TCM315 Fall 2022: Introduction to Open Quantum Systems

Lecture 8: Holomorphic representation of the occupation number dynamics

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CONTENTS

1. INTRODUCTION

In these notes I recall some basic mathematical definitions and properties of the holomorphic Boson representation. The main sources are the § 6 of [3] and the book [1].

2. ELEMENTARY FACTS

Definition. *An* ***holomorphic function*** *is a complex-valued function f of one or more complex variables*

$$f \colon \mathbb{C}^d \mapsto \mathbb{C}$$

($d = 1, 2, \dots$) that is, at every point of its domain, complex differentiable in a neighborhood of the point

$$\mathrm{d}f(z) = \mathrm{d}z(\partial f)(z)$$

and

$$\partial_{\bar{z}} f(z) = 0$$

The existence of a complex derivative in a neighborhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal, locally, to its own Taylor series (analytic). We adhere here to the convention

$$z = x + \imath\, y \qquad\qquad \text{for}\, x\,,y\,\in\,\mathbb{R}$$

2.1. Gaussian measure

We fix the convention for the Gaussian measure in $\mathbb{C}$ based on the identity ([3] pag. 136)

$$\int_{\mathbb{R}^2}\frac{\mathrm{d}x\mathrm{d}y}{2\,\pi}\,e^{-a\,\frac{x^2+y^2}{2}} = \int_{\mathbb{C}}\frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\,\imath\,\pi}e^{-a\,z\bar{z}} = \frac{1}{a} \tag{1}$$

Proof.
To verify the identity we notice that in polar coordinate the Gaussian integral hence becomes

$$\int_{\mathbb{R}^2}\frac{\mathrm{d}x\mathrm{d}y}{2\,\pi}\,e^{-a\,\frac{x^2+y^2}{2}} = \int_0^{2\,\pi}\mathrm{d}\theta\int_0^{\infty}r\,e^{-a\,\frac{r^2}{2}} = \frac{1}{a}\int_0^{2\,\pi}\mathrm{d}\theta$$

Furthermore

$$\mathrm{d}z\mathrm{d}\bar{z} = \det\begin{bmatrix}\mathrm{d}x & \imath\,\mathrm{d}y\\ \mathrm{d}x & -\imath\,\mathrm{d}y\end{bmatrix} = 2\,\imath\,\mathrm{d}x\,\mathrm{d}y$$

Hence (1) follows via a rescaling e.g.

$$z \to \frac{z}{\sqrt{2}} \qquad \& \qquad \bar{z}\to\frac{\bar{z}}{\sqrt{2}}$$

□

Once we are equipped with these convention we define the Gaussian measure as

$$\mathrm{dg}_{z,\bar{z}} = \mathrm{d}z\mathrm{d}\bar{z}\frac{e^{-\bar{z}z}}{2\,\imath\,\pi}$$

2.2. Hilbert space

An Hilbert is a complete vector space with an inner product. The inner product adapted to the holomorphic representation is

$$\langle\, f\,,g\,\rangle = \int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}\bar{f}(z)g(z) = \int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}f(\bar{z})\,g(z)$$

We restrict the attention to holomorphic functions square integrable with respect to the Gaussian measure

$$\int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}f(\bar{z})\,f(z) = \int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}|f(z)|^2\,<\,\infty$$

2.3. Canonical commutation relations (CCR)

In order to implement **canonical commutation relation** we require that

- the **creation** operator $\mathrm{a}^{\dagger}$ must be the adjoint of the **annihilation** a with respect to the inner product

- the commutation relation

$$\left[\mathrm{a}\,,\mathrm{a}^{\dagger}\right]=1$$

must hold true.

