TCM315 Fall 2022: Introduction to Open Quantum Systems

Lecture 9: Jaynes–Cummings–Gaudin model

Course handouts are designed as a study aid and are not meant to replace the recommended textbooks. Handouts may contain typos and/or errors. The students are encouraged to verify the information contained within and to report any issue to the lecturer.

CONTENTS

1. INTRODUCTION

A recommended review on the Jaynes–Cummings model is [5]. The present notes draw from [6]. The model is also leisurely discussed in chapters 4 and again 10 of [2]. The holomorphic formulation of quantum mechanics is introduced in a manner adapted to a physicist readership e.g. in chapter 6 of [7].

2. DERIVATION OF THE SPIN BOSON GAUDIN MODEL

The non-relativistic Hamilton operator describing an atom interacting with a single mode of an electromagnetic field is

$$\mathbb{H} = \frac{1}{2\,m}\|\mathbb{P} - \mathrm{c}\,\boldsymbol{A}(\mathbb{Q})\|^2 + U(\mathbb{Q}) + \sum_{k=1}^{\mathcal{N}} \omega_k\ \mathrm{a}_k^{\dagger}\,\mathrm{a}_k \tag{1}$$

We denote by $\mathbb{P}$ and $\mathbb{Q}$ are the atom momentum and position operators, whereas c is the charge of the atom and m the mass of the atom. The second quantization operators $\{\mathrm{a}_l^{\dagger}\}_{l=1}^{\mathcal{N}}$ and $\{\mathrm{a}_l\}_{l=1}^{\mathcal{N}}$ are creation and annihilation operators of the quantized electromagnetic field.

$$\left[\mathrm{a}_l\,,\mathrm{a}_k^{\dagger}\right] = \delta_{l\,k}\,\mathbb{1}$$

We assume that the vector potential operator takes the form

$$\boldsymbol{A}(\mathbb{Q}) = \sum_{k=1}^{\mathcal{N}}\left(f_k(\mathbb{Q})\ \mathrm{a}_k + \bar{f}_k(\mathbb{Q})\ \mathrm{a}_k^{\dagger}\right)$$

2.1. Simplifying assumptions

The ineffectiveness of natural thinking comes from being overwhelmed by an infinity of possibilities and facts. In order to go on, you have to know what to leave out; this is the essence of effective thinking. (Kurt Gödel, as quoted in Hao Wang's biography "Reflections on Kurt Gödel", MIT Press, 1987)

The dynamics generated by the the full a Hamiltonian is not integrable. A set of physically relevant further approximations allow us to simplify the model while still capturing relevant physics. To this goal

- we neglect multi-photon processes corresponding to the square of the vector potential.
- We assume the external atomic potential to be zero

$$U = 0$$

- We neglect any kinetic effect besides transitions between two atomic levels separated in energy by an amount corresponding the spectral content of the electromagnetic field

$$\frac{1}{2\,m}\|\mathbb{P}\|^2 \to \frac{\Omega}{2}\sigma_3$$

where Ω is the transition frequency and

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is a Pauli matrix, generating the su(2) Lie algebra

$$[\sigma_i\,, \sigma_j] = 2\,\imath\,\epsilon_{i\,j\,k}\sigma_k$$

with

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \& \qquad \sigma_2 = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}$$

$\epsilon_{i\,j\,k}$ being the fully anti-symmetric symbol.

- In accordance with the neglecting the kinetic energy, we evaluate the atom - electromagnetic field interaction in the origin and retain only effect associated to transitions in the electronic state in the so called **rotating wave approximation**:

$$\frac{1}{2\,m}\left(\mathbb{P}^\dagger \boldsymbol{A}(\mathbb{Q}) + \boldsymbol{A}^\dagger(\mathbb{Q})\mathbb{P}\right) \to \sum_{k=1}^{\mathcal{N}}\left(g_k\ \mathrm{a}_k\ \sigma_+ + \bar{g}_k\ \mathrm{a}_k^\dagger\ \sigma_-\right)$$

with g, $\bar{g}$ constant

$$\sigma_+ = \frac{\sigma_1 + \imath\,\sigma_2}{2} \qquad \& \qquad \sigma_+ = \frac{\sigma_1 - \imath\,\sigma_2}{2}$$

In conclusion the **effective Hamilton** operator we want to study is the spin-boson (Gaudin) model [1, 3, 4]

$$\mathbb{H} = \frac{\Omega}{2}\sigma_3 + \sum_{k=1}^{\mathcal{N}}\omega_k\,\mathrm{a}_k^\dagger\,\mathrm{a}_k + \sum_{k=1}^{\mathcal{N}}\left(g_k\ \mathrm{a}_k\ \sigma_+ + \bar{g}_k\ \mathrm{a}_k^\dagger\ \sigma_-\right) \tag{2}$$

In the case $\mathcal{N} = 1$ the spin-boson model is usually referred to as the **Jaynes–Cummings** model [5].

