boldsymbol{v}\right)^\dagger \boldsymbol{\rho} \left(\mathbb{W}_j^\dagger \mathbb{V}_i^\dagger \boldsymbol{v}\right) = \boldsymbol{v}_{i\,j}^\dagger \boldsymbol{\rho} \boldsymbol{v}_{i\,j} \geq 0$$

The sum (5) thus consists of positive definite term. The fact rules out mutual cancellations. Complete positivity of the inverse thus requires

$$\mathbb{V}_i \mathbb{W}_j = c^{(i\,j)} \mathbb{1}_{\mathcal{H}}$$

Trace preservation then implies

$$\mathbb{W}_i^\dagger \mathbb{W}_j = \sum_{k=1}^{\mathscr{V}} \mathbb{W}_i^\dagger \mathbb{V}_k^\dagger \mathbb{V}_k \mathbb{W}_j = \sum_{k=1}^{\mathscr{V}} \bar{c}^{(i\,k)} c^{(k\,j)} \mathbb{1}_{\mathcal{H}} = b^{(i\,j)} \mathbb{1}_{\mathcal{H}}$$

with $b^{(i\,j)}$ the elements of a positive definite matrix with unit trace

$$\sum_{i=1}^{\mathscr{W}} b^{(i\,i)} = 1$$

We now recall that the polar representation of a non-singular matrix (see e.g. § 2.1.10 of [5]) yields

$$\mathbb{W}_i = \mathbb{U}_i \mathbb{P}_i$$

where $\mathbb{U}_i$ is unitary and $\mathbb{P}_i$ is positive definite. We find

$$\mathbb{P}_i^2 = b^{(i\,i)} \mathbb{1}_{\mathcal{H}}$$

and therefore

$$\mathbb{W}_i = \frac{1}{\sqrt{b^{(i\,i)}}} \mathbb{U}_i$$

This latter relation implies

$$\mathbb{1}_{\mathcal{H}} = \sum_{i=1}^{\mathcal{W}} \frac{1}{\sqrt{b^{(i\,i)}}} \mathbb{U}_i^\dagger \mathbb{U}_i \frac{1}{\sqrt{b^{(i\,i)}}} = \mathbb{1}_{\mathcal{H}} \sum_{i=1}^{\mathscr{W}} \frac{1}{b^{(i\,i)}}$$

Since the $b^{(i\,i)}$'s must be positive numbers less than one, the result is possible only if $\mathcal{W} = 1$ and $b^{(1\,1)} = 1$ □

The proposition explains why it is appropriate to talk completely positive semi-groups rather than flows. Flows enjoy the group property and therefore include the inverse as element of the family. In order to any element of the family to enjoy complete positivity we need to restrict the attention to semi-groups.

2.3. Schrödinger and Heisenberg's pictures of a completely positive semi-group

The map (1) is the open-quantum system counter-part of the development map governing time evolution of self-adjoint operators in Heisenberg's picture closed quantum systems. Schrödinger's picture follows from duality. Namely, for any $\mathbb{A} \in \mathcal{B}(\mathcal{H})$ and a given state operator $\boldsymbol{\rho}$ the chain of identity

$$\operatorname{Tr}\left(\boldsymbol{\rho}\,\mathbb{A}_t\right) := \operatorname{Tr}\left(\boldsymbol{\rho}\,\boldsymbol{\Phi}_{ts}\left(\mathbb{A}\right)\right) = \operatorname{Tr}\left(\boldsymbol{\Phi}_{ts}^{\dagger}\left(\boldsymbol{\rho}\right)\,\mathbb{A}\right) := \operatorname{Tr}\left(\boldsymbol{\rho}_t\,\mathbb{A}\right) \tag{6}$$

holds true. In (6) $\mathbb{A}$ and $\boldsymbol{\rho}$ must be respectively identified as the value of $\mathbb{A}_t$ and of $\boldsymbol{\rho}_t$ at time $t = s$. In order to verify (6) we observe that Choi representation of completely positive maps yields

$$\boldsymbol{\Phi}_{ts}\left(\mathbb{A}\right) = \sum_{i=1}^{\mathcal{N}} \mathbb{W}_{t\,s}^{(i)\dagger}\mathbb{A}\mathbb{W}_{t\,s}^{(i)} \tag{7}$$

for some $\mathcal{N} \geq 1$ and $\left\{\mathbb{W}_{ts}^{(i)}\right\}_{i=1}^{\mathcal{N}}$, the claim then follows when use the cyclic property of the trace

