TCM315 Fall 2022: Introduction to Open Quantum Systems

Lecture 12: Completely positive universal dynamical maps

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1. INTRODUCTION

Chapter 3 of [5] offers a crisp presentation of the algebraic properties of maps governing time evolution in open quantum systems. We refer to [4] for explicit examples of completely positive dynamical maps. Finally, chapter 8 of [3] pedagogically introduces the link between universal dynamical maps and quantum information theory.

2. EVOLUTION OF A CONSTITUENT OF A COMPOSITE SYSTEM

We start by recalling the unitary evolution postulate for the state operator of a closed system.
The state operator of a closed system evolves according the Liouville–von Neumann equation:

$$\imath \frac{\mathrm{d}\boldsymbol{\rho}_t}{\mathrm{d}t} = [\mathbb{H}_t\,, \boldsymbol{\rho}_t] \tag{1}$$

The Liouville von Neumann equation relates the state operator $\boldsymbol{\rho}_t$ at time t to its value $\boldsymbol{\rho}_{t_0}$ at time t_0

$$\boldsymbol{\rho}_t = \mathbb{U}_{t,t_0}\boldsymbol{\rho}_{t_0}\,\mathbb{U}^{\dagger}_{t,t_0}$$

via the fundamental solution of the Schrödinger equation

$$\begin{aligned} \imath \partial_t \mathbb{U}_{t,t_0} &= \mathbb{H}_t \mathbb{U}_{t,t_0} \\ \mathbb{U}_{t_0,t_0} &= \mathbb{1}_{\mathcal{H}} \end{aligned}$$

2.1. Universal dynamical map

We now suppose that at time t_0 the state operator of the closed system has the tensor product form

$$\boldsymbol{\rho}_{t_0} = \boldsymbol{\rho}_{t_0}^{(S)} \otimes \boldsymbol{\rho}_{t_0}^{(E)} \tag{2}$$

We refer to the constituents, S and E respectively as "system" and "environment". We also assume that Hamilton operator $\mathbb{H}$ in (1) includes "system" and "environment" interactions

$$\mathbb{H} = \mathbb{H}^{(S)} \otimes \mathbb{1}_{\mathcal{H}_E} + \mathbb{1}_{\mathcal{H}_S} \otimes \mathbb{H}^{(E)} + g\,\mathbb{H}^{(I)}$$

Unitary evolution does not preserve the tensor product form (2) of the initial data. We turn to inquire what is the effective dynamics of the system if we trace out the degrees of freedom of the environment.

2.1.1. Partial trace

Let $\{\boldsymbol{e}_i\}_{i=1}^{\dim \mathcal{H}_E}$ an orthonormal system spanning the Hilbert space of the environment $\mathcal{H}_E$ and such that

$$\boldsymbol{\rho}_{t_0}^{(E)} = \sum_{i=1}^{\dim \mathcal{H}_E} r_i\, \boldsymbol{e}_i \boldsymbol{e}_i^{\dagger}$$

We define the partial trace

$$\boldsymbol{\rho}_t^{(S)} = \operatorname{Tr}_{\mathcal{H}_E}\left(\mathbb{U}_{t,t_0}\boldsymbol{\rho}_{t_0}^{(S)} \otimes \boldsymbol{\rho}_{t_0}^{(E)}\mathbb{U}_{t,t_0}\right) = \sum_{m=1}^{\dim \mathcal{H}_E} \boldsymbol{e}_m^{\dagger}\mathbb{U}_{t,t_0}\boldsymbol{\rho}_{t_0}^{(S)} \otimes \boldsymbol{\rho}_{t_0}^{(E)}\mathbb{U}_{t,t_0}^{\dagger}\boldsymbol{e}_m \tag{3}$$

If we insert the expression of the state operator of the environment we get

$$\boldsymbol{\rho}_t^{(S)} = \sum_{n,m=1}^{\dim \mathcal{H}_E} \boldsymbol{e}_m^{\dagger}\mathbb{U}_{t,t_0}\boldsymbol{\rho}_{t_0}^{(S)} \otimes r_n\, \boldsymbol{e}_n \boldsymbol{e}_n^{\dagger}\mathbb{U}_{t,t_0}^{\dagger}\boldsymbol{e}_m$$

which after defining

$$\mathbb{M}_{t,t_0}^{(m,n)} = \sqrt{r_n}\, \boldsymbol{e}_m^{\dagger}\mathbb{U}_{t,t_0}\, \boldsymbol{e}_n$$

admits the compact equivalent **Kraus representation** of the **quantum dynamical map**

$$\boldsymbol{\rho}_t^{(S)} = \sum_{m,n=1}^{\dim \mathcal{H}_E} \mathbb{M}_{t,t_0}^{(m,n)}\boldsymbol{\rho}_{t_0}\mathbb{M}_{t,t_0}^{(m,n)\dagger} \tag{4}$$

We emphasize that the collection of operators $\left\{\mathbb{M}_{t,t_0}^{(m,n)}\right\}_{m,n=1}^{\dim \mathcal{H}_E}$ depend upon the initial state of the environment but are **independent of the initial state** of the system. For this reason (4) is referred to as **universal dynamical map**.

2.2. Properties of the reduced state operator

We now verify that the partial trace define a reduced state operator (4) enjoying all the mathematical properties required by its physical interpretation.

2.2.1. Self-adjoint

The Kraus representation (4) readily guarantees that reduced state operactor be self-adjoint

$$\boldsymbol{\rho}_t^{(S)} = \boldsymbol{\rho}_t^{(S)\dagger}$$

2.2.2. Unital

Probability conservation

$$\operatorname{Tr} \boldsymbol{\rho}_t^{(S)} = 1$$

is than insured by the completeness relation

$$\sum_{m,n=1}^{\dim \mathcal{H}_E} \mathbb{M}_{t,t_0}^{(m,n)\dagger} \mathbb{M}_{t,t_0}^{(m,n)} = \mathbb{1}_{\mathcal{H}_S} \tag{5}$$