3. HOLOMORPHIC REPRESENTATION OF THE SPIN-BOSON MODEL

In the holomorphic representation, the state vector of the spin-boson model is

$$\boldsymbol{\psi}_t(z_1, z_2, \dots, z_{\mathcal{N}}) = \psi_t^{(1)}(z_1, z_2, \dots, z_{\mathcal{N}})\boldsymbol{e}_1 + \psi_t^{(2)}(z_1, z_2, \dots, z_{\mathcal{N}})\boldsymbol{e}_2$$

where $\{\boldsymbol{e}\}_{i=1}^2$ is the canonical basis of $\mathbb{C}^2$. The Hamilton operator acts on the state vector as

$$\mathbb{H}\boldsymbol{\psi}_t(z) = \left(\frac{\Omega}{2}\sigma_3 + \sum_{k=1}^{\mathcal{N}} \omega_k\, z_k \frac{\mathrm{d}}{\mathrm{d}z_k} + \sum_{k=1}^{\mathcal{N}} \left(g_k \frac{\mathrm{d}}{\mathrm{d}z_k}\,\sigma_+ + \bar{g}_k\, z\, \sigma_-\right)\right)\boldsymbol{\psi}_t(z)$$

where

$$\begin{aligned}
&\sigma_+\, \boldsymbol{e}_1 = \sigma_-\, \boldsymbol{e}_2 = 0 \\
&\sigma_+\, \boldsymbol{e}_2 = \boldsymbol{e}_1 \\
&\sigma_-\, \boldsymbol{e}_1 = \boldsymbol{e}_2
\end{aligned}$$

4. SOLUTION OF THE JAYNES–CUMMINGS DYNAMICS

We focus first for simplicity sake on the $\mathcal{N} = 1$, the Jaynes–Cummings model. We get the vector equation

$$\begin{aligned}
&\imath\partial_t \mathcal{U}_t(z|\bar{z}) = \left(\frac{\Omega}{2}\sigma_3 + \omega\, z\, \frac{\mathrm{d}}{\mathrm{d}z} + g\, \frac{\mathrm{d}}{\mathrm{d}z}\, \sigma_+ + \bar{g}\, z\, \sigma_-\right)\mathcal{U}_t(z|\bar{z}) \\
&\mathcal{U}_0(z|\bar{z}) = e^{z\bar{z}}\, 1_2
\end{aligned}$$

The propagator is a 2×2 matrix. From the propagator we reconstruct the evolution of any state vector by applying the integral relation

$$\boldsymbol{\psi}_t(z) = \int_{\mathbb{C}} \mathrm{dg}_{w,\bar{w}}\, \mathcal{U}_t(z|\bar{w})\, \boldsymbol{\psi}_0(w) \tag{3}$$

4.1. Decomposition into "free" and "detuned" components

We write the Hamilton operator as

$$\mathbb{H} = \mathbb{H}_0 + \mathbb{H}_1$$

with addends

$$\mathbb{H}_0 = \frac{\omega}{2}\sigma_3 + \omega\, z\, \frac{\mathrm{d}}{\mathrm{d}z} \tag{4a}$$

$$\mathbb{H}_1 = \frac{\delta}{2}\sigma_3 + g\, \frac{\mathrm{d}}{\mathrm{d}z}\, \sigma_+ + \bar{g}\, z\, \sigma_- \tag{4b}$$

having denoted

$$\delta = \Omega - \omega$$

We refer to the Hamilton operators (4a) and (4a) as respectively the "free" and "detuned" components. The advantage of the decomposition is that

$$0 = [\mathbb{H}_0\, , \mathbb{H}_1] = \left[\mathbb{H}_0\, , g\, \frac{\mathrm{d}}{\mathrm{d}z}\, \sigma_+ + \bar{g}\, z\, \sigma_-\right]$$

