TCM315 Fall 2022: Introduction to Open Quantum Systems

Lecture 11: Master equation for a central oscillator

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CONTENTS

1. CENTRAL OSCILLATOR MODEL

We consider the Hamilton operator

$$\mathbb{H}=\omega\,\mathrm{a}^{\dagger}\,\mathrm{a}+\sum_{k=1}^{\mathcal{N}}\omega_{k}\,\mathrm{b}_{k}^{\dagger}\,\mathrm{b}_{k}+\sum_{k=1}^{\mathcal{N}}\left(\bar{g}_{k}\,\mathrm{a}\,\mathrm{b}_{k}^{\dagger}+g_{k}\,\mathrm{b}_{k}\,\mathrm{a}^{\dagger}\right)$$

The Hamilton operator describes a multipartite system consisting of a "central" oscillator coupled to a "phonon environment" of other oscillators.

1.1. Holomorphic representation

In the holomorphic representation the equation for the propagator is

$$\imath\,\partial_{t}\mathcal{U}_{t}(z,\boldsymbol{w}|\bar{z},\bar{\boldsymbol{w}})=\left(\omega\,z\,\partial_{z}+\sum_{k=1}^{\mathcal{N}}\omega_{k}\,w_{k}\,\partial_{w_{k}}+\sum_{k=1}^{\mathcal{N}}\left(\bar{g}_{k}\,w_{k}\partial_{z}+g_{k}\,z\,\partial_{w_{k}}\right)\right)\mathcal{U}_{t}(z,\boldsymbol{w}|\bar{z},\bar{\boldsymbol{w}})$$
$$\mathcal{U}_{0}(z,\boldsymbol{w}|\bar{z},\bar{\boldsymbol{w}})=e^{z\bar{z}+\sum_{k=1}^{\mathcal{N}}w_{k}\bar{w}_{k}}$$

More compactly we write

$$\imath\,\partial_{t}\mathcal{U}_{t}(z,\boldsymbol{w}|\bar{z},\bar{\boldsymbol{w}})=[z,\boldsymbol{w}^{\top}]\mathsf{H}\begin{bmatrix}\partial_{z}\\ \boldsymbol{\partial_{w}}\end{bmatrix}\mathcal{U}_{t}(z,\boldsymbol{w}|\bar{z},\bar{\boldsymbol{w}})$$

where

$$\mathsf{H}=\begin{bmatrix}\omega & \boldsymbol{g}^{\dagger}\\ \boldsymbol{g} & \Omega\end{bmatrix}\qquad\&\qquad\Omega=\begin{bmatrix}\omega_{1} & 0 & \dots\\ 0 & \omega_{2} & \ddots\\ \vdots & \ddots & \ddots\end{bmatrix}$$

We look for a solution of the form

$$\mathscr{U}_t(z, \boldsymbol{w}|\bar{z}, \bar{\boldsymbol{w}}) = \frac{1}{Z_t} \exp\left([z, \boldsymbol{w}^\top] \mathsf{K}_t \begin{bmatrix} \bar{z} \\ \bar{\boldsymbol{w}} \end{bmatrix} \right)$$

We get into

$$\imath \left([z, \boldsymbol{w}^\top] \dot{\mathsf{K}}_t \begin{bmatrix} \bar{z} \\ \bar{\boldsymbol{w}} \end{bmatrix} - \frac{\dot{Z}_t}{Z_t} \right) = [z, \boldsymbol{w}^\top] \mathsf{H} \mathsf{K}_t \begin{bmatrix} \bar{z} \\ \bar{\boldsymbol{w}} \end{bmatrix}$$

having defined

$$\mathsf{K}_t = \begin{bmatrix} k_t & \boldsymbol{x}_t^\top \\ \bar{\boldsymbol{y}}_t & \mathscr{K}_t \end{bmatrix}$$