To verify this latter identity, we notice that

$$\mathbb{M}_{t,t_0}^{(m,n)\dagger} \mathbb{M}_{t,t_0}^{(m,n)} = \sum_{m,n}^{\dim \mathcal{H}_E} \sqrt{r_n}\, \boldsymbol{e}_n^\dagger \mathbb{U}_{t,t_0}^\dagger \boldsymbol{e}_m \sqrt{r_n}\, \boldsymbol{e}_m^\dagger \mathbb{U}_{t,t_0} \boldsymbol{e}_n = \sum_{n}^{\dim \mathcal{H}_E} r_n\, \boldsymbol{e}_n^\dagger \mathbb{U}_{t,t_0}^\dagger \left(\mathbb{1}_{\mathcal{H}_S} \otimes \sum_{m}^{\dim \mathcal{H}_E} \boldsymbol{e}_m \boldsymbol{e}_m^\dagger \right) \mathbb{U}_{t,t_0} \boldsymbol{e}_n$$

$$= \sum_{n}^{\dim \mathcal{H}_E} r_n\, \boldsymbol{e}_n^\dagger \mathbb{U}_{t,t_0}^\dagger \mathbb{U}_{t,t_0} \boldsymbol{e}_n = \sum_{n}^{\dim \mathcal{H}_E} r_n\, \boldsymbol{e}_n^\dagger \mathbb{1}_{\mathcal{H}_S} \otimes \mathbb{1}_{\mathcal{H}_E} \boldsymbol{e}_n = \mathbb{1}_{\mathcal{H}_S} \sum_{n}^{\dim \mathcal{H}_E} r_n \boldsymbol{e}_n^\dagger \boldsymbol{e}_n = \mathbb{1}_{\mathcal{H}_S} \sum_{n}^{\dim \mathcal{H}_E} r_n = \mathbb{1}_{\mathcal{H}_S}$$

having used

$$\mathbb{U}_{t,t_0}^\dagger \mathbb{U}_{t,t_0} = \mathbb{1}_{\mathcal{H}_S \otimes \mathcal{H}_E} = \mathbb{1}_{\mathcal{H}_S} \otimes \mathbb{1}_{\mathcal{H}_E}$$

2.2.3. Positive definite

For any $\boldsymbol{v} \in \mathcal{H}_S$ the identity

$$\boldsymbol{v}^\dagger \boldsymbol{\rho}_t^{(S)} \boldsymbol{v} = \sum_{m,n=1}^{\dim \mathcal{H}_E} \boldsymbol{v}^\dagger \mathbb{M}_{t,t_0}^{(m,n)} \boldsymbol{\rho}_{t_0} \mathbb{M}_{t,t_0}^{(m,n)\dagger} \boldsymbol{v}$$

once we insert the diagonal representation of the state operator

$$\boldsymbol{\rho}_{t_0}^{(S)} = \sum_{i=0}^{\dim \mathcal{H}_S} s_i \mathfrak{s}_i \mathfrak{s}_i^\dagger$$

reduces to

$$\boldsymbol{v}^\dagger \boldsymbol{\rho}_t^{(S)} \boldsymbol{v} = \sum_{i=0}^{\dim \mathcal{H}_S} s_i \sum_{m,n=1}^{\dim \mathcal{H}_E} \boldsymbol{v}^\dagger \mathbb{M}_{t,t_0}^{(m,n)} \mathfrak{s}_i \mathfrak{s}_i^\dagger\, \mathbb{M}_{t,t_0}^{(m,n)\dagger} \boldsymbol{v} = \sum_{i=0}^{\dim \mathcal{H}_S} s_i \sum_{m,n=1}^{\dim \mathcal{H}_E} \left| \mathfrak{s}_i^\dagger \mathbb{M}_{t,t_0}^{(m,n)\dagger} \boldsymbol{v} \right|^2 \geq 0$$

3. KRAUS REPRESENTATION AND QUANTUM CHANNELS

Universal dynamical maps in Kraus' form (4) play an important role in quantum information [3]

Definition. ***quantum channel:*** *let $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$ the Hilbert space of a bipartite Hilbert space consisting of a system and an ancilla. Given any unitary evolution $\mathbb{U}$ and projective measurement $\mathbb{P}$ on $\mathcal{H}$, one calls* ***quantum channel*** *the partial trace*

$$\mathcal{E}(\boldsymbol{\rho}^{(S)}) := \operatorname{Tr}_{\mathcal{H}_A} \left(\mathbb{P}\mathbb{U} \left(\boldsymbol{\rho}^{(S)} \otimes \boldsymbol{\rho}^{(A)} \right) \mathbb{U}^\dagger \mathbb{P} \right)$$

In the previous section we already proved that

Proposition. ***(Kraus representation.)*** *For any quantum channel in $\mathcal{H}_S$, there exists a collection of operations $\{\mathbb{M}_l\}_{l=1}^N$ such that*

$$\mathcal{E}(\boldsymbol{\rho}^{(S)}) = \sum_{l=1}^{N} \mathbb{M}_l \, \boldsymbol{\rho}^{(S)} \, \mathbb{M}_l^\dagger$$

In particular, for a probability conserving channel (or non-dissipative) the corresponding effects satisfy a completeness relation

$$\sum_{l=1}^{N} \mathbb{M}_l^\dagger \mathbb{M}_l = \mathbb{1}_{\mathbb{H}_S}$$

The Krauss representation admits also a converse

Proposition. ***(Stinespring dilation - purification)*** *For any quantum channel*

$$\mathcal{E}(\boldsymbol{\rho}) = \sum_{j=1}^{\dim \mathcal{H}_S} \mathbb{M}_j \, \boldsymbol{\rho} \, \mathbb{M}_j^\dagger$$

on an Hilbert space $\mathcal{H}_S$ there exists an ancilla Hilbert space $\mathcal{H}_A$ a state operator $\boldsymbol{\rho}^{(A)}$ and a unitary operator $\mathbb{U}$ on $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$ with $\dim \mathcal{H} \leq 2 \dim \mathcal{H}_S$ such that

$$\mathcal{E}(\boldsymbol{\rho}) = \mathrm{Tr}_A \left(\mathbb{U} \left(\boldsymbol{\rho} \otimes \boldsymbol{\rho}^{(A)} \right) \mathbb{U} \right)$$

Furthermore, it is always possible to choose $\mathcal{H}_A$, $\mathbb{U}$ and $\boldsymbol{\rho}^{(A)}$ is such a way that $\boldsymbol{\rho}^{(A)}$ is a pure state.

Proof.
As any mixed state is the convex linear combination of pure states it is sufficient to prove the proposition for a pure state $\boldsymbol{s} \in \mathcal{H}_S$:

$$\boldsymbol{\rho}^{(S)} = \boldsymbol{s}\boldsymbol{s}^\dagger$$

In such a case the proof is given by the same construction we used for an indirect measurement. Indeed we may choose

$$\boldsymbol{\rho}^{(A)} = \boldsymbol{f}_1 \boldsymbol{f}_1^\dagger$$

where $\{\boldsymbol{f}_j\}_{j=1}^{d_A}$ is an orthonormal basis of the ancilla Hilbert space $\mathcal{H}_A$. As there can be at most $\dim \mathcal{H}_S$ linearly independent effects on $\mathcal{H}_S$ it follows immediately that $d_A \leq \dim \mathcal{H}_S$ and therefore $d \leq 2 \dim \mathcal{H}_S$. Furthermore, it is possible to show that it is always possible to find of an unitary map $\mathbb{U}$

$$\boldsymbol{\psi} = \mathbb{U} \left(\boldsymbol{s} \otimes \boldsymbol{f}_1 \right)$$

whose outcome is amenable to the form

$$\boldsymbol{\psi} = \sum_{j=1}^{\dim \mathcal{H}_S} \left(\mathbb{M}_j \boldsymbol{s} \right) \otimes \boldsymbol{f}_j \tag{6}$$