This is because the identities

$$[\sigma_3\, , \sigma_+] = 2\, \sigma_+ \qquad \& \qquad [\sigma_3\, , \sigma_-] = -2\, \sigma_-$$

yield

$$\left[\frac{\omega}{2}\sigma_3+\omega\,z\,\frac{\mathrm{d}}{\mathrm{d}z}\,,g\,\frac{\mathrm{d}}{\mathrm{d}z}\,\sigma_+\right]=\frac{\omega}{2}g\,\frac{\mathrm{d}}{\mathrm{d}z}\,[\sigma_3\,,\sigma_+]+\omega\,g\,\sigma_+\left[z\,\frac{\mathrm{d}}{\mathrm{d}z}\,,\frac{\mathrm{d}}{\mathrm{d}z}\right]=\omega\,g\,\sigma_+\frac{\mathrm{d}}{\mathrm{d}z}-\omega\,g\,\sigma_+\frac{\mathrm{d}}{\mathrm{d}z}=0$$
$$\left[\frac{\omega}{2}\sigma_3+\omega\,z\,\frac{\mathrm{d}}{\mathrm{d}z}\,,\bar{g}\,z\,\sigma_-\right]=\frac{\omega}{2}\bar{g}\,z\,\frac{\mathrm{d}}{\mathrm{d}z}\,[\sigma_3\,,\sigma_-]+\omega\,\bar{g}\,\sigma_-\left[z\,\frac{\mathrm{d}}{\mathrm{d}z}\,,z\right]=-\omega\,\bar{g}\,\sigma_-z+\omega\,\bar{g}\,\sigma_-z=0$$

This means that we can look for a solution in the form

$$\mathcal{U}_t(z|\bar{z})=e^{-\imath\,\frac{\sigma_3}{2}\,\omega\,t}\int_{\mathbb{C}}\mathrm{d}\mathrm{g}_{w,\bar{w}}\,e^{z\,e^{-\imath\,\omega\,t}\,\bar{w}}\mathcal{K}_t(w|\bar{z})$$

where now

$$\imath\partial_t\mathcal{K}_t(z|\bar{z})=\left(\frac{\delta}{2}\sigma_3+g\,\frac{\mathrm{d}}{\mathrm{d}z}\,\sigma_++\bar{g}\,z\,\sigma_-\right)\mathcal{K}_t(z|\bar{z})$$
$$\mathcal{K}_0(z|\bar{z})=1_2\,e^{z\bar{z}}$$

4.2. Solution of the detuned dynamics

We look for a solution in the form

$$\mathcal{K}_t(z|\bar{z})=\hbar_t^{(0)}(z|\bar{z})\,1_2+\hbar_t^{(3)}(z|\bar{z})\sigma_3+\hbar_t^{(+)}(z|\bar{z})\sigma_++\hbar_t^{(-)}(z|\bar{z})\sigma_-$$

with

$$\hbar_0^{(0)}(z|\bar{z})=e^{z\bar{z}}=\sum_{n=0}^{\infty}\frac{z^n\,\bar{z}^n}{n!}\tag{5a}$$
$$\hbar_0^{(3)}(z|\bar{z})=\hbar_0^{(-)}(z|\bar{z})=\hbar_0^{(-)}(z|\bar{z})=0\tag{5b}$$

We thus get into the equations

$$0=\mathrm{Tr}\left(\left(\imath\partial_t-\mathbb{H}_1\right)\mathcal{K}_t(z|\bar{z})\right)=2\,\imath\,\partial_t\hbar_t^{(0)}(z|\bar{z})-g\,\partial_z\hbar_t^{(-)}(z|\bar{z})-\delta\,\hbar_t^{(3)}(z|\bar{z})-z\,\bar{g}\,\hbar_t^{(+)}(z|\bar{z})$$
$$0=\mathrm{Tr}\left(\sigma_3\left(\imath\partial_t-\mathbb{H}_1\right)\mathcal{K}_t(z|\bar{z})\right)=2\,\imath\,\partial_t\hbar_t^{(3)}(z|\bar{z})-g\,\partial_z\hbar_t^{(-)}(z|\bar{z})-\delta\,\hbar_t^{(0)}(z|\bar{z})+z\,\bar{g}\,\hbar_t^{(+)}(z|\bar{z})$$
$$0=\mathrm{Tr}\left(\sigma_-\left(\imath\partial_t-\mathbb{H}_1\right)\mathcal{K}_t(z|\bar{z})\right)=\imath\,\partial_t\hbar_t^{(+)}(z|\bar{z})-\frac{\delta}{2}\,\hbar_t^{(+)}(z|\bar{z})+g\,\partial_z\left(\hbar_t^{(3)}(z|\bar{z})-\hbar_t^{(0)}(z|\bar{z})\right)$$
$$0=\mathrm{Tr}\left(\sigma_+\left(\imath\partial_t-\mathbb{H}_1\right)\mathcal{K}_t(z|\bar{z})\right)=\imath\,\partial_t\hbar_t^{(-)}(z|\bar{z})+\frac{\delta}{2}\,\hbar_t^{(-)}(z|\bar{z})-\bar{g}\,z\left(\hbar_t^{(3)}(z|\bar{z})+\hbar_t^{(0)}(z|\bar{z})\right)$$