The partial differential equation foliates into the system

$$\imath\, \dot{\mathsf{K}}_t = \mathsf{H}\, \mathsf{K}_t \qquad \mathsf{K}_0 = 1$$

$$\imath\, \frac{\dot{Z}_t}{Z_t} = 0 \qquad Z_0 = 1$$

1.1.1. Analysis of the solution

The solution is

$$\mathsf{K}_t = e^{-\imath\, H\, t}$$

By construction K_t is an unitary matrix. Therefore

$$\mathsf{K}_{-t} \mathsf{K}_t = \mathsf{K}_t^\dagger \mathsf{K}_t = \begin{bmatrix} \bar{k}_t & \boldsymbol{y}_t^\top \\ \bar{\boldsymbol{x}}_t & \mathscr{K}_t^\dagger \end{bmatrix} \begin{bmatrix} k_t & \boldsymbol{x}_t^\top \\ \bar{\boldsymbol{y}}_t & \mathscr{K}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We obtain

$$k_t \bar{k}_t + \boldsymbol{y}_t^\top \bar{\boldsymbol{y}}_t = 1 \tag{1}$$

$$k_t\, \boldsymbol{x}_t^\top + \boldsymbol{y}_t^\top \mathscr{K}_t = 0 \tag{2}$$

$$\bar{\boldsymbol{x}}_t\, k_t + \mathscr{K}_t^\dagger \bar{\boldsymbol{y}}_t = 0 \tag{3}$$

$$\bar{\boldsymbol{x}}_t \boldsymbol{x}_t^\top + \mathscr{K}_t^\dagger \mathscr{K}_t = \mathsf{1} \tag{4}$$

whence

$$\bar{\boldsymbol{x}}_t = -\frac{1}{k_t} \mathscr{K}_t^\dagger \bar{\boldsymbol{y}}_t \qquad \Rightarrow \qquad \boldsymbol{x}_t^\top = -\frac{1}{\bar{k}_t} \boldsymbol{y}_t^\top \mathscr{K}_t \tag{5}$$

The second equation is thus immediately satisfied whereas insertion in the fourth yields

$$\left(1 + \frac{\bar{\boldsymbol{y}}_t \boldsymbol{y}_t^\top}{k_t \bar{k}_t} \right) \mathscr{K}_t = \mathscr{K}_t^{\dagger -1}$$

Sherman-Morrison formula together with (1) yield

$$\left(1 + \frac{\bar{\boldsymbol{y}}_t \boldsymbol{y}_t^\top}{k_t \bar{k}_t} \right)^{-1} = 1 - \bar{\boldsymbol{y}}_t \boldsymbol{y}_t^\top$$

and therefore

$$\mathscr{K}_t^\dagger = \mathscr{K}_t^{-1} (1 - \bar{\boldsymbol{y}}_t \boldsymbol{y}_t^\top)$$

and

$$\bar{\boldsymbol{x}}_t = -\bar{k}_t\, \mathscr{K}_t^{-1} \bar{\boldsymbol{y}}_t \tag{6}$$

2. THERMAL INITIAL DATA

We suppose that initially the environment is in a thermal state.

$$\boldsymbol{\rho} = \frac{1}{Z} e^{-\beta \mathbb{H}}$$

The holopmorphic representation of a thermal state is

$$R_\beta(\boldsymbol{w}, \bar{\boldsymbol{w}}) = \frac{1}{Z} \exp\left(\sum_{i=1}^{\mathcal{N}} w_i e^{-\beta\,\omega_i} \bar{w}_i \right) \equiv \frac{1}{Z} \exp\left(\boldsymbol{w}^\dagger \mathscr{B}_\beta \boldsymbol{w} \right)$$

namely

$$R(\boldsymbol{w}, \bar{\boldsymbol{w}}) = \frac{1}{Z} \prod_{i=1}^{\mathcal{N}} \boldsymbol{c}^\dagger_{\bar{w}_i} e^{-\beta \mathbb{H}} \prod_{i=1}^{\mathcal{N}} \boldsymbol{c}_{\bar{w}_i} = \frac{1}{Z} \prod_{i=1}^{\mathcal{N}} \left(\boldsymbol{c}^\dagger_{\bar{w}_i} \sum_{j=0}^{\infty} \frac{e^{-\beta\,\omega_i\, j} (\bar{w}_i\, \mathrm{a}^\dagger)^j}{j!} \phi_0 \right) = \frac{1}{Z} \prod_{i=1}^{\mathcal{N}} e^{w_i e^{-\beta\,\omega_i} \bar{w}_i}$$

The partition function is

$$Z = \int \prod_{i=1}^{\mathcal{N}} \mathrm{d}\mathrm{g}_{w_i, \bar{w}_i} e^{w_i e^{-\beta\,\omega_i} \bar{w}_i} = \int \prod_{i=1}^{\mathcal{N}} \frac{\mathrm{d}w_i \mathrm{d}\bar{w}_i}{2\,\pi\,\imath} e^{-w_i \bar{w}_i + w_i e^{-\beta\,\omega_i} \bar{w}_i} = \prod_{i=1}^{\mathcal{N}} \frac{1}{(1 - e^{-\beta\,\omega_i})}$$