These requirements imply that if we surmise the one-to-one correspondence

$$z \Leftrightarrow \mathrm{a}^{\dagger}$$

then

$$\left\langle\, f\,,\mathrm{a}^{\dagger}\,g\,\right\rangle=\int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}f(\bar{z})\,z\,g(z)=\langle\,\mathrm{a}\,f\,,g\,\rangle$$

must also hold true. Indeed the chain of identities holds since

$$\mathrm{dg}_{z,\bar{z}}\,z=-(\partial_{\bar{z}}\mathrm{dg}_{z,\bar{z}})$$

and therefore

$$\left\langle\, f\,,\mathrm{a}^{\dagger}\,g\,\right\rangle=\int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}f(\bar{z})\,z\,g(z)=\int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}(\partial f)(\bar{z})\,g(z)=\int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}\overline{(\partial f)}(z)\,g(z)=\langle\,\mathrm{a}\,f\,,g\,\rangle$$

We therefore thus arrive at the identification

$$\mathrm{a}\Leftrightarrow\partial_{z}$$

The holomorphic representation of the commutation relation is therefore

$$[\partial_{z}\,,z]\,f(z)=f(z)$$

Remark. *In the above discussion we tacitly assumed that*

$$\int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}|z\,f(z)|^{2}\;<\;\infty$$
$$\int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}|\partial_{z}\,f(z)|^{2}\;<\;\infty$$

These conditions impose a restrictions on the set abstracts states admitting an holomorphic representation and consequently on the Hilbert space $\mathcal{H}$. We refer to [2] for a thorough discussion of functional analysis aspects.

* *

3. OCCUPATION NUMBER BASIS

The holomorphic representation of deals with square integrable holomorphic functions of one or more complex variables. Monomials of the Taylor expansion form an orthonormal basis in $\mathbb{L}^{2}(\mathbb{C},\mathrm{dg})$

$$\int_{\mathbb{C}}\mathrm{dg}_{z,\bar{z}}\frac{\bar{z}^{l}\,z^{k}}{\sqrt{l!}\sqrt{k!}}=\delta_{l\,k} \tag{2}$$

Physically the degree of each monomial is in one to one correspondence with the occupation number of a state in abstract Hilbert space formulation. We refer to this basis as the canonical basis of the holomorphic representation

3.1. Completeness relation

Any orthonormal basis in $\mathcal{H}$ must satisfy the completeness relation

$$\mathbb{1}=\sum_{l=0}^{\infty}\boldsymbol{e}_{l}\boldsymbol{e}_{l}^{\dagger}$$

meaning that for any $\boldsymbol{v} \in \mathcal{H}$

$$\boldsymbol{v} = \sum_{l=0}^{\infty} \boldsymbol{e}_l \, \langle \, \boldsymbol{e}_l \, , \boldsymbol{v} \, \rangle$$

In the holomorphic representation the completeness relation takes the form

$$\sum_{l=0}^{\infty} \frac{z^l \, \bar{z}^l}{l!} = e^{z \, \bar{z}}$$

The completeness relation is defined with respect to the **inner product** which defines the Hilbert space. Specifically, the identity

$$f(z) = \int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{w,\bar{w}} e^{z \, \bar{w}} f(w)$$

holds true for any holomorphic function

$$f(z) = \sum_{l=0} \frac{f_l}{l!} z^l$$

4. RELATION BETWEEN ABSTRACT HILBERT SPACE AND HOLOMORPHIC REPRESENTATION

In the abstract formulation, a coherent state of the Heisenberg algebra is

$$\boldsymbol{c}_b = e^{b \, \mathrm{a}^{\dagger}} \varphi_0$$

with φ_0 the normalized vacuum state

$$\varphi_0^{\dagger} \varphi_0 = 1$$

Proposition. *Coherent states enjoy the completeness relation with respect to the Gaussian measure*

$$\int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{z,\bar{z}} \boldsymbol{c}_z \boldsymbol{c}_z^{\dagger} = 1_{\mathcal{H}}$$