$$\operatorname{Tr}\left(\boldsymbol{\rho}\,\boldsymbol{\Phi}_{ts}\left(\mathbb{A}\right)\right) = \sum_{i=1}^{\mathcal{N}} \operatorname{Tr}\left(\boldsymbol{\rho}\,\mathbb{W}_{t\,s}^{(i)\dagger}\mathbb{A}\mathbb{W}_{t\,s}^{(i)}\right) = \sum_{i=1}^{\mathcal{N}} \operatorname{Tr}\left(\mathbb{W}_{t\,s}^{(i)}\boldsymbol{\rho}\,\mathbb{W}_{t\,s}^{(i)\dagger}\mathbb{A}\right) \equiv \operatorname{Tr}\left(\boldsymbol{\rho}_t\,\mathbb{A}\right)$$

2.4. Constraints imposed by probability conservation on the generator and its adjoint

Probability conservation

$$1 = \operatorname{Tr}\left(\boldsymbol{\rho}_t\right) = \operatorname{Tr}\left(\boldsymbol{\rho}\boldsymbol{\Phi}_{ts}(\mathbb{1})\right) \tag{8}$$

immediately translates into the constraint that the generator we defined via (3) and its adjoint must satisfy.

2.4.1. Generator

Differentiating (8) yields the identity

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\left(\boldsymbol{\rho}_t\right) = \operatorname{Tr}\left(\boldsymbol{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}_{ts}(\mathbb{1})\right) = \operatorname{Tr}\left(\boldsymbol{\rho}\mathbb{L}_t(\mathbb{1})\right)$$

which must hold independently of the initial data $\boldsymbol{\rho}$. As a consequence we get into the constraint

$$\mathbb{L}_t(\mathbb{1}) = 0$$

2.4.2. Adjoint of the generator

Similarly from

$$0 = \frac{\mathrm{d}\operatorname{Tr}\boldsymbol{\rho}_t}{\mathrm{d}t} = \operatorname{Tr}\frac{\mathrm{d}\boldsymbol{\rho}_t}{\mathrm{d}t} = \operatorname{Tr}\mathbb{L}_t^{\dagger}(\boldsymbol{\rho}_t)$$

we therefore conclude that for any state operator $\boldsymbol{\rho}_t$

$$\operatorname{Tr}\left(\mathbb{L}_t^{\dagger}(\boldsymbol{\rho}_t)\right) = 0$$

2.5. Lindblad's result

Historically [2], Lindblad [4] worked in the Heisenberg picture (i.e. with operators in $\mathcal{B}(\mathcal{H})$ and with $\mathcal{H}$ not necessarily finite dimensional) and derived the equation that a **time autonomous** completely positive semigroup

$$\mathbf{\Phi}_{ts} = \mathbf{\Phi}_{t-s}$$

must satisfy.

Theorem. *(**Lindblad**) A linear operator $\mathbb{L}\colon \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is the generator of a **completely positive time autonomous semi-group if and only if** for any $\mathbb{A} \in \mathcal{B}(\mathcal{H})$*

$$\mathbb{L}\left(\mathbb{A}\right) = \imath\left[\mathbb{H}\,,\mathbb{A}\right] + \sum_{l\,=\,1}^{\mathcal{N}}\left(\mathbb{V}^{(l)\dagger}\,\mathbb{A}\,\mathbb{V}^{(l)} - \frac{\mathbb{V}^{(l)\dagger}\,\mathbb{V}^{(l)}\,\mathbb{A} + \mathbb{A}_{t}\,\mathbb{V}^{(l)\dagger}\,\mathbb{V}^{(l)}}{2}\right) \tag{9}$$

where $\mathbb{H}\,,\mathbb{V}^{(l)}\,,\mathbb{V}^{(l)\dagger}\,,\mathbb{V}^{(l)\dagger}\,\mathbb{V}^{(l)} \in \mathcal{B}(\mathcal{H})$ for all $l = 1, 2, \dots, \mathcal{N}$ with $\mathcal{N}$ an arbitrary integer. The dual result in the Schrödinger picture is that for any trace class $\boldsymbol{\rho}$

$$\mathbb{L}^{\dagger}\left(\boldsymbol{\rho}\right) = -\imath\left[\mathbb{H}\,,\boldsymbol{\rho}\right] + \sum_{l\,=\,1}^{\mathcal{N}}\left(\mathbb{V}^{(l)}\,\boldsymbol{\rho}\,\mathbb{V}^{(l)\dagger} - \frac{\mathbb{V}^{(l)\dagger}\,\mathbb{V}^{(l)}\,\boldsymbol{\rho} + \boldsymbol{\rho}\,\mathbb{V}^{(l)\dagger}\,\mathbb{V}^{(l)}}{2}\right) \tag{10}$$