3. PARTIAL TRACE

We wish to compute, the reduced state operator of the central boson

$$\begin{aligned} \mathscr{R}_t(z, \bar{z}) &= \mathrm{Tr}_{\mathcal{H}_E} \left(\mathbb{U}_t \boldsymbol{\rho}_S \otimes \boldsymbol{\rho}_E \mathbb{U}_t^\dagger \right) \equiv \mathrm{Tr}_{\mathcal{H}_E} \left(\mathbb{U}_t \boldsymbol{\rho}_S \otimes \boldsymbol{\rho}_E \mathbb{U}_{-t} \right) \\ &= \int \mathrm{d}\mathrm{g}_{a,\bar{a}} \mathrm{d}\mathrm{g}_{b,\bar{b}} \mathrm{d}\mathrm{g}_{\boldsymbol{w},\bar{\boldsymbol{w}}} \mathrm{d}\mathrm{g}_{\boldsymbol{v}\bar{\boldsymbol{v}}} \mathrm{d}\mathrm{g}_{\boldsymbol{u},\bar{\boldsymbol{u}}}\, \mathscr{U}_t(z, \boldsymbol{w} | \bar{a}, \bar{\boldsymbol{v}})\, \mathscr{R}_0(a, \bar{b})\, R_E(\boldsymbol{v}, \bar{\boldsymbol{u}})\, \mathscr{U}_{-t}(b, \boldsymbol{u} \,|\, \bar{z}, \bar{\boldsymbol{w}}) \end{aligned}$$

The partial trace allows us to define the reduced propagator

$$\mathscr{R}_t(z, \bar{z}) = \int \mathrm{d}\mathrm{g}_{a,\bar{a}} \mathrm{d}\mathrm{g}_{b,\bar{b}}\, \mathscr{G}_t(z, \bar{z} | \bar{a}, b)\, \mathscr{R}_0(a, \bar{b})$$

where now

$$\mathscr{G}_t(z, \bar{z} | \bar{a}, b) = \int \mathrm{d}\mathrm{g}_{\boldsymbol{w},\bar{\boldsymbol{w}}} \mathrm{d}\mathrm{g}_{\boldsymbol{v}\bar{\boldsymbol{v}}} \mathrm{d}\mathrm{g}_{\boldsymbol{u},\bar{\boldsymbol{u}}}\, \mathscr{U}_t(z, \boldsymbol{w} | \bar{a}, \bar{\boldsymbol{v}})\, R_E(\boldsymbol{v}, \bar{\boldsymbol{u}})\, \mathscr{U}_{-t}(b, \boldsymbol{u} \,|\, \bar{z}, \bar{\boldsymbol{w}})$$

The reduced propagator subsumes all information about the evolution of the central system independently of the initial datum. In the literature the reduced propagator is commonly referred to as the **dynamical map** of the open quantum system.

3.1. Explicit evaluation of the dynamical map.

We suppose that the initial state of the environment is a thermal state at temperature β^{-1}. We thus write

$$R_E(\boldsymbol{v}, \bar{\boldsymbol{v}}) = \det(1_{\mathcal{N}} - e^{-\beta\,\Omega}) \exp\left(\boldsymbol{v}^\top e^{-\beta\,\Omega}\, \bar{\boldsymbol{v}} \right)$$

We thus arrive at

$$\mathscr{G}_t(z, \bar{z} | \bar{a}, b) = \det(1_{\mathcal{N}} - e^{-\beta\,\Omega}) \int \mathrm{d}\mathrm{g}_{\boldsymbol{w},\bar{\boldsymbol{w}}} \mathrm{d}\mathrm{g}_{\boldsymbol{v}\bar{\boldsymbol{v}}} \mathrm{d}\mathrm{g}_{\boldsymbol{u},\bar{\boldsymbol{u}}}\, \mathscr{U}_t(z, \boldsymbol{w} | \bar{a}, \bar{\boldsymbol{v}})\, \exp\left(\boldsymbol{v}^\top e^{-\beta\,\Omega}\, \bar{\boldsymbol{u}} \right)\, \mathscr{U}_{-t}(b, \boldsymbol{u} \,|\, \bar{z}, \bar{\boldsymbol{w}})$$