To uphold the claim we observe that if $\mathbb{U}$ is an unitary, and $\{\boldsymbol{f}_j\}_{j=1}^{d_A}$ is a complete basis of $\mathcal{H}_A$ whose first element is $\boldsymbol{f}_1$, we can use the completeness relation

$$\mathbb{1}_{\mathcal{H}_A} = \sum_{j=1}^{d_A} \boldsymbol{f}_j \boldsymbol{f}_j^\dagger$$

to write

$$\boldsymbol{\psi} = \mathbb{1}_{\mathcal{H}_S} \otimes \mathbb{1}_{\mathcal{H}_A} \mathbb{U}\left(\boldsymbol{s} \otimes \boldsymbol{f}_1\right) = \sum_{j=1}^{d_A} \mathbb{1}_{\mathcal{H}_S} \otimes \boldsymbol{f}_j \boldsymbol{f}_j^{\dagger} \mathbb{U}\left(\boldsymbol{s} \otimes \boldsymbol{f}_1\right) = \sum_{j=1}^{d_A} \left(\boldsymbol{f}_j^{\dagger} \mathbb{U}\left(\boldsymbol{s} \otimes \boldsymbol{f}_1\right)\right) \otimes \boldsymbol{f}_j$$

We arrive at (6) by defining

$$\mathbb{M}_j \boldsymbol{s} = \boldsymbol{f}_j^{\dagger} \mathbb{U}\left(\boldsymbol{s} \otimes \boldsymbol{f}_1\right)$$

The immediate consequence is that

$$\operatorname{Tr}_{\mathcal{H}_A} \boldsymbol{\psi} \boldsymbol{\psi}^{\dagger} = \sum_{j=1}^{\dim \mathcal{H}_S} \boldsymbol{f}_j^{\dagger} \boldsymbol{\psi} \boldsymbol{\psi}^{\dagger} \boldsymbol{f}_j = \sum_{j=1}^{\dim \mathcal{H}_S} \mathbb{M}_j \, \boldsymbol{s} \, \boldsymbol{s}^{\dagger} \mathbb{M}_j^{\dagger}$$

thus proving the claim. □

4. POSITIVITY VERSUS COMPLETE POSITIVITY

In what follow, we consider only **linear maps $\boldsymbol{\Phi}$** acting on Hilbert space operators.

A dynamical map in Kraus form maps a positive definite operator into a positive definite operator. In fact, it enjoys a further property with no classical counter-part: *complete positivity*. Complete positivity characterizes the behavior of the map when extended to embedding Hilbert spaces defined by tensor product with any ancillary Hilbert space $\mathbb{C}^k$.

The following example shows that the extension of a positive map $\boldsymbol{\Phi}$ to an embedding Hilbert space by tensor product does not necessary preserve positivity.

4.1. Partial adjoint of an entangled state operator

Let us consider a bipartite system formed by two qubits. The Hilbert space $\mathbb{H}$ of the bipartite system is four dimensional. An entangled state in $\mathbb{H}$ is

$$\boldsymbol{\psi} = \frac{\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2}{\sqrt{2}} \tag{7}$$

The state is also **maximally entangled** because the Schmidt number (see § 2.7 of [3]), counting the tensor products of orthogonal states in the linear combination, coincides with the dimension of the (smallest) constituent Hilbert spaces. The state operator specified by (7) is

$$\boldsymbol{\psi}\boldsymbol{\psi}^{\dagger} = \frac{\mathbb{E}_{1\,1} \otimes \mathbb{E}_{1\,1} + \mathbb{E}_{2\,1} \otimes \mathbb{E}_{2\,1} + \mathbb{E}_{1\,2} \otimes \mathbb{E}_{1\,2} + \mathbb{E}_{2\,2} \otimes \mathbb{E}_{2\,2}}{2} \tag{8}$$

where

$$\mathbb{E}_{1\,1} = \boldsymbol{e}_1 \boldsymbol{e}_1^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbb{E}_{1\,2} = \boldsymbol{e}_1 \boldsymbol{e}_2^{\dagger} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbb{E}_{2\,1} = \mathbb{E}_{2\,1}^{\dagger} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbb{E}_{2\,2} = \boldsymbol{e}_2 \boldsymbol{e}_2^{\dagger} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

As we evaluate the tensor product

$$\mathbb{E}_{1\,1} \otimes \mathbb{E}_{1\,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \mathbb{E}_{1\,1} = \begin{bmatrix} \mathbb{E}_{1\,1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we proceed in the same way for the remaining ones. We finally re-write (8) as

$$\boldsymbol{\psi}\boldsymbol{\psi}^{\dagger} = \begin{bmatrix} \mathbb{E}_{1\,1} & \mathbb{E}_{1\,2} \\ \mathbb{E}_{2\,1} & \mathbb{E}_{2\,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The resulting matrix is a projector and a state operator: by construction is **positive definite**.

Let $\mathbf{\Phi}$ be the map over $\mathcal{M}_2(\mathbb{C})$ turning a matrix into its adjoint

$$\mathbf{\Phi}\,(\mathbb{A}) = \mathbb{A}^\dagger$$

Clearly, $\mathbf{\Phi}$ is positive since for any **positive definite** matrix $\mathbb{A}$ the chain of equalities

$$\mathrm{Det}\,(\mathbb{A} - a\,\mathbb{1})^* = \mathrm{Det}\left(\mathbb{A}^\dagger - a^*\,\mathbb{1}\right) = \mathrm{Det}\left(\mathbb{A}^\dagger - a\,\mathbb{1}\right)$$

holds true. Similarly, let us denote by $\mathbb{1}_2$ the identity map on $\mathcal{M}_2(\mathbb{C})$. We can then define the map acting on $\mathcal{M}_4\,(\mathbb{C})$ as the the tensor product $\mathcal{I}_2 \otimes \mathbf{\Phi}$. We obtain

$$(\mathcal{I}_2 \otimes \mathbf{\Phi})(\boldsymbol{\psi}\boldsymbol{\psi}^\dagger) = \begin{bmatrix} \mathbf{\Phi}(\mathbb{E}_{1\,1}) & \mathbf{\Phi}(\mathbb{E}_{1\,2}) \\ \mathbf{\Phi}(\mathbb{E}_{2\,1}) & \mathbf{\Phi}(\mathbb{E}_{2\,2}) \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{1\,1} & \mathbb{E}_{2\,1} \\ \mathbb{E}_{1\,2} & \mathbb{E}_{2\,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A straightforward calculation shows that

$$\mathrm{Det}(\mathbb{B} - b\,\mathbb{1}) = 0$$

has one **negative root** for $b = -1$. The upshot is that, **the tensor product of two positive definite maps may not be positive definite**.

4.2. Completely positive maps.

In order to rule out the occurrence of phenomena like the one described in the example of § 4 4.1, it is expedient to introduce a refined notion of positivity

Definition. $\mathbf{\Phi}\colon \mathcal{M}_n\,(\mathbb{C}) \mapsto \mathcal{M}_m\,(\mathbb{C})$ *is* p-***positive*** *if and only if the map* $\boldsymbol{\mathscr{F}}\colon \mathcal{M}_{p\,\times\,n}(\mathbb{C}) \mapsto \mathcal{M}_{p\,\times\,m}(\mathbb{C})$ *defined by*

$$\boldsymbol{\mathscr{F}} = \mathcal{I}_p \otimes \mathbf{\Phi}$$

, is positive.