Note that as required

$$\mathbb{L}\left(\mathbb{1}\right) = \operatorname{Tr}\left(\mathbb{L}^{\dagger}\left(\boldsymbol{\rho}_{t}\right)\right) = 0$$

The fact that the sums in (9) and (10) extend on an arbitrary number $\mathcal{N}$ of terms physically reflects the fact that an universal dynamical map evolves the (sub)-system as if its environment was continuously performing a generalized measurement on it without recording the result.

2.6. Derivation of the Lindblad–Gorini–Kossakowski–Sudarshan equations

Gorini, Kossakowski, and Sudarshan in [3] derived a result equivalent to Lindblad's working in $\mathcal{M}_{d}(\mathbb{C})$. We prove here an extension of their result for **time non-autonomous** completely positive semi-groups

Theorem. *A linear operator $\mathbb{L}_{t}\colon \mathcal{M}_{d}\left(\mathbb{C}\right) \mapsto \mathcal{M}_{d}\left(\mathbb{C}\right)$ is the generator of a completely positive semi-group on $\mathcal{M}_{d}\left(\mathbb{C}\right)$ if acting upon any $\mathbb{A} \in \mathcal{M}_{d}(\mathbb{C})$ it admits the representation*

$$\mathbb{L}_{t}\left(\mathbb{A}\right) = \imath\left[\mathbb{H}_{t}\,,\mathbb{A}\right] + \sum_{k=1}^{d^{2}-1} \mathscr{g}_{t}^{(k)}\left(\mathbb{G}_{t}^{(k)\dagger}\,\mathbb{A}\,\mathbb{G}_{t}^{(k)} - \frac{\mathbb{G}_{t}^{(k)\dagger}\,\mathbb{G}_{t}^{(k)}\,\mathbb{A} + \mathbb{A}_{t}\,\mathbb{G}_{t}^{(k)\dagger}\,\mathbb{G}_{t}^{(k)}}{2}\right) \tag{11}$$

or equivalently its adjoint acts on any state operator $\boldsymbol{\rho}$ as

$$\mathbb{L}_{t}^{\dagger}\left(\boldsymbol{\rho}\right) = -\imath\left[\mathbb{H}_{t}\,,\boldsymbol{\rho}\right] + \sum_{k=1}^{d^{2}-1} \mathscr{g}_{t}^{(k)}\left(\mathbb{G}_{t}^{(k)}\,\boldsymbol{\rho}\,\mathbb{G}_{t}^{(k)\dagger} - \frac{\mathbb{G}_{t}^{(k)\dagger}\,\mathbb{G}_{t}^{(k)}\,\boldsymbol{\rho} + \boldsymbol{\rho}\,\mathbb{G}_{t}^{(k)\dagger}\,\mathbb{G}_{t}^{(k)}}{2}\right) \tag{12}$$

where

- $\mathbb{H}_{t} = \mathbb{H}_{t}^{\dagger}$
- $\left\{\mathbb{G}_{t}^{(l)}\right\}_{l=0}^{d^{2}-1}$ *forms for all t a basis of $\mathbb{M}_{d}\left(\mathbb{C}\right)$ orthonormal with respect to the Hilbert–Schmidt scalar product.*
- $\mathscr{g}_{t}^{(l)} \geq 0$ *for all t and l*

Proof.

- **Necessary condition**: if (9) holds the solution is completely positive.