The integral is Gaussian and can be explicitly evaluated. The evaluation requires, however, decomposing the quadratic forms in the arguments of the exponential functions into degrees of freedom of the environment and those of the central system. The result takes the form

$$\mathscr{G}_t(z,\bar{z}|\bar{a},b)=N_t\exp\left(z\,m_t^{(1,1)}\bar{z}+z\,m_t^{(1,2)}\bar{a}+b\,m_t^{(2,1)}\bar{z}+b\,m_t^{(2,2)}\bar{a}\right)$$

Straightforward but somewhat tedious algebra yields the explicit expression of the coefficients $m_t^{(i,j)}$, $i,j=1,2$ and of the normalization prefactor N_t in terms of the parameters of the environment. Some inverse engineering permits, however, to establish the relation that they need to satisfy.

To start with, we observe that trace preservation imposes that for any initial value of the state operator

$$1=\int \mathrm{dg}_{z,\bar{z}}\mathscr{R}_t(z,\bar{z})=\int \mathrm{dg}_{z,\bar{z}}\mathrm{dg}_{a,\bar{a}}\mathrm{dg}_{b,\bar{b}}\,\mathscr{G}_t(z,\bar{z}|\bar{a},b)\,\mathscr{R}_0(a,\bar{b})=\int \mathrm{dg}_{a,\bar{a}}\mathscr{R}_0(a,\bar{a})$$

The holomorphic representation of the Dirac δ thus requires

$$\int \mathrm{dg}_{z,\bar{z}}\mathscr{G}_t(z,\bar{z}|\bar{a},b)=e^{\bar{a}b}$$

so that

$$\int \mathrm{dg}_{b,\bar{b}}\,e^{\bar{a}b}\,\mathscr{R}_0(a,\bar{b})=\mathscr{R}_0(a,\bar{a})$$

If we perform the Gaussian integral specifying the trace of the dynamical map

$$N_t\int \mathrm{dg}_{z,\bar{z}}\exp\left(z\,m_t^{(1,1)}\bar{z}+z\,m_t^{(1,2)}\bar{a}+b\,m_t^{(2,1)}\bar{z}+b\,m_t^{(2,2)}\bar{a}\right)=\frac{N_t}{1-m_t^{(1,1)}}\exp\left(\frac{b\,m_t^{(2,1)}m_t^{(1,2)}\bar{a}}{1-m_t^{(1,1)}}+b\,m_t^{(2,2)}\bar{a}\right)$$

we than get the conditions

$$N_t=1-m_t^{(1,1)} \tag{7a}$$

$$m_t^{(2,2)}=1-\frac{m_t^{(2,1)}m_t^{(1,2)}}{1-m_t^{(1,1)}} \tag{7b}$$

Next, we observe that the state operator must be self adjoint

$$(\mathscr{R}_t(z,\bar{z}))^{\dagger}=\mathscr{R}_t(z,\bar{z})$$

The condition implies

$$m_t^{(1,1)},m_t^{(2,2)}\in\mathbb{R}$$

and

$$\overline{m_t^{(1,2)}}=m_t^{(2,1)}$$

4. MASTER EQUATION

The time derivative of the dynamical map yields

$$\partial_t\mathscr{G}_t(z,\bar{z}|\bar{a},b)=\left(z\,\dot{m}_t^{(1,1)}\bar{z}+z\,\dot{m}_t^{(1,2)}\bar{b}+a\,\dot{m}_t^{(2,1)}\bar{z}+a\,\dot{m}_t^{(2,2)}\bar{b}-\frac{\dot{m}_t^{(1,1)}}{1-m_t^{(1,1)}}\right)\mathscr{G}_t(z,\bar{z}|\bar{a},b)$$

and consequently

$$\partial_t\mathscr{R}_t(z,\bar{z})=\int \mathrm{dg}_{a,\bar{a}}\mathrm{dg}_{b,\bar{b}}\left(z\,\dot{m}_t^{(1,1)}\bar{z}+z\,\dot{m}_t^{(1,2)}\bar{b}+a\,\dot{m}_t^{(2,1)}\bar{z}+a\,\dot{m}_t^{(2,2)}\,\bar{b}-\frac{\dot{m}_t^{(1,1)}}{1-m_t^{(1,1)}}\right)\mathscr{G}_t(z,\bar{z}|\bar{a},b)\,\mathscr{R}_t(a,\bar{b})$$