Proof.
Let $\{\varphi_n\}_{n=0}^{\infty}$ the basis of the boson harmonic oscillator

$$\int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{z,\bar{z}} \boldsymbol{c}_z \boldsymbol{c}_z^{\dagger} = \sum_{m,n=0}^{\infty} \int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{z,\bar{z}} \frac{(z \ \mathrm{a}^{\dagger})^m}{m!} \varphi_0 \varphi_0^{\dagger} \frac{(\bar{z} \ \mathrm{a})^n}{n!} = \sum_{n=0}^{\infty} \frac{(\mathrm{a}^{\dagger})^n}{\sqrt{n!}} \varphi_0 \varphi_0^{\dagger} \frac{(\mathrm{a})^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} \varphi_n \varphi_n^{\dagger}$$

□

Similarly we observe that

$$\mathrm{a} \, e^{b \, \mathrm{a}^{\dagger}} \varphi_0 = b \, e^{b \, \mathrm{a}^{\dagger}} \varphi_0$$

4.1. Mapping $\mathcal{H} \mapsto \mathbb{L}^2(\mathbb{C}, \mathrm{g})$

Definition. *A normal ordered abstract operator is*

$$\mathbb{O} = \sum_{l,k} O_{l\,k} (\mathrm{a}^{\dagger})^l \, \mathrm{a}^k$$

Coherent states specify the mapping from the abstract Hilbert space to the holomorphic representation

$$\boldsymbol{c}_z^\dagger \mathbb{O} \boldsymbol{c}_z = \sum_{l,k} O_{l\,k} z^l \bar{z}^k \varphi_0^\dagger e^{\bar{z}\,\mathrm{a}} e^{z\,\mathrm{a}^\dagger} \varphi_0 = \sum_{l,k} O_{l\,k} \bar{z}^l\, z^k e^{\bar{z}z}$$

We readily see that the identity in $\mathcal{H}$ correspond to

$$O_{mn} = \begin{cases} 1 & m = n = 0 \\ 0 & m\,, n > 0 \end{cases}$$

We analyze how the action of an abstract operator onto a Hilbert space state translates into the holomorphic formalism

$$\tilde{\psi} = \mathbb{O}\psi$$

We identify matrix elements and vector components by inserting the completeness relation

$$\mathbb{O}\psi = \int_{\mathbb{C}} \mathrm{dg}_{z,\bar{z}} \boldsymbol{c}_z \boldsymbol{c}_z^\dagger \mathbb{O} \int_{\mathbb{C}} \mathrm{dg}_{w,\bar{w}} \boldsymbol{c}_w \boldsymbol{c}_w^\dagger \psi = \int_{\mathbb{C}} \mathrm{dg}_{z,\bar{z}} \boldsymbol{c}_z \int_{\mathbb{C}} \mathrm{dg}_{w,\bar{w}} K_{\mathbb{O}}(\bar{z}, w) \psi_{\boldsymbol{c}}(\bar{w})$$

where in the holomorphic representation

Definition. *the quantity*

$$K_{\mathbb{O}}(\bar{z}, w) = \boldsymbol{c}_z^\dagger \mathbb{O} \boldsymbol{c}_w$$

is the ***kernel*** *of the operator* $\mathbb{O}$.

whereas

$$\psi_{\boldsymbol{c}}(\bar{w}) = \boldsymbol{c}_w^\dagger \psi$$

is the representation of the abstract state ψ. The above definitions then imply

$$\tilde{\psi} = \int_{\mathbb{C}} \frac{\mathrm{d}z \mathrm{d}\bar{z}}{2\,\imath\,\pi} e^{-z\bar{z}}\, \boldsymbol{c}_z\, \tilde{\psi}_{\boldsymbol{c}}(\bar{z})$$
$$\tilde{\psi}_{\boldsymbol{c}}(\bar{z}) = \int_{\mathbb{C}} \frac{\mathrm{d}w \mathrm{d}\bar{w}}{2\,\pi} e^{-w\bar{w}} K_{\mathbb{O}}(\bar{z}, w) \psi_{\boldsymbol{c}}(\bar{w})$$