  The idea of the proof is to construct the solution of

$$\begin{aligned}&\frac{\mathrm{d}\boldsymbol{\Phi}_{ts}(\mathbb{A})}{\mathrm{d}t} = \mathbb{L}_t\left(\boldsymbol{\Phi}_{ts}(\mathbb{A})\right)\\&\boldsymbol{\Phi}_{ss}(\mathbb{A}) = \mathbb{A}\end{aligned}$$

  by composing by means of Trotter' s formula (see appendix A) the solutions of two operator differential equations **separately** generating a completely positive map. To this goal, we couch the generator $\mathbb{L}_t$ into the form

$$\mathbb{L}_t(\mathbb{A}_t) = \imath\,[\mathbb{K}_t\,,\mathbb{A}_t] + \sum_{l=1}^{d^2-1} g_t^{(k)} \mathbb{G}_t^{(l)\dagger}\mathbb{A}_t\mathbb{G}_t^{(l)} \tag{13}$$

  with $\mathbb{K}_t$ an operator satisfying

$$\mathbb{K}_t = \mathbb{H}_t + \imath \sum_{l=1}^{d^2-1} g_t^{(k)} \mathbb{G}_t^{(l)\dagger}\mathbb{G}_t^{(l)}$$

  We start to accomplish our program by observing that the solution of

$$\begin{aligned}&\frac{\mathrm{d}\boldsymbol{\Psi}_{ts}^{(0)}(\mathbb{A})}{\mathrm{d}t} = \imath\left[\mathbb{K}_t\,,\boldsymbol{\Psi}_{ts}^{(0)}(\mathbb{A})\right]\\&\boldsymbol{\Psi}_{ss}^{(0)}(\mathbb{A}) = \mathbb{A}\end{aligned}$$

  is completely positive because we can explicitly write it in the Choi form

$$\boldsymbol{\Psi}_t^{(0)}(\mathbb{A}) = \mathscr{K}_{ts}\,\mathbb{A}\,\mathscr{K}_{ts}^{\dagger}$$

  where $\mathscr{K}_{ts}$ solves

$$\mathscr{K}_{ts} = \mathbb{1} + \int_s^t du\,\mathbb{K}_u\mathscr{K}_{us}$$

  In the **time autonomous case** this latter equation admits the solution

$$\mathscr{K}_{ts} = \mathscr{K}_{t-s} = e^{\imath\,\mathbb{K}\,(t-s)}$$

  Similarly, the linear map solution of

$$\begin{aligned}&\frac{\mathrm{d}\boldsymbol{\Psi}_{ts}^{(1)}(\mathbb{A})}{\mathrm{d}t} = \sum_{l=1}^{d^2-1} g_t^{(k)} \mathbb{G}_t^{(l)\dagger}\boldsymbol{\Psi}_{ts}^{(1)}(\mathbb{A})\mathbb{G}_t^{(l)}\\&\boldsymbol{\Psi}_{ss}^{(1)}(\mathbb{A}) = \mathbb{A}\end{aligned}$$

  is completely positive because it is constructed as the exponential map of a completely positive map.

  We now compose the two maps to define

$$\boldsymbol{X}_{ts}(\mathbb{A}) = \boldsymbol{\Psi}_{ts}^{(1)}\left(\boldsymbol{\Psi}_{ts}^{(0)}(\mathbb{A})\right)$$

  and then apply Trotter's formula to construct the solution of the full equation as

$$\boldsymbol{\Phi}_{ts}(\mathbb{A}) = \lim_{n\nearrow\infty} \boldsymbol{X}^n_{s+\frac{t-s}{n}\,s}(\mathbb{A})$$

  In this expression, we construct the n-th power of the map $\boldsymbol{X}_{s+\frac{t-s}{n}\,s}(\mathbb{A})$ recursively by setting

$$\begin{aligned}&\boldsymbol{X}^2_{s+\frac{t-s}{n}\,s}(\mathbb{A}) = \boldsymbol{X}_{s+\frac{t-s}{n}\,s}\left(\boldsymbol{X}_{s+\frac{t-s}{n}\,s}(\mathbb{A})\right)\\&\boldsymbol{X}^3_{s+\frac{t-s}{n}\,s}(\mathbb{A}) = \boldsymbol{X}_{s+\frac{t-s}{n}\,s}\left(\boldsymbol{X}^2_{s+\frac{t-s}{n}\,s}(\mathbb{A})\right)\\&\vdots\\&\boldsymbol{X}^n_{s+\frac{t-s}{n}\,s}(\mathbb{A}) = \boldsymbol{X}_{s+\frac{t-s}{n}\,s}\left(\boldsymbol{X}^{n-1}_{s+\frac{t-s}{n}\,s}(\mathbb{A})\right)\end{aligned}$$

- **Sufficient condition**: if the solution is completely positive and unital (9) must hold true.