It is important to clarify what the tensor product $\mathcal{I}_p \otimes \mathbf{\Phi}$ of maps means. If $\{\mathbb{O}_{i,j}\}_{i,j=1}^{k}$ is a collection of bounded operators acting on the finite dimensional Hilbert space $\mathcal{H}_S$, we consider extensions of the linear map $\mathbf{\Phi}$ induced by acting with the identity on the ancilla

$$\mathcal{I}_k \otimes \mathbf{\Phi}\colon \begin{bmatrix} \mathbb{O}_{1\,1} & \dots & \mathbb{O}_{1\,k} \\ \vdots & \vdots & \vdots \\ \mathbb{O}_{k\,1} & \dots & \mathbb{O}_{k\,k} \end{bmatrix} \mapsto \begin{bmatrix} \mathbf{\Phi}(\mathbb{O}_{1\,1}) & \dots & \mathbf{\Phi}(\mathbb{O}_{1\,k}) \\ \vdots & \vdots & \vdots \\ \mathbf{\Phi}(\mathbb{O}_{k\,1}) & \dots & \mathbf{\Phi}(\mathbb{O}_{k\,k}) \end{bmatrix}$$

Complete positivity means that for for any $k \in \mathbb{N}$ the extension linear map also maps positive operators into positive operators on the tensor product Hilbert space. This notion can be further strengthened in the form

Definition. $\mathbf{\Phi}\colon \mathcal{M}_n\,(\mathbb{C}) \mapsto \mathcal{M}_m\,(\mathbb{C})$ *when* $\mathcal{I}_p \otimes \mathbf{\Phi}$*, is* ***completely positive*** *when it is* p-***positive*** *for all positive integers* p*.*

4.2.1. Square matrices whose elements are operators

The notion of complete positivity entails the consideration of matrices whose elements are themselves matrices, and in the most general case, bounded operators. In this respect, we recall (in the words of [6]) that if $\mathbb{F}$ is a field (i.e. $\mathbb{R}$, $\mathbb{C}$ etc.), then the set $\mathcal{M}_n(\mathbb{F})$ of all $n \times n$ matrices over $\mathbb{F}$ forms a **ring** (non-commutative if $n \geq 2$), because its elements can be added, subtracted and multiplied, and all the ring axioms (associativity, distributivity, etc.) hold. The upshot is that we can manipulate matrix of operators as ordinary matrices.

Let us consider $\mathcal{B}(\mathcal{H})$ the space of bounded operators over an Hilbert space $\mathcal{H}$ and $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ the set of $n \times n$ matrices over $\mathcal{B}(\mathcal{H})$. Upon denoting by $\{\mathbb{E}_{i\,j}\}_{i,j=1}^{n}$ the canonical basis of $\mathbb{C}^n$, we can use the tensor product notation to compactly write any element of $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$

$$\mathcal{M}_n(\mathcal{B}(\mathcal{H})) \ni \mathbb{M} = \begin{bmatrix} \mathbb{O}_{1\,1} & \dots & \mathbb{O}_{1\,k} \\ \vdots & \vdots & \vdots \\ \mathbb{O}_{k\,1} & \dots & \mathbb{O}_{k\,k} \end{bmatrix} \sim \sum_{i,j=1}^{n} \mathbb{E}_{i\,j} \otimes \mathbb{O}_{i\,j}$$

We now notice that if $\mathbb{M}$ is positive definite

$$0 \leq \boldsymbol{V}^{\dagger} \mathbb{M} \boldsymbol{V} \qquad \forall \boldsymbol{V} \in \mathcal{H}^{\otimes n}$$

then it must be possible to find a collection $\left\{\mathbb{A}_i^{(\ell)}\right\}_{i=1}^{n}$, $\ell = 1, \dots, L$ of elements of $\mathcal{B}(\mathcal{H})$ such that

$$\mathbb{O}_{i\,j} = \sum_{\ell=1}^{L} \mathbb{A}_i^{(\ell)\dagger} \mathbb{A}_j^{(\ell)}$$

Namely, it is non restrictive to write

$$\boldsymbol{V} = \sum_{i=1}^{n} \boldsymbol{e}_i \otimes \boldsymbol{v}_i$$

where the $\boldsymbol{v}_i$'s are a collection of elements of $\mathcal{H}$. Then the positivity condition becomes

$$\begin{aligned} \boldsymbol{V}^{\dagger} \mathbb{M} \boldsymbol{V} &= \left(\sum_{i=1}^{n} \boldsymbol{e}_i \otimes \boldsymbol{v}_i\right)^{\dagger} \left(\sum_{a,b=1}^{n} \mathbb{E}_{a\,b} \otimes \mathbb{O}_{a\,b}\right) \left(\sum_{j=1}^{n} \boldsymbol{e}_j \otimes \boldsymbol{v}_j\right) \\ &= \sum_{a,b=1}^{n} \boldsymbol{v}_a^{\dagger} \mathbb{O}_{a\,b} \boldsymbol{v}_b = \sum_{a,b=1}^{n} \sum_{\ell=1}^{L} \boldsymbol{v}_a^{\dagger} \mathbb{A}_a^{(\ell)\dagger} \mathbb{A}_b^{(\ell)} \boldsymbol{v}_b = \sum_{\ell=1}^{L} \left\| \sum_{a=1}^{n} \mathbb{A}_a^{(\ell)} \boldsymbol{v}_a \right\|^2 \geq 0 \end{aligned}$$

4.2.2. The Jamiołkowski representation

The observations of the foregoing section can be further developed when $\mathcal{B}(\mathcal{H}) = \mathbb{C}^d$. In such a case, any operator admits an expansion on the canonical basis of $\mathbb{C}^d$

$$\mathbb{O} = \sum_{i\,j=1}^{d} \mathbb{E}_{i\,j} \operatorname{Tr}\left(\mathbb{E}_{i\,j}^{\dagger} \mathbb{O}\right)$$

Correspondingly, we reconstruct the action a linear map on an operator from the action on the elements of the canonical basis:

$$\boldsymbol{\Phi}(\mathbb{O}) = \sum_{i\,j=1}^{d} \boldsymbol{\Phi}(\mathbb{E}_{i\,j}) \operatorname{Tr}\left(\mathbb{E}_{i\,j}^{\dagger} \mathbb{O}\right)$$

Thus, for any collection of d^2 linear independent operators on $\mathcal{H}$, the extension by the identity map of $\boldsymbol{\Phi}$ to an operator on $\mathbb{C}^d \otimes \mathbb{C}^d$ reads

$$\sum_{l,k}^{d} \mathbb{E}_{l\,k} \otimes \boldsymbol{\Phi}(\mathbb{O}_{l\,k}) = \sum_{l,k}^{d} \sum_{i\,j=1}^{d} \mathbb{E}_{l\,k} \otimes \boldsymbol{\Phi}(\mathbb{E}_{i\,j}) \operatorname{Tr}\left(\mathbb{E}_{i\,j}^{\dagger} \mathbb{O}_{l\,k}\right)$$