Upon defining

$$x_t(z|\bar{z})=\hbar_t^{(0)}(z|\bar{z})+\hbar_t^{(3)}(z|\bar{z})$$
$$y_t(z|\bar{z})=\hbar_t^{(0)}(z|\bar{z})-\hbar_t^{(3)}(z|\bar{z})$$

the system of four equations decouples into two systems of two equations

$$\imath\,\partial_t y_t(z|\bar{z})-\bar{g}\,z\,\hbar_t^{(+)}(z|\bar{z})+\frac{\delta}{2}\,y_t(z|\bar{z})=0$$
$$\imath\,\partial_t\hbar_t^{(+)}(z|\bar{z})-\frac{\delta}{2}\,\hbar_t^{(+)}(z|\bar{z})-g\,\partial_z\,y_t(z|\bar{z})=0$$

and

$$\imath\,\partial_t x_t(z|\bar{z})-g\,\partial_z\hbar_t^{(-)}(z|\bar{z})-\frac{\delta}{2}\,x_t(z|\bar{z})=0$$
$$\imath\,\partial_t\hbar_t^{(-)}(z|\bar{z})+\frac{\delta}{2}\,\hbar_t^{(-)}(z|\bar{z})-\bar{g}\,z\,x_t(z|\bar{z})=0$$

4.2.1. Solution of the first pair of equations

Holomorphic functions coincide with their Taylor expansion

$$\mathcal{y}_t(z|\bar{z}) = \sum_{n=0}^{\infty} \mathcal{y}_t^{(n)}(\bar{z}) z^n$$
$$\mathcal{k}_t^{(+)}(z|\bar{z}) = \sum_{n=0}^{\infty} \mathcal{k}_t^{(+,n)}(\bar{z}) z^n$$

we arrive at the hierarchy of equations

$$\imath\, \partial_t \mathcal{y}_t^{(n)}(\bar{z}) - \bar{g}\, \mathcal{k}_t^{(+,n-1)}(\bar{z}) + \frac{\delta}{2}\, \mathcal{y}_t^{(n)}(\bar{z}) = 0$$
$$\imath\, \partial_t \mathcal{k}_t^{(+,n-1)}(\bar{z}) - \frac{\delta}{2}\, \mathcal{k}_t^{(+,n-1)}(\bar{z}) - g\, n\, \mathcal{y}_t^{(n)}(\bar{z}) = 0$$

We distinguish two situations

- For $n = 0$ we get

$$\mathcal{y}_t^{(0)}(\bar{z}) = e^{\imath \frac{\delta}{2} t} \mathcal{y}_t^{(0)}(\bar{z})$$

- For $n \geq 1$

$$\mathcal{k}_t^{(+,n-1)}(\bar{z}) = \frac{1}{\bar{g}} \left(\imath\, \partial_t \mathcal{y}_t^{(n)}(\bar{z}) + \frac{\delta}{2}\, \mathcal{y}_t^{(n)}(\bar{z}) \right)$$

yields

$$\partial_t^2 \mathcal{y}_t^{(n)}(\bar{z}) + \left(n\, |g|^2 + \frac{\delta^2}{2} \right)\, \mathcal{y}_t^{(n)}(\bar{z}) = 0$$

The boundary condition (5a) imposes

$$\mathcal{y}_t^{(n)}(\bar{z}) = \frac{\bar{z}^n}{n!} \cos(\nu_n\, t) + \mathcal{c}^{(n)}(\bar{z}) \sin(\nu_n\, t)$$