  A linear map of $\mathcal{M}_d(\mathbb{C})$ into itself is completely positive if and only if is amenable to Choi form ; Let $\{\mathbb{F}_k\}_{k=0}^{d^2-1}$ be a basis of $\mathcal{M}_d(\mathbb{C})$ orthonormal with respect to the Hilbert–Schmidt inner product such that

$$\mathbb{F}_0 = \frac{1}{\sqrt{d}}\mathbb{1} \tag{14}$$

In this basis the Choi normal form (7) of the completely positive map becomes

$$\mathbf{\Phi}_{ts}(\mathbb{A}) = \sum_{l,k=0}^{d^2-1} c_{ts}^{(l\,k)} \mathbb{F}_l\, \mathbb{A}\, \mathbb{F}_k^\dagger$$

with

$$c_{ts}^{(l\,k)} := \sum_{i\,\geq\,1} \operatorname{Tr}\left(\mathbb{F}_l^\dagger \mathbb{W}_{ts}^{(i)}\right) \operatorname{Tr}\left(\mathbb{W}_{t\,s}^{(i)\dagger}\mathbb{F}_k\right)$$

For any t, s the $c_{ts}^{(l\,k)}$'s form the entries of a **positive definite** $d^2 \times d^2$ matrix $\mathbb{C}_{ts}$. For any $\boldsymbol{v} \in \mathbb{C}^{d^2}$ we get into

$$\sum_{l,k=0}^{d^2-1} \bar{v}_l\, c_{ts}^{(l\,k)}\, v_k = \sum_{l,k=0}^{d^2-1} \bar{v}_l \sum_{i\,\geq\,1} \operatorname{Tr}\left(\mathbb{F}_l^\dagger \mathbb{W}_{ts}^{(i)}\right) \operatorname{Tr}\left(\mathbb{W}_{t\,s}^{(i)\dagger}\mathbb{F}_k\right) v_k = \sum_{i\,\geq\,1} \left\| \sum_{k=1}^{d^2-1} \operatorname{Tr}\left(\mathbb{W}_{t\,s}^{(i)\dagger}\mathbb{F}_k\right) v_k \right\|^2$$

We also recall here that positive definiteness implies

$$c_{ts}^{(l\,k)} = \bar{c}_{ts}^{(k\,l)}$$

In order to determine the generator it is sufficient to evaluate the derivative of the semi-group at $t = s$. To this goal we require

$$\mathbf{\Phi}_{t\,t}(\mathbb{A}) = \mathbb{A}$$

whence we get the conditions

$$c_{tt}^{(l\,k)} = d\, \delta_{l,0}\, \delta_{k,0}$$

We thus construct the **generator** of the linear flow

$$\mathbb{L}_t(\mathbb{A}) = \lim_{\varepsilon \searrow 0} \frac{\mathbf{\Phi}_{t+\varepsilon\, t}(\mathbb{A}) - \mathbb{A}}{t} = \lim_{\varepsilon \searrow 0} \frac{c_{t+\varepsilon\, t}^{(0\,0)} - d}{\varepsilon} \frac{\mathbb{A}}{d}$$
$$+ \lim_{\varepsilon \searrow 0} \sum_{l=1}^{d^2-1} \frac{c_{t+\varepsilon\, t}^{(l\,0)}\, \mathbb{F}_l\, \mathbb{A} + c_{t+\varepsilon\, t}^{(0\,l)}\, \mathbb{A}\, \mathbb{F}_l^\dagger}{\sqrt{d}\,\varepsilon} + \lim_{\varepsilon \searrow 0} \sum_{l,k=1}^{d^2-1} \frac{c_{t+\varepsilon\, t}^{(l\,k)}\, \mathbb{F}_l\, \mathbb{A}\, \mathbb{F}_k^\dagger}{\varepsilon}$$

To neaten the notation we define

$$\ell_t^{(l\,k)} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} c_{t+s\,t}^{(l\,k)}$$

and then write

$$\mathbb{L}_t(\mathbb{A}) = \ell_t^{(0\,0)} \frac{\mathbb{A}}{d} + \sum_{l=1}^{d^2-1} \frac{\ell_t^{(l\,0)} \mathbb{F}_l\, \mathbb{A} + \ell_t^{(0\,l)} \mathbb{A}\, \mathbb{F}_l^\dagger}{\sqrt{d}} + \sum_{l,k=1}^{d^2-1} \ell_t^{(l\,k)}\, \mathbb{F}_l\, \mathbb{A}\, \mathbb{F}_k^\dagger \tag{15}$$