The following observation is then useful in order to assess whether a linear map is completely positive. If we identify the $\mathbb{O}_{l\,k}$'s with the elements of the canonical basis of $\mathcal{M}_d(\mathbb{C})$, the extension operator

$$\boldsymbol{\Psi}\boldsymbol{\Psi}^{\dagger} = \frac{1}{d} \sum_{i\,j=1}^{d} \mathbb{E}_{i\,j} \otimes \mathbb{E}_{i\,j} \tag{9}$$

is a pure state on $\mathcal{H} \otimes \mathcal{H}$. Namely, we may identify

$$\boldsymbol{\Psi} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \boldsymbol{e}_i \otimes \boldsymbol{e}_i^{\dagger} \in \mathcal{H} \otimes \mathcal{H}$$

with the state vector obtained by **fully entangling** the elements of the canonical basis of $\mathbb{C}^d$. The "self-outer" product then yields state vector with

$$\boldsymbol{\Psi}\boldsymbol{\Psi}^{\dagger} = \frac{1}{d} \sum_{i,j=1}^{d} (\boldsymbol{e}_i \otimes \boldsymbol{e}_i)\,(\boldsymbol{e}_j^{\dagger} \otimes \boldsymbol{e}_j^{\dagger}) = \frac{1}{d} \sum_{i,j=1}^{d} \boldsymbol{e}_i \boldsymbol{e}_j^{\dagger} \otimes \boldsymbol{e}_i \boldsymbol{e}_j^{\dagger} = \frac{1}{d} \sum_{i,j=1}^{d} \mathbb{E}_{ij} \otimes \mathbb{E}_{ij}$$

Clearly, (9) is positive definite by construction

$$\boldsymbol{V}^{\dagger}\boldsymbol{\Psi}\boldsymbol{\Psi}^{\dagger}\boldsymbol{V} = \left\| \boldsymbol{V}^{\dagger}\boldsymbol{\Psi} \right\|^2$$

The above considerations suggest, and in fact it is possible to prove that in order to assess complete positivity of a linear map on operators on a Hilbert space of dimension d it is sufficient to check that

$$\sum_{i\,j=1}^{d} \mathbb{E}_{i\,j} \otimes \boldsymbol{\Phi}(\mathbb{E}_{i\,j}) \sim \begin{bmatrix} \boldsymbol{\Phi}(\mathbb{E}_{1\,1}) & \dots & \boldsymbol{\Phi}(\mathbb{E}_{1\,d}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Phi}(\mathbb{E}_{d\,1}) & \dots & \boldsymbol{\Phi}(\mathbb{E}_{d\,d}) \end{bmatrix} \tag{10}$$

is positive definite. The purification entailed by (9) is called Jamiołkowski's representation and the assessment of complete positivity based on (10) Jamiołkowski's criterion [2].

5. LINEAR SELF-ADJOINTNESS AND TRACE PRESERVING MAPS

We now turn to address the following question: if we construct the most general linear map $\boldsymbol{\Phi}\colon \mathcal{M}_d(\mathbb{C}) \mapsto \mathcal{M}_d(\mathbb{C})$ preserving

- self-adjointness
- trace
- positivity

do these property alone sufficient to recover the Kraus form that we obtained by tracing out environment degrees of freedom from a unitary evolution starting from initial data in tensor product form?

To address this question we observe that a linear map acting on state operators in $\mathcal{M}_d(\mathbb{C})$

$$\boldsymbol{\rho}' = \boldsymbol{\Phi}(\boldsymbol{\rho})$$

once expressed in canonical basis components takes the form

$$\mathrm{R}'_{ab} = \sum_{ij=1}^{\mathrm{d}^2} \mathrm{F}_{abij}\mathrm{R}_{ij}$$

where

$$\mathrm{R}_{ab} = \mathrm{Tr}(\mathbb{E}_{ab}^{\dagger}\boldsymbol{\rho})$$

If we instead choose a self-adjoint basis $\{\boldsymbol{\sigma}_i\}_{i=0}^{d^2-1}$ of $\mathcal{M}_d(\mathbb{C})$ orthonormal with respect to the Hilbert-Schmidt inner product, the state operator is specified by a set $\{x_i\}_{i=0}^{d^2-1}$

$$\boldsymbol{\rho} = \sum_{i=0}^{d^2-1} x_i \boldsymbol{\sigma}_i$$

The most general linear map involves d^4 constants F_{abij}, and each can be complex, so there are $2\,d^4$ real constants involved in equation. Using just an orthonormal basis $\{\mathbb{E}_k\}_{k=1}^{d^2}$of $\mathcal{M}_d(\mathbb{C})$, we can write

$$\boldsymbol{\rho}' = \sum_{lk=1}^{d^2} \mathrm{C}_{lk} \sum_{i=0}^{d^2-1} x_i \mathbb{E}_l \boldsymbol{\sigma}_i \mathbb{E}_k^{\dagger}$$

5.1. Self-adjoint property

If we require the transformation to be self-adjointness preserving

$$\mathrm{C}_{lk} = \bar{\mathrm{C}}_{kl}$$

i.e the C_{lk} are the component of a self adjoint matrix $\mathbb{C}$. The spectral theorem then insures as

$$\mathbb{C} = \sum_{a=1}^{d^2} \kappa_a \boldsymbol{c}_a \boldsymbol{c}_a^\dagger$$

with the $\{\boldsymbol{c}_a\}_{a=1}^{d^2}$ forming an orthonormal basis of $\mathbb{C}^{d^2}$. Therefore upon introducing the canonical basis $\{\boldsymbol{e}_l\}_{l=1}^{d^2}$ of $\mathbb{C}^{d^2}$

$$\mathrm{C}_{lk} = \sum_{a=1}^{d^2} \kappa_a (\boldsymbol{e}_l^\dagger \boldsymbol{c}_a)\,(\boldsymbol{c}_a^\dagger \boldsymbol{e}_k)$$

so that upon inserting in the expression of $\boldsymbol{\rho}'$

$$\boldsymbol{\rho}' = \sum_{a=1}^{d^2} \sum_{i=0}^{d^2-1} \kappa_a\, x_i \left(\sum_{l=1}^{d^2} (\boldsymbol{e}_l^\dagger \boldsymbol{c}_a)\, \mathbb{E}_l \right) \boldsymbol{\sigma}_i \left(\sum_{k=1}^{d^2} \mathbb{E}_k^\dagger\, (\boldsymbol{c}_a^\dagger \boldsymbol{e}_k) \right)$$