4.2. Mapping $\mathbb{L}^2(\mathbb{C},\mathrm{g}) \mapsto \mathcal{H}$

If the kernel admits the representation

$$K_{\mathbb{O}}(\bar{z}, z) = \sum_{l,k} O_{lk}\, \bar{z}^l\, z^k\, e^{\bar{z}z}$$

then the chain of identities

$$\int_{\mathbb{C}} \mathrm{dg}_{w,\bar{w}} K_{\mathbb{O}}(\bar{z}, w) \psi_{\boldsymbol{c}}(\bar{w}) = \int_{\mathbb{C}} \mathrm{dg}_{w,\bar{w}} \sum_{l,k} O_{lk}\, \bar{z}^l\, w^k\, e^{\bar{z}w} \psi_{\boldsymbol{c}}(\bar{w})$$
$$= \sum_{l,k} O_{lk}\, \bar{z}^l\, \partial_{\bar{z}}^k \int_{\mathbb{C}} \mathrm{dg}_{w,\bar{w}}\, e^{\bar{z}w} \psi_{\boldsymbol{c}}(\bar{w}) = \sum_{l,k} O_{lk}\, \bar{z}^l\, \partial_{\bar{z}}^k \psi_{\boldsymbol{c}}(\bar{z})$$

leads to the normal ordered operator

$$\mathbb{O} = \sum_{l,k} O_{lk}\, z^l\, \partial_z^k$$

which is the holomorphic representation of the abstract operator

$$\mathbb{O} = \sum_{lk} O_{lk} (\mathrm{a}^\dagger)^l\, \mathrm{a}^k = \int_{\mathbb{C}} \mathrm{dg}_{z,\bar{z}} \boldsymbol{c}_z \int_{\mathbb{C}} \mathrm{dg}_{w,\bar{w}} K_{\mathbb{O}}(\bar{z}, w) \boldsymbol{c}_w^\dagger$$

5. FORCED HARMONIC OSCILLATOR

We now turn to study

$$\imath\, \partial_t \, \mathrm{U}_{t,t_o} = \left(\omega \; \mathrm{a}^\dagger \, \mathrm{a} + g_t \, \mathrm{a}^\dagger + \bar{g}_t \, \mathrm{a}\right) \mathrm{U}_{t,t_o}$$
$$\mathrm{U}_{t_o,t_o} = 1_{\mathcal{H}}$$

The flow equations in the holomorphic representation become

$$\imath\, \partial_t \mathcal{U}_{t,t_o}(z|\bar{z}) = (\omega\, z\, \partial_z + g_t\, z + \bar{g}_t \partial_z)\, \mathcal{U}_{t,t_o}(z|\bar{z})$$
$$\mathcal{U}_{t_o,t_o}(z|\bar{z}) = e^{z\,\bar{z}}$$

5.1. Solution by Gaussian Ansatz

We look for a solution in the form

$$\mathcal{U}_{t,t_o}(z|\bar{z}) = e^{z\, x_t\, \bar{z} + z\, \bar{y}_t + \bar{z}\, y_t + w_t}$$

We get into

$$\begin{aligned} &\imath\, \dot{x}_t = \omega\, x_t && x_{t_0} = 1 \\ &\imath\, \dot{\bar{y}}_t = \omega\, \bar{y}_t + g_t && \bar{y}_{t_0} = 0 \\ &\imath\, \dot{y}_t = \bar{g}_t\, x_t && y_{t_0} = 0 \\ &\imath\, \dot{w}_t = \bar{g}_t\, \bar{y}_t && w_{t_0} = 0 \end{aligned}$$