having defined

$$\nu_n = \sqrt{n\, |g|^2 + \frac{\delta^2}{4}} \tag{6}$$

Upon inserting the coefficients in the Taylor expansions we get

$$\mathcal{y}_t(z|\bar{z}) = e^{\imath \frac{\delta}{2} t} + \sum_{n=1}^{\infty} z^n \left(\frac{\bar{z}^n}{n!} \cos(\nu_n\, t) + \mathcal{c}^{(n)}(\bar{z})\, \sin(\nu_n\, t) \right)$$
$$\mathcal{k}_t^{(+)}(z|\bar{z}) = \frac{1}{\bar{g}} \sum_{n=0}^{\infty} \frac{z^n\, \bar{z}^{n+1}}{(n+1)!} \left(\frac{\delta}{2} \cos(\nu_{n+1}\, t) - \imath\, \nu_{n+1} \sin(\nu_{n+1}\, t) \right)$$
$$+ \frac{1}{\bar{g}} \sum_{n=0}^{\infty} z^n\, \mathcal{c}^{(n+1)}(\bar{z}) \left(\frac{\delta}{2}\, \sin(\nu_{n+1}\, t) + \imath\, \nu_{n+1} \cos(\nu_{n+1}\, t) \right)$$

The condition

$$\mathcal{k}_0^{(+)}(z|\bar{z}) = 0$$

then yields

$$\mathcal{c}^{(n+1)}(\bar{z}) = \frac{\imath\, \bar{z}^{n+1}}{(n+1)!\, \nu_{n+1}} \frac{\delta}{2} \qquad \forall\, n \geq 0$$

After straightforward algebra, we arrive at

$$\mathscr{k}_t^{(+)}(z|\bar{z}) = \sum_{n=0}^{\infty} \frac{-\imath\, g\, z^n\, \bar{z}^{n+1}}{n!\, \nu_{n+1}} \sin(\nu_{n+1}\, t)$$

whereas

$$\mathscr{y}_t(z|\bar{z}) = e^{\imath \frac{\delta}{2} t} + \sum_{n=1}^{\infty} \frac{z^n\, \bar{z}^n}{n!} \left(\cos(\nu_n\, t) + \frac{\imath\, \delta}{2\, \nu_n} \sin(\nu_n\, t) \right) = \sum_{n=0}^{\infty} \frac{z^n\, \bar{z}^n}{n!} \left(\cos(\nu_n\, t) + \frac{\imath\, \delta}{2\, \nu_n} \sin(\nu_n\, t) \right)$$

We are entitled to write the last equality because

$$\frac{\imath\, \delta}{2\, \nu_0} \sin(\nu_0\, t) = \frac{\imath\, \delta}{|\delta|} \sin\left(\frac{|\delta|}{2}\, t \right) = \imath\, \sin\left(\frac{\delta}{2}\, t \right)$$

4.2.2. Solution of the second pair of equations

The second pair of equations can be treated in the same fashion. The coefficients of the Taylor expansions satisfy

$$\imath\, \partial_t \mathscr{x}_t^{(n-1)}(\bar{z}) - g\, n\, \mathscr{k}_t^{(-,n)}(\bar{z}) - \frac{\delta}{2}\, \mathscr{x}_t^{(n-1)}(\bar{z}) = 0$$

$$\imath\, \partial_t \mathscr{k}_t^{(-,n)}(\bar{z}) + \frac{\delta}{2}\, \mathscr{k}_t^{(-,n)}(\bar{z}) - \bar{g}\, \mathscr{x}_t^{(n-1)}(\bar{z}) = 0$$

- For $n = 0$ the requirement of analytic expansion readily yields

  $$\imath\, \partial_t \mathscr{k}_t^{(-,0)}(\bar{z}) + \frac{\delta}{2}\, \mathscr{k}_t^{(-,0)}(\bar{z}) = 0$$

  whose only admissible solution in consideration of (5b) is

  $$\mathscr{k}_t^{(-,0)}(\bar{z}) = 0$$

- For $n \geq 1$ we use

  $$\mathscr{x}_t^{(n-1)}(\bar{z}) = \frac{1}{\bar{g}} \left(\imath\, \partial_t \mathscr{k}_t^{(-,n)}(\bar{z}) + \frac{\delta}{2}\, \mathscr{k}_t^{(-,n)}(\bar{z}) \right)$$

  to derive

  $$\partial_t^2 \mathscr{k}_t^{(-,n)}(\bar{z}) + \nu_n\, \mathscr{k}_t^{(-,n)}(\bar{z}) = 0$$

  with ν_n specified by (6). The form of the solution compatible with (5b) is

  $$\mathscr{k}_t^{(-,n)}(\bar{z}) = \mathscr{k}_0^{(-,n)}(\bar{z}) \sin(\nu_n\, t)$$