The definition

$$\mathbb{V}_a = \sum_{l=1}^{d^2} (\boldsymbol{e}_l^\dagger \boldsymbol{c}_a)\, \mathbb{E}_l$$

allows us to couch the result into the compact form

$$\boldsymbol{\rho}' = \sum_{a=1}^{d^2} \kappa_a \mathbb{V}_a \boldsymbol{\rho} \mathbb{V}_a^\dagger = \sum_{a=1}^{d^2} \kappa_a \sum_{i=0}^{d^2-1} x_i \mathbb{V}_a \boldsymbol{\sigma}_i \mathbb{V}_a^\dagger \tag{11}$$

The set of operators $\{\mathbb{V}_a\}_{a=1}^{d^2}$ also form an orthonormal basis of $\mathcal{M}_d(\mathbb{C})$. Namely

$$\mathrm{Tr}\left(\mathbb{V}_a^\dagger \mathbb{V}_b\right) = \sum_{l,k=1}^{d^2} (\boldsymbol{c}_a^\dagger \boldsymbol{e}_l)\,(\boldsymbol{e}_k^\dagger \boldsymbol{c}_b)\, \mathrm{Tr}\left(\mathbb{E}_l^\dagger \mathbb{E}_k\right) = \sum_{l,k=1}^{d^2} (\boldsymbol{c}_a^\dagger \boldsymbol{e}_l)\,(\boldsymbol{e}_k^\dagger \boldsymbol{c}_b) \delta_{lk} = \boldsymbol{c}_a^\dagger \left(\sum_{l=1}^{d} \boldsymbol{e}_l \boldsymbol{e}_l^\dagger \right) \boldsymbol{c}_b = \delta_{ab}$$

5.2. Trace preservation

The condition

$$\mathrm{Tr}\, \boldsymbol{\rho}' = 1$$

imposes

$$1 = \sum_{a=1}^{d^2} \kappa_a\, \mathrm{Tr}\left(\mathbb{V}_a \boldsymbol{\rho} \mathbb{V}_a^\dagger\right)$$

The condition is satisfied if

$$\mathbb{1}_d = \sum_{a=1}^{d^2} \kappa_a \mathbb{V}_a^\dagger \mathbb{V}_a = \sum_{lk=1}^{d^2} \mathrm{C}_{lk} \mathbb{E}_k^\dagger \mathbb{E}_l$$

and also implies

$$d = \sum_{a=1}^{d^2} \kappa_a = \mathrm{Tr}\, \mathbb{C}$$

5.3. Positivity

Finally positivity preservation requires that for any $\boldsymbol{v} \in \mathcal{H}$

$$0 \leq \boldsymbol{v}^{\dagger}\boldsymbol{\rho}'\boldsymbol{v} = \sum_{a=1}^{d^2} \kappa_a \boldsymbol{v}^{\dagger}\mathbb{V}_a\boldsymbol{\rho}\mathbb{V}_a^{\dagger}\boldsymbol{v} = \sum_{a=1}^{d^2} \kappa_a \boldsymbol{u}^{(a)\dagger}\boldsymbol{\rho}\boldsymbol{u}^{(a)}$$

with

$$\boldsymbol{u}^{(a)} = \sum_{l=1}^{d^2} (\boldsymbol{e}_l^{\dagger}\boldsymbol{c}_a)\,\mathbb{E}_l\boldsymbol{v}$$

We now see that

$$\kappa_a \geq 0 \qquad \text{for all}\, a = 1, \dots, d^2$$

is sufficient but not necessary to insure positivity.

Example. *A simple example illustrate the situation [4]. Let us consider a linear map acting on the space of two-level system state operators whose general expression in the basis of Pauli matrices (with $\sigma_0 = 1_2$) is*

$$\boldsymbol{\rho} = \frac{1_2 + \sum_{i=1}^{3} x_i\sigma_i}{2}$$

with coordinates $x_i \in \mathbb{R}$, $i = 1, \dots, 3$ and inside the Bloch ball

$$\sum_{i=1}^{3} x_i^2 \leq 1 \tag{12}$$

We consider the mapping specified by identifying the $\mathbb{V}_a$ with the elements of the Pauli basis and

$$\begin{aligned} \varkappa_a &= 1 & a &= 0, \dots, 2 \\ \varkappa_3 &= -1 \end{aligned}$$

The image of a state operator under such mapping is

$$\boldsymbol{\rho}' = \frac{\boldsymbol{\rho} + \sigma_1\boldsymbol{\rho}\sigma_1 + \sigma_2\boldsymbol{\rho}\sigma_2 - \sigma_3\boldsymbol{\rho}\sigma_3}{2}$$

The mapping is trace preserving because

$$\sum_{a=0}^{3} \varkappa_a = 2$$

To verify that the mapping preserves positivity, we evaluate the coordinates of the image state operator in the Pauli basis

$$\begin{aligned} \operatorname{Tr}\sigma_1\boldsymbol{\rho}' &= \frac{2\,\operatorname{Tr}(\sigma_1\boldsymbol{\rho}) + \operatorname{Tr}(\sigma_2\sigma_1\sigma_2\boldsymbol{\rho}) - \operatorname{Tr}(\sigma_3\sigma_1\sigma_3\boldsymbol{\rho})}{2} = \operatorname{Tr}(\sigma_1\boldsymbol{\rho}) + \frac{\imath\operatorname{Tr}(\sigma_2\sigma_3\boldsymbol{\rho}) - \imath\operatorname{Tr}(\sigma_2\sigma_3\boldsymbol{\rho})}{2} = \operatorname{Tr}(\sigma_1\boldsymbol{\rho}) = x_1 \\ \operatorname{Tr}\sigma_2\boldsymbol{\rho}' &= \frac{2\,\operatorname{Tr}(\sigma_2\boldsymbol{\rho}) + \operatorname{Tr}(\sigma_1\sigma_2\sigma_1\boldsymbol{\rho}) - \operatorname{Tr}(\sigma_3\sigma_2\sigma_3\boldsymbol{\rho})}{2} = \operatorname{Tr}(\sigma_2\boldsymbol{\rho}) + \frac{\imath\operatorname{Tr}(\sigma_3\sigma_1\boldsymbol{\rho}) - \imath\operatorname{Tr}(\sigma_3\sigma_1\boldsymbol{\rho})}{2} = \operatorname{Tr}(\sigma_2\boldsymbol{\rho}) = x_2 \\ \operatorname{Tr}\sigma_3\boldsymbol{\rho}' &= \frac{\operatorname{Tr}(\sigma_1\sigma_3\sigma_1\boldsymbol{\rho}) + \operatorname{Tr}(\sigma_2\sigma_3\sigma_2\boldsymbol{\rho})}{2} = \frac{\imath\operatorname{Tr}(\sigma_1\sigma_2\boldsymbol{\rho}) + \imath\operatorname{Tr}(\sigma_1\sigma_2\boldsymbol{\rho})}{2} = -\operatorname{Tr}(\sigma_3\boldsymbol{\rho}) = -x_3 \end{aligned}$$

and (12) is thus satisfied.

* *

As a consequence **positivity alone does not imply Kraus' form of the dynamical map**. We are thus in position to answer the question we set out to address at the beginning of this section. In order to recover Kraus' form we need a stronger requirement than positivity. This requirement is indeed complete positivity.