The requirement that $\mathbb{L}_t$ should be unital implies

$$0 = \ell_t^{(0\,0)} \frac{\mathbb{1}}{d} + \sum_{l=1}^{d^2-1} \frac{\ell_t^{(l\,0)} \mathbb{F}_l + \ell_t^{(0\,l)} \mathbb{F}_l^\dagger}{\sqrt{d}} + \sum_{l,k=1}^{d^2-1} \ell_t^{(l\,k)}\, \mathbb{F}_l\, \mathbb{F}_k^\dagger$$

Thus, if we write (15) in the form

$$\mathbb{L}_t\,(\mathbb{A}) = \ell_t^{(0\,0)}\frac{\mathbb{A}}{d} + \sum_{l=1}^{d^2-1}\frac{(\ell_t^{(l\,0)}\mathbb{F}_l - \ell_t^{(0\,l)}\,\mathbb{F}_l^{\dagger})\,\mathbb{A} + \mathbb{A}\,(\ell_t^{(0\,l)}\,\mathbb{F}_l^{\dagger} - \ell_t^{(l\,0)}\mathbb{F}_l)}{2\,\sqrt{d}}$$
$$+ \sum_{l=1}^{d^2-1}\frac{(\ell_t^{(l\,0)}\mathbb{F}_l + \ell_t^{(0\,l)}\,\mathbb{F}_l^{\dagger})\,\mathbb{A} + \mathbb{A}\,(\ell_t^{(0\,l)}\mathbb{F}_l^{\dagger} + \ell_t^{(l\,0)}\mathbb{F}_l)}{2\,\sqrt{d}} + \sum_{l,k=1}^{d^2-1}\ell_t^{(l\,k)}\,\mathbb{F}_l\,\mathbb{A}\,\mathbb{F}_k^{\dagger}$$

and use the “unitality” condition, we get

$$\mathbb{L}_t\,(\mathbb{A}) = \sum_{l=1}^{d^2-1}\frac{(\ell_t^{(l\,0)}\mathbb{F}_l - \ell_t^{(0\,l)}\,\mathbb{F}_l^{\dagger})\,\mathbb{A} + \mathbb{A}\,(\ell_t^{(0\,l)}\,\mathbb{F}_l^{\dagger} - \ell_t^{(l\,0)}\mathbb{F}_l)}{2\,\sqrt{d}} + \sum_{l,k=1}^{d^2-1}\ell_t^{(l\,k)}\left(\mathbb{F}_l\,\mathbb{A}\,\mathbb{F}_k^{\dagger} - \frac{\mathbb{F}_k^{\dagger}\mathbb{F}_l\,\mathbb{A} + \mathbb{A}\mathbb{F}_k^{\dagger}\mathbb{F}_l}{2}\right)$$

It is then expedient to define the self-adjoint operator

$$\mathbb{H}_t = \sum_{l=1}^{d^2-1}\frac{(\ell_t^{(l\,0)}\mathbb{F}_l - \ell_t^{(0\,l)}\,\mathbb{F}_l^{\dagger})}{2\,\sqrt{d}\,\imath}$$

in order to reduce (15) to the simpler form

$$\mathbb{L}_t\,(\mathbb{A}) = \imath\,[\mathbb{H}_t\,,\mathbb{A}] + \sum_{l,k=1}^{d^2-1}\ell_t^{(l\,k)}\left(\mathbb{F}_l\,\mathbb{A}\,\mathbb{F}_k^{\dagger} - \frac{\mathbb{F}_l\,\mathbb{F}_k^{\dagger}\,\mathbb{A} + \mathbb{A}\,\mathbb{F}_l\,\mathbb{F}_k^{\dagger}}{2}\right) \tag{16}$$

Finally, we notice that the $\ell_t^{(l\,k)}$ for $l, k = 1, \dots, d^2 - 1$ are also the entries of a $(d^2 - 1) \times (d^2 - 1)$ positive definite matrix.