The solution is

$$\begin{aligned} x_t &= e^{-\imath\,\omega\,(t-t_0)} \\ \bar{y}_t &= -\imath \int_{t_0}^t \mathrm{d}s\, e^{-\imath\,\omega\,(t-s)}\, g_s = e^{-\imath\,\omega\,t} \bar{h}_t \\ y_t &= -\imath \int_{t_0}^t \mathrm{d}s\, \bar{g}_s\, e^{-\imath\,\omega\,(s-t_o)} = -\imath\, e^{\imath\,\omega\,t_o} h_t \\ w_t &= -\int_{t_o}^t \mathrm{d}s\, \bar{g}_s\, e^{-\imath\,\omega\,s} \bar{h}_s = -\int_{t_o}^t \mathrm{d}s\, \dot{h}_s \bar{h}_s \end{aligned}$$

5.2. Transition probabilities

We define the **transition probability** between two states as the probability that a system starting at time zero from an eigenstate of the unperturbed system is found at time t in another eigenstate of the unperturbed Hamiltonian

$$\mathrm{P}(n \to m) = |\langle\, \varphi_m\,, \mathrm{U}_{t,t_0}\, \varphi_n\,\rangle|^2 = \frac{1}{m!\,n!} \left| \partial_\jmath^m \partial_{\bar{\jmath}}^n \big|_{\jmath=\bar{\jmath}=0} \mathcal{Z}_{t,t_o}(\jmath, \bar{\jmath}) \right|^2$$

where

$$\mathcal{Z}_{t,t_o}(\jmath, \bar{\jmath}) = e^{-\int_{t_o}^t \mathrm{d}s\, \dot{h}_s \bar{h}_s} \int_{\mathbb{C}} \mathrm{dg}_{z,\bar{z}} \int_{\mathbb{C}} \mathrm{dg}_{w,\bar{w}}\, e^{\jmath\,\bar{z}}\, e^{z\,x_t\,\bar{w} + z\,\bar{y}_t + \bar{w}\,y_t}\, e^{\bar{\jmath}\,w}$$

Upon recalling the holomorphic representation of the δ function, we conclude that the integrals leave the kernel invariant in form

$$\mathcal{Z}_{t,t_o}(\jmath, \bar{\jmath}) = e^{-\int_{t_o}^t \mathrm{d}s\, \dot{h}_s \bar{h}_s} e^{\jmath\, x_t\, \bar{\jmath} + \jmath\, \bar{y}_t + \bar{\jmath}\, y_t}$$

We readily get

$$\mathrm{P}(0 \to m) = \frac{|h_t|^{2\,m}}{m!} e^{-|h_t|^2}$$

whereas more involved but conceptually straightforward algebra yields

$$\mathrm{P}(n \to m) = \frac{|h_t|^{2\,n+2\,m}}{m!\,n!}\, e^{-|h_t|^2} \left| \sum_{l=0}^{n\wedge m} \frac{(-1)^l\, m!\, n!\, |h_t|^{-2\,l}}{l!\,(m-l)!\,(n-l)!} \right|^2 \tag{3}$$

5.3. Qualitative analysis of transition probabilities

The first observation is that (3) is symmetric in the quantum numbers m, n. If we fix the initial state, (3), however, attributes different probabilities to transitions toward lower or higher energy states owing to the dependence of the sum upon $m \wedge n$, the minimum between m and n.

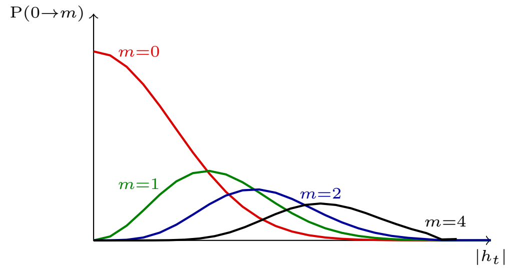
5.4. Transitions involving the ground states