6. CHOI'S THEOREM

Choi [1] proved that completely positive maps are always amenable to an explicit canonical form. This form is indeed Kraus' form.

Theorem. *(**Choi 1975.**) Let $\mathbf{\Phi}\colon \mathcal{M}_n(\mathbb{C}) \mapsto \mathcal{M}_m(\mathbb{C})$ is completely positive if and only if*

$$\mathbf{\Phi}(\mathbb{A}) = \sum_{i=1}^{\ell} \mathbb{W}_i \mathbb{A} \mathbb{W}_i^{\dagger} \tag{13}$$

for all positive definite $\mathbb{A} \in : \mathcal{M}_n(\mathbb{C})$ and $\{\mathbb{V}_i\}_{i=1}^{\ell}$ a collection of $m \times n$ matrices with complex entries.

Proof.

- If $\mathbf{\Phi}(\cdot)$ admits the representation (13) then $\mathbf{\Phi}(\cdot)$ is completely positive. In order to prove complete positivity, we need to prove that for any m and for any matrix with operator entries such that for any vector $\oplus_{i=1}^{m} \boldsymbol{v}_i$

$$\left(\sum_{i=1}^{n} \boldsymbol{e}_i \otimes \boldsymbol{v}_i\right)^{\dagger} \left(\sum_{a,b=1}^{n} \mathbb{E}_{a\,b} \otimes \mathbb{O}_{a\,b}\right) \left(\sum_{j=1}^{n} \boldsymbol{e}_j \otimes \boldsymbol{v}_j\right) = \begin{bmatrix} \boldsymbol{v}_1^{\dagger} & \dots & \boldsymbol{v}_m^{\dagger} \end{bmatrix} \begin{bmatrix} \mathbb{O}_{1\,1} & \dots & \mathbb{O}_{1\,k} \\ \vdots & \vdots & \vdots \\ \mathbb{O}_{k\,1} & \dots & \mathbb{O}_{k\,k} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_m \end{bmatrix} \geq 0$$

  the inequality

$$0 \leq \begin{bmatrix} \boldsymbol{v}_1^{\dagger} & \dots & \boldsymbol{v}_m^{\dagger} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}(\mathbb{O}_{1\,1}) & \dots & \mathbf{\Phi}(\mathbb{O}_{1\,k}) \\ \vdots & \vdots & \vdots \\ \mathbf{\Phi}(\mathbb{O}_{k\,1}) & \dots & \mathbf{\Phi}(\mathbb{O}_{k\,k}) \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_m \end{bmatrix} = \sum_{i=1}^{\ell} \begin{bmatrix} (\mathbb{W}_i^{\dagger}\boldsymbol{v}_1)^{\dagger} & \dots & (\mathbb{W}_i^{\dagger}\boldsymbol{v}_m)^{\dagger} \end{bmatrix} \begin{bmatrix} \mathbb{O}_{1\,1} & \dots & \mathbb{O}_{1\,k} \\ \vdots & \vdots & \vdots \\ \mathbb{O}_{k\,1} & \dots & \mathbb{O}_{k\,k} \end{bmatrix} \begin{bmatrix} \mathbb{W}_i^{\dagger}\boldsymbol{v}_1 \\ \vdots \\ \mathbb{W}_i^{\dagger}\boldsymbol{v}_m \end{bmatrix}$$

  holds true. We immediately recognize that because of the arbitrariness of $\oplus_{i=1}^{m} \boldsymbol{v}_i$ the condition is satisfied.

- If $\mathbf{\Phi}(\cdot)$ is completely positive then it admits the representation (13): by Jamiołkowski's criterion we know that for any $\boldsymbol{V} \in \mathcal{H} \otimes \mathcal{H}$ the inequality

$$0 \leq \sum_{l,k=1}^{d} \boldsymbol{V}^{\dagger} \left(\mathcal{I}_d \otimes \mathbf{\Phi}\right) \left(\mathbb{E}_{lk} \otimes \mathbb{E}_{lk}\right) \boldsymbol{V}$$

  must hold true, which becomes

$$0 \leq \boldsymbol{V}^{\dagger} \left(\sum_{l,k=1}^{d} \mathbb{E}_{l\,k} \otimes \mathbf{\Phi}(\mathbb{E}_{l\,k})\right) \boldsymbol{V}$$

  As $\boldsymbol{V}$ is arbitrary, we can choose it

$$\boldsymbol{V} = \sum_{i=1}^{d} \boldsymbol{f}_i \otimes \boldsymbol{g}_i$$

  where for now we only assume the vectors $\{\boldsymbol{f}_i\}_{i=1}^{d}$, $\{\boldsymbol{g}_i\}_{i=1}^{d}$ in $\mathbb{C}^d$ to be linear independent. We now use the representation of a self-adjoint linear map (11) to couch the inequality into the form

$$0 \leq \sum_{a=1}^{d^2} \kappa_a \sum_{i=1}^{d} \boldsymbol{f}_i^{\dagger} \otimes \boldsymbol{g}_i^{\dagger} \left(\sum_{l,k=1}^{d} \mathbb{E}_{l\,k} \otimes \mathbb{V}_a \mathbb{E}_{l\,k} \mathbb{V}_a^{\dagger}\right) \sum_{j=1}^{d} \boldsymbol{f}_j \otimes \boldsymbol{g}_j = \sum_{a=1}^{d^2} \kappa_a \sum_{i,j=1}^{d} \sum_{l,k=1}^{d} \boldsymbol{f}_i^{\dagger} \mathbb{E}_{l\,k} \boldsymbol{f}_j \left(\boldsymbol{g}_i^{\dagger} \mathbb{V}_a \mathbb{E}_{l\,k} \mathbb{V}_a^{\dagger} \boldsymbol{g}_j\right)$$

  Upon recalling that elements of the canonical basis of $\mathcal{M}_d(\mathbb{C})$ are dual products of elements of the canonical basis of $\mathbb{C}^d$

$$\mathbb{E}_{l\,k} = \boldsymbol{e}_l \boldsymbol{e}_k^{\dagger}$$

we get into

$$0 \leq \sum_{a=1}^{d^2} \kappa_a \sum_{i,j=1}^{d} \left(\boldsymbol{g}_i^\dagger \, \mathbb{V}_a \bar{\boldsymbol{f}}_i \right) \left(\boldsymbol{f}_j^\top \mathbb{V}_a^\dagger \boldsymbol{g}_j \right)$$

We observe that by definition

$$\sum_{i=1}^{d} \boldsymbol{g}_i^\dagger \, \mathbb{V}_a \bar{\boldsymbol{f}}_i = \sum_{i=1}^{d} \sum_{l=1}^{d^2} (\boldsymbol{e}_l^\dagger \boldsymbol{c}_a) \, \boldsymbol{g}_i^\dagger \, \mathbb{E}_l \bar{\boldsymbol{f}}_i$$