Our goal is to couch the above integral into the form

$$\partial_t \mathscr{R}_t(z,\bar z) = \mathbb{L}_t \mathscr{R}_t(z,\bar z)$$

where $\mathbb{L}_t$ is a differential operator acting on the state operator at time t. To this goal we observe that

$$\begin{aligned}
\partial_z \mathscr{G}_t(z,\bar z|\bar a,b) &= \left(\boldsymbol{m}_t^{(1,1)}\bar z + \boldsymbol{m}_t^{(1,2)}\bar b\right)\mathscr{G}_t(z,\bar z|\bar a,b)\\
\partial_{\bar z} \mathscr{G}_t(z,\bar z|\bar a,b) &= \left(z\,\boldsymbol{m}_t^{(1,1)} + a\,\boldsymbol{m}_t^{(2,1)}\right)\mathscr{G}_t(z,\bar z|\bar a,b)\\
\partial_z\partial_{\bar z} \mathscr{G}_t(z,\bar z|\bar a,b) &= \boldsymbol{m}_t^{(1,1)}\,\mathscr{G}_t(z,\bar z|\bar a,b) + \left(z\,\boldsymbol{m}_t^{(1,1)} + a\,\boldsymbol{m}_t^{(2,1)}\right)\left(\boldsymbol{m}_t^{(1,1)}\,\bar z + \boldsymbol{m}_t^{(1,2)}\bar b\right)\mathscr{G}_t(z,\bar z|\bar a,b)
\end{aligned}$$

yield the identities

$$\begin{aligned}
\bar b\,\mathscr{G}_t(z,\bar z|\bar a,b) &= \frac{1}{\boldsymbol{m}_t^{(1,2)}}\left(\partial_z - \boldsymbol{m}_t^{(1,1)}\bar z\right)\mathscr{G}_t(z,b|\bar z,\bar a)\\
a\,\mathscr{G}_t(z,\bar z|\bar a,b) &= \frac{1}{\boldsymbol{m}_t^{(2,1)}}\left(\partial_{\bar z} - z\,\boldsymbol{m}_t^{(1,1)}\right)\mathscr{G}_t(z,\bar z|\bar a,b)\\
a\,\bar b\,\mathscr{G}_t(z,\bar z|\bar a,b) &= \frac{1}{\boldsymbol{m}_t^{(2,1)}\,\boldsymbol{m}_t^{(1,2)}}\left(\partial_z\partial_{\bar z} - \boldsymbol{m}_t^{(1,1)} - z\,\boldsymbol{m}_t^{(1,1)^2}\bar z\right)\mathscr{G}_t(z,\bar z|\bar a,b)\\
&- \frac{\boldsymbol{m}_t^{(1,1)}}{\boldsymbol{m}_t^{(2,1)}\boldsymbol{m}_t^{(1,2)}}\left(z\left(\partial_z - \boldsymbol{m}_t^{(1,1)}\bar z\right) + \bar z\left(\partial_{\bar z} - z\,\boldsymbol{m}_t^{(1,1)}\right)\right)\mathscr{G}_t(z,\bar z|\bar a,b)
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
\partial_t \mathscr{R}_t(z,\bar z) &= z\,\dot{\boldsymbol{m}}_t^{(1,1)}\bar z\,\mathscr{R}_t(z,\bar z) + \frac{\dot{\boldsymbol{m}}_t^{(1,2)}}{\boldsymbol{m}_t^{(1,2)}}\,z\left(\partial_z - \bar z\,\boldsymbol{m}_t^{(1,1)}\right)\mathscr{R}_t(z,\bar z) + \frac{\dot{\boldsymbol{m}}_t^{(2,1)}}{\boldsymbol{m}_t^{(2,1)}}\,\bar z\left(\partial_{\bar z} - z\,\boldsymbol{m}_t^{(1,1)}\right)\mathscr{R}_t(z,\bar z)\\
&+ \frac{\dot{\boldsymbol{m}}_t^{(2,2)}}{\boldsymbol{m}_t^{(2,1)}\,\boldsymbol{m}_t^{(1,2)}}\left(\partial_z\partial_{\bar z} - \boldsymbol{m}_t^{(1,1)} + z\,\boldsymbol{m}_t^{(1,1)^2}\bar z\right)\mathscr{R}_t(z,\bar z) - \frac{\dot{\boldsymbol{m}}_t^{(2,2)}\,\boldsymbol{m}_t^{(1,1)}}{\boldsymbol{m}_t^{(2,1)}\,\boldsymbol{m}_t^{(1,2)}}(z\,\partial_z + \bar z\,\partial_{\bar z})\mathscr{R}_t(z,\bar z) - \frac{\dot{\boldsymbol{m}}_t^{(1,1)}}{1-\boldsymbol{m}_t^{(1,1)}}\mathscr{R}_t(z,\bar z)
\end{aligned}$$