$$\ell_t^{(l\,k)} = \boldsymbol{e}_l^{\dagger}\,\mathsf{L}\,\boldsymbol{e}_k$$

This is because $\mathbb{C}_{t+\varepsilon,t}$ is positive definite for all t. Furthermore at $\varepsilon = 0$ the only non-vanishing entry is $c_{0\,0}(t,t) = d$ which does not contribute as well as all elements of the the first row and of the first column of $\mathbb{C}_{t+\varepsilon,t}$ to the definition of the $(d^2-1)\times(d^2-1)$ square matrix $\mathbb{L}$. Hence the sub-matrix of the increments must be positive definite. The spectral theorem then gives

$$\mathsf{L} = \sum_{i=1}^{d^2-1}\mathscr{g}_t^{(i)}\boldsymbol{v}_t^{(i)}\boldsymbol{v}_t^{(i)\dagger}$$

where $\left\{\boldsymbol{v}_t^{(i)}\right\}_{l=1}^{d^2-1}$ are the eigenvectors of the self-adjoint and positive definite matrix L. The projection on the canonical basis is

$$\ell_t^{(l\,k)} = \sum_{i=1}^{d^2-1}\mathscr{g}_t^{(i)}(\boldsymbol{e}_l^{\dagger}\boldsymbol{v}_t^{(i)})\,(\boldsymbol{v}_t^{(i)\dagger}\boldsymbol{e}_k)$$

We thus can rewrite the generator as

$$\mathbb{L}_t\,(\mathbb{A}) = \imath\,[\mathbb{H}_t\,,\mathbb{A}] + \sum_{l,k=1}^{d^2-1}\sum_{i=1}^{d^2-1}\mathscr{g}_t^{(i)}(\boldsymbol{e}_l^{\dagger}\boldsymbol{v}_t^{(i)})\left(\mathbb{F}_l\,\mathbb{A}\,\mathbb{F}_k^{\dagger} - \frac{\mathbb{F}_l\,\mathbb{F}_k^{\dagger}\,\mathbb{A} + \mathbb{A}\,\mathbb{F}_l\,\mathbb{F}_k^{\dagger}}{2}\right)(\boldsymbol{v}_t^{(i)\dagger}\boldsymbol{e}_k)$$

$$\mathbb{G}_t^{(i)} = \sum_{l=1}^{d^2-1}(\boldsymbol{e}_l^{\dagger}\boldsymbol{v}_t^{(i)})\,\mathbb{F}_l$$

It is straightforward to verify that together with

$$\mathbb{G}_t^{(0)} = \frac{1}{\sqrt{d}}\mathbb{1}_d$$

the new set of operators $\left\{\mathbb{G}_t^{(i)}\right\}_{i=0}^{d^2-1}$ form an orthonormal basis with respect with respect to the Hilbert-Schmidt inner product

$$\operatorname{Tr}\left(\mathbb{G}_t^{(0)}\mathbb{G}_t^{(i)}\right) = \sum_{l=1}^{d^2-1}(\boldsymbol{e}_l^\dagger \boldsymbol{v}_t^{(i)})\operatorname{Tr}(\mathbb{F}_l) = 0 \qquad i = 1,\dots,d^2-1$$

$$\operatorname{Tr}\left(\mathbb{G}_t^{(j)\dagger}\mathbb{G}_t^{(i)}\right) = \sum_{l,k=1}^{d^2-1}(\boldsymbol{v}_t^{(j)\dagger}\boldsymbol{e}_k)(\boldsymbol{e}_l^\dagger \boldsymbol{v}_t^{(i)})\operatorname{Tr}\left(\mathbb{F}_k^\dagger\mathbb{F}_l\right) = \sum_{l,k=1}^{d^2-1}(\boldsymbol{v}_t^{(j)\dagger}\boldsymbol{e}_k)(\boldsymbol{e}_l^\dagger \boldsymbol{v}_t^{(i)})\delta_{l\,k} = \delta_{i\,j} \qquad i,j = 1,\dots,d^2-1$$

□

Finally, we observe that the form (11) of the generator involves d^2-1 terms. It is however possible to re-define the sum in (16) so that it ranges over a set $\{\mathbb{M}_i\}_{i=1}^{\mathcal{N}}$ operators satisfying the only constraint

$$\sum_{i=1}^{\mathcal{N}}\mathbb{M}_i^\dagger\mathbb{M}_i = \mathbb{1}_d$$

required by a generalized measurement.