The transition probability from the ground state to any state m is

$$\mathrm{P}(0 \to m) = \frac{|h_t|^{2\,m}}{m!} e^{-|h_t|^2} \tag{4}$$

and in view of the $m \leftrightarrow n$ symmetry of (3) we get into

$$\mathrm{P}(0 \to m) = \mathrm{P}(m \to 0)$$



Plot of the behavior of (4 versus $|h_t|$ for different values of m

5.4.1. Survival probability

The probability to find the system after a finite time in the same energy eigenstate in called the **survival probability**. From (3) we read

$$\mathrm{P}(n \to n) = |h_t|^{4\,n}\, e^{-|h_t|^2} \left| \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l |h_t|^{-2\,l}}{(n-l)!} \right|^2$$

5.4.2. Weak coupling analysis

We derived the dynamics of the forced harmonic oscillator as (rough) weak coupling approximation. By weak coupling we mean here the asymptotic expression of (3)

$$|h_t| \ll 1 \tag{5}$$

and retaining only the leading order term. We observe that for $k \geq 1$

- transitions towards lower energy states occur with probability

$$\mathrm{P}(n \to n-k) \approx \binom{n}{k} \frac{|h_t|^{2\,k}}{k!}$$

  If $n \gg k$ Stirling's approximation for the binomal coefficient

$$\binom{n}{k} \approx \frac{n^k}{k!}$$

  allows us to further simplify the result to

$$\mathrm{P}(n \to n-k) \approx n^k \frac{|h_t|^{2\,k}}{(k!)^2}$$

  In particular for $k = 1$ we get into

$$\mathrm{P}(n \to n-1) \approx n\,|h_t|^2$$

Remark. *It is important to emphasize that this last formula holds true if (5) is such that*

$$n\,|h_t|^2 \ll 1$$

in other words we cannot use it to estimate probabilities of states of arbitrary high energy if $|h_t|$ is very small but ***fixed****.*

* *

- Transitions toward higher energy states

$$\mathrm{P}(n \to n+k) \approx \binom{n+k}{k}\frac{|h_t|^{2\,k}}{k!} \overset{k \ll n}{\approx} (n+k)^k \frac{|h_t|^{2\,k}}{(k!)^2}$$

and in particular for $k = 1$

$$\mathrm{P}(n \to n+1) \approx (n+1)\,|h_t|^2$$

- At small coupling, we attain an informative approximation about the survival probability by expanding up to the first sub-leading term

$$\mathrm{P}(n \to n) \approx |h_t|^{4\,n}\,e^{-|h_t|^2}\left|(-1)^n\,|h_t|^{-2\,n} + n\,(-1)^{n-1}\,|h_t|^{-2\,(n-1)}\right|^2 \approx (1-|h_t|^2)\left(1-2\,n|h_t|^2\right)$$

whence we conclude

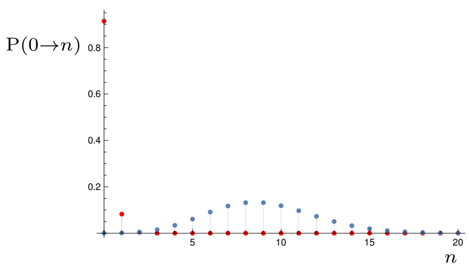
$$\mathrm{P}(n \to n) \approx 1 - (2\,n+1)|h_t|^2$$

Qualitatively the result indicates that at small coupling probability is approximately conserved by restricting the attention to nearest neighbor energy levels:

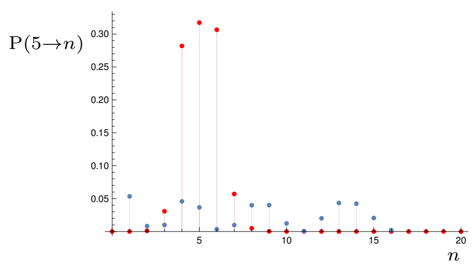
$$\mathrm{P}(n \to n) + \mathrm{P}(n \to n-1) + \mathrm{P}(n \to n+1) \approx 1$$

5.4.3. Qualitative behavior at finite coupling

As $|h_t|$ increases, the number of final states for which the transition probability is not negligible, increases