and after reordering terms

$$\begin{aligned}
\partial_t \mathscr{R}_t(z,\bar z) &= z\,\bar z\,\mathscr{R}_t(z,\bar z)\boldsymbol{m}_t^{(1,1)}\left(\frac{\dot{\boldsymbol{m}}_t^{(1,1)}}{\boldsymbol{m}_t^{(1,1)}} - \frac{\dot{\boldsymbol{m}}_t^{(1,2)}}{\boldsymbol{m}_t^{(1,2)}} - \frac{\dot{\boldsymbol{m}}_t^{(2,1)}}{\boldsymbol{m}_t^{(2,1)}} + \frac{\dot{\boldsymbol{m}}_t^{(2,2)}\,\boldsymbol{m}_t^{(1,1)}}{\boldsymbol{m}_t^{(2,1)}\,\boldsymbol{m}_t^{(1,2)}}\right) + z\,\partial_z\mathscr{R}_t(z,\bar z)\left(\frac{\dot{\boldsymbol{m}}_t^{(1,2)}}{\boldsymbol{m}_t^{(1,2)}} - \frac{\dot{\boldsymbol{m}}_t^{(2,2)}\,\boldsymbol{m}_t^{(1,1)}}{\boldsymbol{m}_t^{(2,1)}\,\boldsymbol{m}_t^{(1,2)}}\right)\\
&+ \bar z\,\partial_{\bar z}\mathscr{R}_t(z,\bar z)\left(\frac{\dot{\boldsymbol{m}}_t^{(2,1)}}{\boldsymbol{m}_t^{(2,1)}} - \frac{\dot{\boldsymbol{m}}_t^{(2,2)}\,\boldsymbol{m}_t^{(1,1)}}{\boldsymbol{m}_t^{(2,1)}\boldsymbol{m}_t^{(1,2)}}\right) + \frac{\dot{\boldsymbol{m}}_t^{(2,2)}}{\boldsymbol{m}_t^{(2,1)}\,\boldsymbol{m}_t^{(1,2)}}\partial_z\partial_{\bar z}\mathscr{R}_t(z,\bar z) - \left(\frac{\dot{\boldsymbol{m}}_t^{(2,2)}\boldsymbol{m}_t^{(1,1)}}{\boldsymbol{m}_t^{(2,1)}\,\boldsymbol{m}_t^{(1,2)}} + \frac{\dot{\boldsymbol{m}}_t^{(1,1)}}{1-\boldsymbol{m}_t^{(1,1)}}\right)\mathscr{R}_t(z,\bar z)
\end{aligned}$$

This expression can be further simplified using (7b) to arrive at a manifestly trace preserving form.

4.1. Master equation in the ladder operator formalism

In order to recover the corresponding expression in the abstract operator we recall that

$$\boldsymbol{\rho}_t = \int \mathrm{dg}_{z,\bar z}\mathrm{dg}_{w,\bar w}e^{\bar z\,\mathrm{a}^\dagger}\varphi_0\mathscr{R}_t(z,\bar w)\varphi_0^\dagger e^{w\,\mathrm{a}}$$

Next we avail us of the identities:

- by integrating by parts on the Gaussian measure we get

$$\int \mathrm{dg}_{z,\bar z}\mathrm{dg}_{w,\bar w}e^{\bar z\,\mathrm{a}^\dagger}\varphi_0\,z\,\bar w\,\mathscr{R}_t(z,\bar w)\varphi_0^\dagger e^{w\,\mathrm{a}} = \int \mathrm{dg}_{z,\bar z}\mathrm{dg}_{w,\bar w}\left(\partial_{\bar z}e^{\bar z\,\mathrm{a}^\dagger}\right)\varphi_0\mathscr{R}_t(z,\bar w)\varphi_0^\dagger\left(\partial_w e^{w\,\mathrm{a}}\right) = \mathrm{a}^\dagger\,\boldsymbol{\rho}_t\boldsymbol{a}$$