We thus arrive at

$$\mathscr{k}_t^{(-)}(z|\bar{z}) = \sum_{n=1}^{\infty} z^n\, \mathscr{k}_0^{(-,n)}(\bar{z}) \sin(\nu_n\, t)$$

$$\mathscr{x}_t(z|\bar{z}) = \frac{1}{\bar{g}} \sum_{n=0}^{\infty} z^n\, \mathscr{k}_0^{(-,n+1)}(\bar{z}) \left(\frac{\delta}{2} \sin(\nu_{n+1}\, t) + \imath\, \nu_{n+1} \cos(\nu_{n+1}\, t) \right)$$

The initial condition on $\mathscr{x}_t$ fixes the unknown constants:

$$\mathscr{k}_0^{(-,n+1)}(\bar{z}) = \frac{-\imath\, \bar{g}\, \bar{z}^n}{n!\, \nu_{n+1}} \qquad \forall\, n \geq 0$$

The conclusion is

$$\mathscr{k}_t^{(-)}(z|\bar{z}) = \sum_{n=0}^{\infty} \frac{-\imath\, \bar{g}\, z^{n+1}\, \bar{z}^n}{n!\, \nu_{n+1}} \sin(\nu_{n+1}\, t)$$

$$\mathscr{x}_t(z|\bar{z}) = \sum_{n=0}^{\infty} \frac{z^n\, \bar{z}^n}{n!} \left(-\frac{\imath\, \delta}{2\, \nu_{n+1}} \sin(\nu_{n+1}\, t) + \cos(\nu_{n+1}\, t) \right)$$

4.3. Expression of the propagator

The propagator is

$$\mathcal{U}_t(z|\bar{z}) = e^{-\imath \frac{\omega t}{2}\sigma_3} \sum_{n=0}^{\infty} \frac{e^{-\imath \omega n t}(z\bar{z})^n}{n!} \begin{bmatrix} \cos(\nu_{n+1}t) + \frac{\delta \sin(\nu_{n+1} t)}{2 \imath \nu_{n+1}} & \frac{g \bar{z} \sin(\nu_{n+1} t)}{\imath \nu_{n+1}} \\ e^{-\imath \omega t} \frac{\bar{g} z \sin(\nu_{n+1} t)}{\imath \nu_{n+1}} & \cos(\nu_n t) - \frac{\delta \sin(\nu_n t)}{2 \imath \nu_n} \end{bmatrix}$$

with ν_n given by (6). Upon observing that

$$e^{-\imath \frac{\omega t}{2}\sigma_3} = \begin{bmatrix} e^{-\imath \frac{\omega t}{2}} & 0 \\ 0 & e^{\imath \frac{\omega t}{2}} \end{bmatrix}$$

the propagator becomes

$$\mathcal{U}_t(z|\bar{z}) = \sum_{n=0}^{\infty} \frac{e^{-\imath \omega n t}(z\bar{z})^n}{n!} \begin{bmatrix} e^{-\imath \frac{\omega t}{2}} \left(\cos(\nu_{n+1}t) + \frac{\delta \sin(\nu_{n+1} t)}{2 \imath \nu_{n+1}}\right) & e^{-\imath \frac{\omega t}{2}} \frac{g \bar{z} \sin(\nu_{n+1} t)}{\imath \nu_{n+1}} \\ e^{-\frac{\imath \omega t}{2}} \frac{\bar{g} z \sin(\nu_{n+1} t)}{\imath \nu_{n+1}} & e^{\imath \frac{\omega t}{2}} \left(\cos(\nu_n t) - \frac{\delta \sin(\nu_n t)}{2 \imath \nu_n}\right) \end{bmatrix}$$

whence we readily verify that

$$\mathcal{U}_{-t}(z|\bar{z}) = \left(\mathcal{U}_t(z|\bar{z})\right)^{\dagger}$$

5. INDICATORS OF THE JAYNES-CUMMINGS MODEL

A pure state operator of the Jaynes-Cummings model takes the form

$$\boldsymbol{\rho}_t(z,\bar{z}) = \boldsymbol{\psi}_t(z)\boldsymbol{\psi}_t^{\dagger}(z)$$

where the evolution of the state vector is governed by (3).