Plot of the transition probabilities from the ground state (4) for different values of the arrival state. Red dots correspond to $|h_t| = 0.3$ whereas blue ones refer to $|h_t| = 3$



Plot of the transition probabilities from the energy level $n = 5$ for different values of the arrival state. Red dots correspond to $|h_t| = 0.3$ whereas blue ones refer to $|h_t| = 3$

5.5. Comparison with the classical case

The classical counterpart of the model is the Hamilton function

$$H(z, \bar z) = \omega \bar z z + g_t\, z + \bar g_t\, \bar z$$

in complex coordinates $(z, \bar{z})$. The Hamilton equation of motions for the complex coordinates

$$
\begin{aligned}
\dot{\mathcal{z}}_t &= -\imath(\partial_{\bar{z}} H)(\mathcal{z}_t, \bar{\mathcal{z}}_t) \\
\dot{\bar{\mathcal{z}}}_t &= \imath(\partial_z H)(\mathcal{z}_t, \bar{\mathcal{z}}_t)
\end{aligned}
$$

are decoupled. It is therefore sufficient to solve

$$
\begin{aligned}
\dot{\mathcal{z}}_t &= -\imath\,\omega\,\mathcal{z}_t - \imath\,\bar{\mathcal{g}}_t \\
\mathcal{z}_t &= z_0
\end{aligned}
$$

We find

$$
\mathcal{z}_t = e^{-\imath\omega\,t} z_0 - \imath \int_0^t \mathrm{d}s\, e^{-\imath\,\omega\,(t-s)} \bar{\mathcal{g}}_s = e^{-\imath\omega\,t}\left(z_0 - \imath\,\bar{h}_t\right) \tag{6}
$$

As the dynamics is non-autonomous, the value of the Hamilton function is not preserved along a trajectory. Motivated by the decomposition (**??**) of the Hamilton operator into **unperturbed** and **interaction** terms, we identify the **energy of the harmonic oscillator at time t** as

$$
E_t = \omega\,|\mathcal{z}_t|^2
$$

Along the trajectory (6) we get

$$
E_t = E_0 - \imath\,\omega\left(z_0 \bar{h}_t - \bar{z}_0 h_t\right) + \omega\,|h_t|^2
$$

where

$$
E_0 = \omega\,|z_0|^2
$$

Next we write

$$
z_0 = \sqrt{\frac{E_0}{\omega}}\, e^{\imath \arg z_0} \qquad \& \qquad h_t = |h_t|\, e^{\imath \arg h_t}
$$

so that the time dependent energy becomes

$$
E_t = E_0 + 2\sqrt{\omega\, E_0}\,|\bar{h}_t| \sin\theta_t + \omega\,|h_t|^2 \tag{7}
$$

with

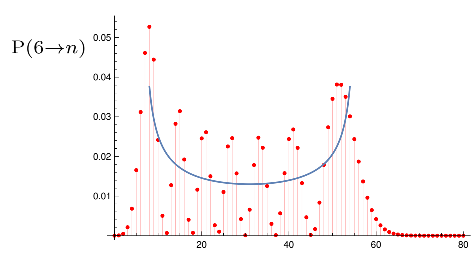
$$
\theta_t = \arg z_0 - \arg h_t
$$

We thus see that the energy (7) oscillates in the region

$$
(\sqrt{E_0} - \sqrt{\omega}\, h_t)^2 \,\leq\, E_t \,\leq\, (\sqrt{E_0} + \sqrt{\omega}\, h_t)^2
$$

Finally, if we consider an ensemble of oscillators with an equiprobable distribution in θ, the corresponding energy probability distribution is

$$
\begin{aligned}
\mathrm{P}(\epsilon \,\leq\, E_t \,<\, \epsilon + \mathrm{d}\epsilon) &= \frac{\mathrm{d}\epsilon}{\pi}\left|\frac{\mathrm{d}\theta_t}{\mathrm{d}\epsilon_t}\right|_{E_t=\epsilon} \\
&= \frac{\mathrm{d}\epsilon}{\pi\sqrt{4\,\omega\,|h_t|^2\,E_0 - (E_0 + \omega\,|h_t|^2 - \epsilon)^2}}
\end{aligned} \tag{8}
$$



Plot of the classical probability density function (8) with $|h_t| = 5$ and $E_0 = 6$ contrasted with the quantum transition probabilities $\mathrm{P}(6 \to n)$ as n varies on the spectrum.