  and

$$\int \mathrm{dg}_{z,\bar z}\mathrm{dg}_{w,\bar w}e^{\bar z\,\mathrm{a}^\dagger}\varphi_0\,\partial_z\,\partial_{\bar w}\,\mathscr{R}_t(z,\bar w)\varphi_0^\dagger e^{w\,\mathrm{a}} = \int \mathrm{dg}_{z,\bar z}\mathrm{dg}_{w,\bar w}\left(\bar z e^{\bar z\,\mathrm{a}^\dagger}\right)\varphi_0\mathscr{R}_t(z,\bar w)\varphi_0^\dagger\left(w\,e^{w\,\mathrm{a}}\right) = \mathrm{a}\,\boldsymbol{\rho}_t\boldsymbol{a}^\dagger$$

- by integrating by parts on the Gaussian measure and by using the property that a coherent state behaves like an eigenstate of the annihilation operator

$$\begin{aligned}&\int \mathrm{d}\mathrm{g}_{z,\bar{z}}\mathrm{d}\mathrm{g}_{w,\bar{w}}e^{\bar{z}\,\mathrm{a}^{\dagger}}\varphi_0\,\partial_z\,z\,\mathscr{R}_t(z,\bar{w})\varphi_0^{\dagger}e^{w\,\mathrm{a}}=\int \mathrm{d}\mathrm{g}_{z,\bar{z}}\mathrm{d}\mathrm{g}_{w,\bar{w}}\left(\bar{z}\,z\,e^{\bar{z}\,\mathrm{a}^{\dagger}}\right)\varphi_0\mathscr{R}_t(z,\bar{w})\varphi_0^{\dagger}e^{w\,\mathrm{a}}\\&=\int \mathrm{d}\mathrm{g}_{z,\bar{z}}\mathrm{d}\mathrm{g}_{w,\bar{w}}\left(z\;\mathrm{a}\;e^{\bar{z}\,\mathrm{a}^{\dagger}}\right)\varphi_0\mathscr{R}_t(z,\bar{w})\varphi_0^{\dagger}e^{w\,\mathrm{a}}=\int \mathrm{d}\mathrm{g}_{z,\bar{z}}\mathrm{d}\mathrm{g}_{w,\bar{w}}\left(\mathrm{a}\;\partial_{\bar{z}}e^{\bar{z}\,\mathrm{a}^{\dagger}}\right)\varphi_0\mathscr{R}_t(z,\bar{w})\varphi_0^{\dagger}e^{w\,\mathrm{a}}=\mathrm{a}\;\mathrm{a}^{\dagger}\;\boldsymbol{\rho}_t\end{aligned}$$

and

$$\begin{aligned}&\int \mathrm{d}\mathrm{g}_{z,\bar{z}}\mathrm{d}\mathrm{g}_{w,\bar{w}}e^{\bar{z}\,\mathrm{a}^{\dagger}}\varphi_0\,\bar{w}\partial_{\bar{w}}\,\mathscr{R}_t(z,\bar{w})\varphi_0^{\dagger}e^{w\,\mathrm{a}}=\int \mathrm{d}\mathrm{g}_{z,\bar{z}}\mathrm{d}\mathrm{g}_{w,\bar{w}}\,e^{\bar{z}\,\mathrm{a}^{\dagger}}\varphi_0\partial_{\bar{w}}\mathscr{R}_t(z,\bar{w})\varphi_0^{\dagger}\left(\partial_w\,e^{w\,\mathrm{a}}\right)\\&=\int \mathrm{d}\mathrm{g}_{z,\bar{z}}\mathrm{d}\mathrm{g}_{w,\bar{w}}\,e^{\bar{z}\,\mathrm{a}^{\dagger}}\varphi_0\mathscr{R}_t(z,\bar{w})\varphi_0^{\dagger}\left(e^{w\,\mathrm{a}}\,w\;\mathrm{a}\right)=\boldsymbol{\rho}_t\;\mathrm{a}^{\dagger}\;\mathrm{a}\end{aligned}$$