5.1. Evolution of an two level system energy eigenstate

In particular, we may consider the evolution of a pure state describing at time zero the two level system in the energy eigenstate whereas any energy eigenstate of the boson oscillator oscillator can be occupied with finite probability

$$\boldsymbol{\psi}_0(z) = f(z)\,\boldsymbol{e}_1 = \sum_{n=0}^{\infty} f_n \frac{z^n}{\sqrt{n!}} \boldsymbol{e}_1$$

bearing in mind that

$$\frac{\omega}{2}\sigma_3 \boldsymbol{e}_1 = \frac{\omega}{2}\boldsymbol{e}_1$$

Remark. *In the holomorphic formalism the "physically correct" way to write a state vector is as the product of an occupation eigenstate*

$$\varphi_n(z) = \frac{z^n}{\sqrt{n!}}$$

times the corresponding probability amplitude e.g. f_n*.*

* *

At any later time the state vector becomes

$$\boldsymbol{\psi}_t(z) = \int_{\mathbb{C}} \mathrm{d}g_{w,\bar{w}} \mathcal{U}_t(z|\bar{w})\, f(w)\, \boldsymbol{e}_1 = e^{-\imath \frac{\omega t}{2}\sigma_3} \sum_{n=0}^{\infty} e^{-\imath \omega n t} f_n \frac{z^n}{\sqrt{n!}} \begin{bmatrix} \cos(\nu_{n+1}t) + \frac{\delta \sin(\nu_{n+1} t)}{2 \imath \nu_{n+1}} \\ e^{-\imath \omega t} \frac{\bar{g}\, z \sin(\nu_{n+1} t)}{\imath \nu_{n+1}} \end{bmatrix}$$

We notice that unitary evolution now couples, "entangles" both eigenstates of the two level system with eigenstates of the boson oscillator:

$$\boldsymbol{e}_2^{\dagger}\boldsymbol{\psi}_t(z) = \sum_{n=0}^{\infty} e^{-\imath\,\omega\,n\,t} f_n \frac{z^{n+1}}{\sqrt{n!}} \frac{\bar{g}\,\sin(\nu_{n+1}\,t)}{\imath\,\nu_{n+1}}$$

If we denote by ϵ_t the dynamical variable describing the energy of the two level system, the survival probability of the initial state at time t is

$$\mathrm{P}\left(\epsilon_t = \frac{\omega}{2}\,|\epsilon_0 = \frac{\omega}{2}\right) = \int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{w,\bar{w}} |\,\mathrm{e}_1^{\dagger}\,\boldsymbol{\psi}_t(w)|^2 = \sum_{n=0}^{\infty} f_n^2 \left(\cos^2(\nu_{n+1}t) + \frac{\delta^2\,\sin^2(\nu_{n+1}\,t)}{4\,\nu_{n+1}^2}\right)$$

whilst

$$\mathrm{P}\left(\epsilon_t = -\frac{\omega}{2}\,|\epsilon_0 = \frac{\omega}{2}\right) = \int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{w,\bar{w}} |\,\mathrm{e}_2^{\dagger}\,\boldsymbol{\psi}_t(w)|^2 = \sum_{n=0}^{\infty} f_n^2\,(n+1)\,|g|^2\,\frac{\sin^2(\nu_{n+1}\,t)}{\nu_{n+1}^2}$$

Upon recalling the definition of ν_n (6), we verify that

$$\mathrm{P}\left(\epsilon_t = \frac{\omega}{2}|\epsilon_0 = \frac{\omega}{2}\right) + \mathrm{P}\left(\epsilon_t = -\frac{\omega}{2}|\epsilon_0 = \frac{\omega}{2}\right) = 1$$

6. TIME EVOLUTION

In order to explore the dynamical implications of unitary evolution it is expedient to make more explicit hypotheses about the initial probability distribution of the energy in the oscillator. We thus assume a Poisson statistics with parameter α

$$|f_n|^2 = \frac{\alpha^n}{n!} e^{-\alpha}$$

The survival probability of the "excited state" of the two level system is then

$$\mathrm{P}\left(\epsilon_t = \frac{\omega}{2}|\epsilon_0 = \frac{\omega}{2}\right) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} e^{-\alpha} \left(\cos^2(\nu_{n+1}t) + \frac{\delta^2\,\sin^2(\nu_{n+1}\,t)}{4\,\nu_{n+1}^2}\right)$$

To simplify this expression we assume resonance i.e. absence of detuning

$$\delta = 0$$

and strong coupling :

$$|g|^2 = 1$$

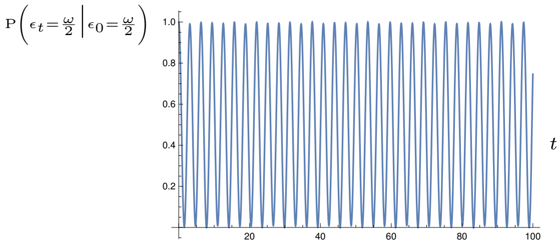
The survival probability reduces to

$$\mathrm{P}\left(\epsilon_t = \frac{\omega}{2}|\epsilon_0 = \frac{\omega}{2}\right) = e^{-\alpha} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \cos^2(t\sqrt{n+1}) \tag{7}$$