We now avail us of the arbitrariness of the $\boldsymbol{f}_i$'s, and $\boldsymbol{g}_i$'s to impose that only one addend in the sum is non vanishing. To do so we may set

$$\boldsymbol{f}_i = \boldsymbol{e}_i \qquad \forall\, i$$

where as usual $\boldsymbol{e}_i$ is the i-th element of the canonical basis of $\mathbb{C}^d$ and

$$\boldsymbol{g}_i = \sum_{l=1}^{d^2} (\boldsymbol{e}_l^\dagger \boldsymbol{c}_b) \mathbb{E}_l \boldsymbol{e}_i$$

with b arbitrary. Then the chain of identities holds true

$$\sum_{i=1}^{d} \boldsymbol{g}_i^\dagger \, \mathbb{V}_a \bar{\boldsymbol{f}}_i = \sum_{i=1}^{d} \sum_{l,k=1}^{d^2} (\boldsymbol{c}_b^\dagger \boldsymbol{e}_l)(\boldsymbol{e}_l^\dagger \boldsymbol{c}_a) \, \boldsymbol{e}_i^\dagger \, \mathbb{E}_b^\dagger \mathbb{E}_l \boldsymbol{e}_i = \sum_{l,k=1}^{d^2} (\boldsymbol{c}_b^\dagger \boldsymbol{e}_k)(\boldsymbol{e}_l^\dagger \boldsymbol{c}_a) \; \mathrm{Tr} \left(\sum_{i=1}^{d} \boldsymbol{e}_i \boldsymbol{e}_i^\dagger \, \mathbb{E}_k^\dagger \mathbb{E}_l \right)$$

$$= \sum_{l,k=1}^{d^2} (\boldsymbol{c}_b^\dagger \boldsymbol{e}_k)(\boldsymbol{e}_l^\dagger \boldsymbol{c}_a) \; \mathrm{Tr} \left(\mathbb{E}_k^\dagger \mathbb{E}_l \right) = \sum_{l,k=1}^{d^2} (\boldsymbol{c}_b^\dagger \boldsymbol{e}_l)(\boldsymbol{e}_l^\dagger \boldsymbol{c}_a) \, \delta_{l\,k} = \sum_{l=1}^{d^2} (\boldsymbol{c}_b^\dagger \boldsymbol{e}_l)(\boldsymbol{e}_l^\dagger \boldsymbol{c}_a) = \boldsymbol{c}_b^\dagger \boldsymbol{c}_a = \delta_{a\,b}$$

In such a case we obtain

$$\kappa_b \, \geq \, 0$$

where $1 \, \leq \, b \, \leq \, d^2$ is arbitrary. Because we can repeat the argument for any $b = 1, \dots, d^2$ we are entitled to conclude that $\boldsymbol{\Phi}$ is completely positive. Once we have shown that all eigenvalues are positive, we can set

$$\mathbb{W}_i = \sqrt{\varkappa_i} \mathbb{V}_i$$

and therefore

$$\sum_{l,k=1}^{d} \mathbb{E}_{l\,k} \, \otimes \, \boldsymbol{\Phi}(\mathbb{E}_{l\,k}) = \sum_{l,k=1}^{d} \mathbb{E}_{l\,k} \, \otimes \, \sum_{a=1}^{d^2} \mathbb{W}_a \mathbb{E}_{l\,k} \mathbb{W}_a^\dagger$$

□

It is instructive to also reproduce Choi's original proof of the implication

$$\text{complete positive map} \implies \boldsymbol{\Phi}(\mathbb{A}) = \sum_a \mathbb{W}_a \mathbb{A} \mathbb{W}_a^\dagger$$

We then see that k-positivity means that there must exist a collection of $L \times k$ operators $\left\{ \mathbb{O}_l^{(\ell)} \right\}_{l=1}^{k}$, $\ell = 1, \dots, L$ such that

$$\begin{bmatrix} \boldsymbol{\Phi}(\mathrm{E}_{1\,1}) & \dots & \boldsymbol{\Phi}(\mathrm{E}_{1\,k}) \\ \vdots & \vdots & \vdots \\ \boldsymbol{\Phi}(\mathrm{E}_{k\,1}) & \dots & \boldsymbol{\Phi}(\mathrm{E}_{k\,k}) \end{bmatrix} = \sum_{\ell=1}^{L} \begin{bmatrix} \mathbb{O}_1^{(\ell)\dagger} \\ \vdots \\ \mathbb{O}_k^{(\ell)\dagger} \end{bmatrix} \begin{bmatrix} \mathbb{O}_1^{(\ell)} & \dots & \mathbb{O}_k^{(\ell)} \end{bmatrix} = \sum_{\ell=1}^{L} \begin{bmatrix} \mathbb{O}_1^{(\ell)\dagger} \mathbb{O}_1^{(\ell)} & \mathbb{O}_1^{(\ell)\dagger} \mathbb{O}_2^{(\ell)} & \dots \\ \mathbb{O}_2^{(\ell)\dagger} \mathbb{O}_1^{(\ell)} & \mathbb{O}_2^{(\ell)\dagger} \mathbb{O}_2^{(\ell)} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

or more compactly

$$\sum_{i\,j=1}^{k} \mathbb{E}_{i\,j} \otimes \mathbf{\Phi}(\mathrm{E}_{i\,j}) = \sum_{i\,j=1}^{k} \mathbb{E}_{i\,j} \otimes \sum_{\ell=1}^{L} \mathbb{O}_i^{(\ell)\dagger}\mathbb{O}_j^{(\ell)}$$

We observe that

$$\begin{bmatrix} \mathbb{O}_1^{(\ell)\dagger} \\ \vdots \\ \mathbb{O}_k^{(\ell)\dagger} \end{bmatrix} \mathbb{E}_{i\,j} \begin{bmatrix} \mathbb{O}_1^{(\ell)} & \dots & \mathbb{O}_k^{(\ell)} \end{bmatrix} = \begin{bmatrix} \mathbb{O}_1^{(\ell)\dagger} \\ \vdots \\ \mathbb{O}_k^{(\ell)\dagger} \end{bmatrix} \boldsymbol{e}_i \boldsymbol{e}_j^{\dagger} \begin{bmatrix} \mathbb{O}_1^{(\ell)} & \dots & \mathbb{O}_k^{(\ell)} \end{bmatrix} = \mathbb{O}_i^{(\ell)\dagger}\mathbb{O}_j^{(\ell)}$$

whence

$$\sum_{i\,j=1}^{k} \mathbb{E}_{i\,j} \otimes \mathbf{\Phi}(\mathrm{E}_{i\,j}) = \sum_{i\,j=1}^{k} \mathbb{E}_{i\,j} \otimes \sum_{\ell=1}^{L} \mathbb{W}_\ell \mathbb{E}_{i\,j} \mathbb{W}_\ell^{\dagger}$$

after the identification

$$\mathbb{W}_\ell^{\dagger} = \begin{bmatrix} \mathbb{O}_1^{(\ell)} & \dots & \mathbb{O}_k^{(\ell)} \end{bmatrix}$$

* * *

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