We obtain

$$\begin{aligned}\partial_t\boldsymbol{\rho}_t=&\,\mathrm{a}^{\dagger}\,\boldsymbol{\rho}_t\,\mathrm{a}\,m_t^{(1,1)}\left(\frac{\dot{m}_t^{(1,1)}}{m_t^{(1,1)}}-\frac{\dot{m}_t^{(1,2)}}{m_t^{(1,2)}}-\frac{\dot{m}_t^{(2,1)}}{m_t^{(2,1)}}+\frac{\dot{m}_t^{(2,2)}\,m_t^{(1,1)}}{m_t^{(2,1)}\,m_t^{(1,2)}}\right)+\mathrm{a}^{\dagger}\,\mathrm{a}\,\boldsymbol{\rho}_t\left(\frac{\dot{m}_t^{(1,2)}}{m_t^{(1,2)}}-\frac{\dot{m}_t^{(2,2)}\,m_t^{(1,1)}}{m_t^{(2,1)}\,m_t^{(1,2)}}\right)\\&+\boldsymbol{\rho}_t\,\mathrm{a}^{\dagger}\,\mathrm{a}\left(\frac{\dot{m}_t^{(2,1)}}{m_t^{(2,1)}}-\frac{\dot{m}_t^{(2,2)}\,m_t^{(1,1)}}{m_t^{(2,1)}m_t^{(1,2)}}\right)+\mathrm{a}\,\boldsymbol{\rho}_t\,\mathrm{a}^{\dagger}\,\frac{\dot{m}_t^{(2,2)}}{m_t^{(2,1)}\,m_t^{(1,2)}}-\left(\frac{\dot{m}_t^{(2,2)}m_t^{(1,1)}}{m_t^{(2,1)}\,m_t^{(1,2)}}+\frac{\dot{m}_t^{(1,1)}}{1-m_t^{(1,1)}}\right)\mathscr{R}_t(z,\bar{z})\end{aligned}$$

Finally, using the commutation relation and (7b), we arrive at manifestly trace preserving form of the master equation

$$\partial_t\boldsymbol{\rho}_t=-\frac{1}{2}\left(\frac{\dot{m}_t^{(2,1)}}{m_t^{(2,1)}}-\frac{\dot{m}_t^{(1,2)}}{m_t^{(1,2)}}\right)\left[\mathrm{a}^{\dagger}\,\mathrm{a}\,,\boldsymbol{\rho}_t\right]+a_t\left(\mathrm{a}^{\dagger}\,\boldsymbol{\rho}_t\,\mathrm{a}-\frac{\mathrm{a}\,\mathrm{a}^{\dagger}\,\boldsymbol{\rho}_t+\boldsymbol{\rho}_t\,\mathrm{a}\,\mathrm{a}^{\dagger}}{2}\right)+b_t\left(\mathrm{a}\,\boldsymbol{\rho}_t\,\mathrm{a}^{\dagger}-\frac{\mathrm{a}^{\dagger}\,\mathrm{a}\,\boldsymbol{\rho}_t+\mathrm{a}^{\dagger}\,\mathrm{a}\,\boldsymbol{\rho}_t}{2}\right)$$

or equivalently

$$\partial_t\boldsymbol{\rho}_t=-\frac{1}{2}\left(\frac{\dot{m}_t^{(2,1)}}{m_t^{(2,1)}}-\frac{\dot{m}_t^{(1,2)}}{m_t^{(1,2)}}\right)\left[\mathrm{a}^{\dagger}\,\mathrm{a}\,,\boldsymbol{\rho}_t\right]+a_t\left(\frac{\left[\mathrm{a}^{\dagger}\,,\boldsymbol{\rho}_t\,\mathrm{a}\right]+\left[\mathrm{a}^{\dagger}\,\boldsymbol{\rho}_t\,,\mathrm{a}\right]}{2}\right)+b_t\left(\frac{\left[\mathrm{a}\,,\boldsymbol{\rho}_t\,\mathrm{a}^{\dagger}\right]+\left[\mathrm{a}\,\boldsymbol{\rho}_t\,,\mathrm{a}^{\dagger}\right]}{2}\right)$$

where

$$\begin{aligned}a_t&=m_t^{(1,1)}\left(\frac{\dot{m}_t^{(1,1)}}{m_t^{(1,1)}}-\frac{\dot{m}_t^{(1,2)}}{m_t^{(1,2)}}-\frac{\dot{m}_t^{(2,1)}}{m_t^{(2,1)}}+\frac{\dot{m}_t^{(2,2)}\,m_t^{(1,1)}}{m_t^{(2,1)}\,m_t^{(1,2)}}\right)\\b_t&=-\frac{1}{m_t^{(2,1)}\,m_t^{(1,2)}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_t^{(2,1)}m_t^{(1,2)}}{1-m_t^{(1,1)}}\end{aligned}$$

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