3. NON COMPLETELY POSITIVE MASTER EQUATION

If we drop hypothesis **i**, we obtain the trace preserving master equation (9) and its adjoint (12) with the only restriction

$$|g_t^{(l)}| < \infty \qquad \forall\, t\,,l$$

in other words finite negative values $g_t^{(l)}$ are compatible with trace preservation. The solution flow is **completely bounded** and it can be proven to reduce to the general form

$$\boldsymbol{\Phi}_{t\,s}(\boldsymbol{\rho}_s) = \sum_{i=1}^{\mathcal{N}^{(+)}}\mathbb{W}_{t\,s}^{(i)}\boldsymbol{\rho}_s\mathbb{W}_{t\,s}^{(i)\dagger} - \sum_{i=\mathcal{N}^{(+)}+1}^{\mathcal{N}}\mathbb{W}_{t\,s}^{(i)}\boldsymbol{\rho}_s\mathbb{W}_{t\,s}^{(i)\dagger}$$

based on rigorous results in operator algebra by Wittstock [11] and Paulsen [6]. Thus a completely bounded flow is always amenable to the difference of two completely positive ones. The representation shows that a completely bounded flow does not preserve by default positivity but may give rise to a (completely) positive evolution for initial data in a certain "positivity domain" [10] and eventually only up to a certain time.

APPENDICES

Appendix A: Trotter formula

We refer to chapter 1 of [?] for a proof of Trotter's formula formula valid in an abstract Banach space $\mathcal{B}(\mathbb{C})$. Here we follow [?] and prove Trotter's in $\mathcal{M}_d(\mathbb{C})$. In $\mathcal{M}_d(\mathbb{C})$ we avail us of the norm

$$\parallel \mathbb{A} \parallel = \sup_{\boldsymbol{x}\in\mathbb{C}^d}\{\parallel \mathbb{A}\boldsymbol{x} \parallel \;|\; \parallel \boldsymbol{x} \parallel = 1\} = \sup_{\boldsymbol{x}\in\mathbb{C}^d}\left\{\sqrt{\langle\, \boldsymbol{x}\,,\mathbb{A}^\dagger\mathbb{A}\boldsymbol{x}\,\rangle}\;|\; \parallel \boldsymbol{x} \parallel = 1\right\} \tag{A1}$$

to control the convergence of series defining functions of matrices. The norm offers the advantage to readily verifying the inequality

$$\parallel \mathbb{A}\,\mathbb{B} \parallel \leq \parallel \mathbb{A} \parallel \parallel \mathbb{B} \parallel \tag{A2}$$

in view of

$$\frac{\langle \mathbb{A}\,\mathbb{B}\,\boldsymbol{x}\,,\mathbb{A}\,\mathbb{B}\,\boldsymbol{x}\rangle}{\langle \boldsymbol{x}\,,\boldsymbol{x}\rangle} = \frac{\langle \mathbb{A}\mathbb{B}\boldsymbol{x}\,,\mathbb{A}\mathbb{B}\boldsymbol{x}\rangle}{\langle \mathbb{B}\boldsymbol{x}\,,\mathbb{B}\boldsymbol{x}\rangle}\frac{\langle \mathbb{B}\boldsymbol{x}\,,\mathbb{B}\boldsymbol{x}\rangle}{\langle \boldsymbol{x}\,,\boldsymbol{x}\rangle}$$

Theorem. *(**Trotter's formula**): for any finite dimensional matrices*

$$e^{\mathbb{A}+\mathbb{B}} = \lim_{N\nearrow\infty} \left(e^{\frac{\mathbb{A}}{N}}\, e^{\frac{\mathbb{B}}{N}} \right)^N$$

Proof.
We sketch the proof observing that precise estimates of reminders can be obtained in the norm (A1). For N sufficiently large we write

$$e^{\frac{\mathbb{A}}{N}}\, e^{\frac{\mathbb{B}}{N}} = \mathbb{1} + \frac{\mathbb{A}+\mathbb{B}}{N} + O\left(\frac{1}{N^2}\right)$$

and consequently

$$\mathbb{Z}_N = \ln\left(e^{\frac{\mathbb{A}}{N}}\, e^{\frac{\mathbb{B}}{N}} \right) = \frac{\mathbb{A}+\mathbb{B}}{N} + O\left(\frac{1}{N^2}\right)$$

We are therefore in the position to conclude that

$$\lim_{N\nearrow\infty} \left(e^{\frac{\mathbb{A}}{N}}\, e^{\frac{\mathbb{B}}{N}} \right)^N = \lim_{N\nearrow\infty} \left(e^{\mathbb{Z}_N} \right)^N = \lim_{N\nearrow\infty} e^{N\,\mathbb{Z}_N} = \lim_{N\nearrow\infty} e^{\mathbb{A}+\mathbb{B}+O\left(\frac{1}{N}\right)} = e^{\mathbb{A}+\mathbb{B}}$$

□

* * *

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