APPENDIX

Appendix A: Proof of that occupation number basis element are orthonormal

To prove (2)

$$\int_{\mathbb{C}} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\,\imath\,\pi} e^{-z\bar{z}} \bar{z}^l\, z^k = (-)^l \int_{\mathbb{C}} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\,\imath\,\pi} \left(\partial_z^l e^{-z\bar{z}}\right)\, z^k$$

and

$$\int_{\mathbb{C}} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\,\imath\,\pi} e^{-z\bar{z}}\, \partial_z^l z^k = \begin{cases} 0 & k\, <\, l \\ l! & k\, =\, l \\ \dfrac{k!}{(k-l)!} \displaystyle\int_{\mathbb{C}} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\,\pi} e^{-z\bar{z}}\, z^{k-l} & k\, >\, l \end{cases}$$

We now notice that in view of the chain of identities

$$\int_{\mathbb{C}} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\,\pi} e^{-z\bar{z}}\, z^{k-l} = \int_{\mathbb{R}^2} \frac{\mathrm{d}x\mathrm{d}y}{\pi} e^{-(x^2+y^2)}\, (x+\imath\, y)^{k-l} = \int_0^\infty \int_0^{2\pi} \frac{r\mathrm{d}r\mathrm{d}\theta}{\pi} e^{-r^2}\, (r\, \cos\theta + \imath\, \sin\theta)^{k-l}$$

the remaining integral reduces to

$$\int_{\mathbb{C}} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\,\pi} e^{-z\bar{z}}\, z^{k-l} = \int_0^\infty \int_0^{2\pi} \frac{r\mathrm{d}r\mathrm{d}\theta}{\pi} e^{-r^2}\, r^{k-l} e^{\imath(k-l)\theta} = \int_0^\infty \mathrm{d}r\, e^{-r} r^{\frac{k-l}{2}} \int_0^{2\,\pi} \frac{\mathrm{d}\theta}{2\,\pi} e^{\imath(k-l)\theta}$$

Finally, the integral identity

$$\int_0^{2\,\pi} \frac{\mathrm{d}\theta}{2\,\pi} e^{\imath(k-l)\theta} = \delta_{l,k}$$

yields the claim.

1. Proof of the completeness relation

The identity is an immediate consequence of the ortho-normality relation (2)

$$\int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{w,\bar{w}} e^{z\,\bar{w}} f(w) = \sum_{l,k=0}^{\infty} \frac{z^l}{l!} f_k \int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{w,\bar{w}}\, \bar{w}^l \frac{w^k}{k!}$$

a. *Formal proof of the completeness relation*

The formal representation is useful in view of path-integral manipulations

$$\int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{w,\bar{w}} e^{z\,\bar{w}} f(w) = \int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{w,\bar{w}} e^{z\,\bar{w}} e^{w\,\partial_u}|_{u=0} f(u)$$

Using the Gaussian integral

$$\int_{\mathbb{C}} \frac{\mathrm{d}z\mathrm{d}\bar{z}}{2\,\imath\,\pi} e^{-\bar{w}w} e^{\bar{w}z+w\jmath} = e^{z\jmath}$$

we get into

$$\int_{\mathbb{C}} \mathrm{d}\mathrm{g}_{w,\bar{w}} e^{z\,\bar{w}} f(w) = e^{z\,\partial_u}|_{u=0} f(u) = f(u+z)|_{u=0} = f(z)$$

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