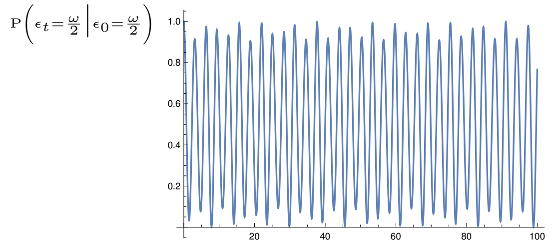
As we vary α the probability exhibits **quantum revivals**. Namely, for small values of α the probability oscillates indicating transitions between the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ states. As α increases the probability lingers for longer and longer times in the meta-stable state

$$\mathrm{P}\left(\epsilon_t = \frac{\omega}{2}|\epsilon_0 = \frac{\omega}{2}\right) = \frac{1}{2}$$

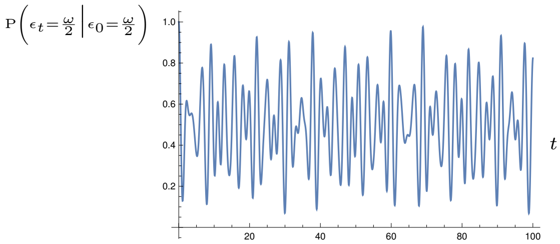
before **reviving** the oscillations. Such behavior is in agreement with **almost periodicity**. The return time-scale grows with α and may well become non-observable if we hold fixed the observation time whilst choosing α sufficiently large. This is indeed the case in illustrated the last plot in the right column if we suppose that the observation horizon is $t = 40$.



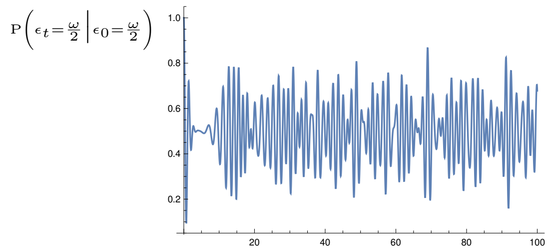
Plot of the probability (7) $\alpha = 0.01$



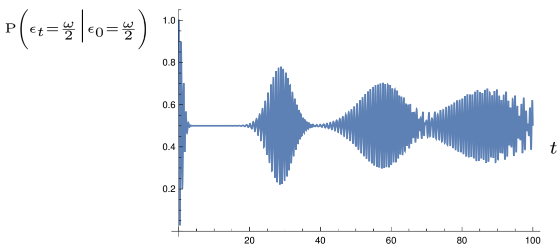
Plot of the probability (7) $\alpha = 0.1$



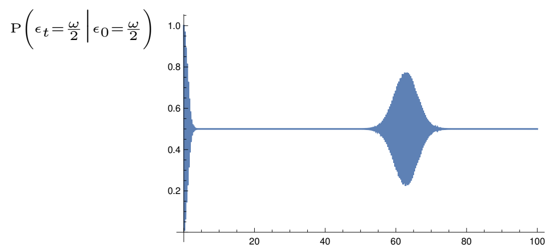
Plot of the probability (7) $\alpha = 1$



Plot of the probability (7) $\alpha = 4$



Plot of the probability (7) $\alpha = 9$



Plot of the probability (7) $\alpha = 99$

* * *

[1] O. Babelon and D. Talalaev. On the Bethe Ansatz for the Jaynes-Cummings-Gaudin model. *Journal of Statistical Mechanics: Theory and Experiment*, 2007:P06013, June 2007, hep-th/0703124.

[2] H.-P. Breuer and F. Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, reprint edition, 2002.

[3] M. Gaudin. *The Bethe Wavefunction*. Cambridge University Press, 2014.

[4] A. J. Leggett, S. Chakravarty, A. Dorsey, M. Fisher, A. Garg, and W. Zwerger. Dynamics of the dissipative two-state system. *Review of Modern Physics*, 59(1):1–85, January 1987.

[5] B. W. Shore and P. L. Knight. The Jaynes-Cummings Model. *Journal of Modern Optics*, 40(7):1195–1238, July 1993.

[6] S. Stenholm. Beyond the Jaynes-Cummings model. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 46:224013, Nov. 2013.

[7] J. Zinn-Justin. *Path Integrals in Quantum Mechanics*. Oxford Graduate Texts. Oxford University Press, 2010.