TCM315 Fall 2022: Introduction to Open Quantum Systems

Lecture 7: Boson and fermion oscillators

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INTRODUCTION

The bosonic harmonic oscillator is probably the simplest model of Quantum Mechanics in an infinite dimensional Hilbert space. The simplifying features of the model endow it with many properties shared by finite dimensional models. We refer to chapter 11 of [4] for a precise mathematical treatment.

DUAL PICTURE OF THE DYNAMICS

The postulates prescribe for the evolution of the expected value of a dynamical variable $\mathscr{A}$ associated to the self-adjoint operator $\mathbb{A}$ the law

$$\mathrm{E}\,\mathscr{A}_t = \mathrm{Tr}\,(\mathbb{A}\boldsymbol{\rho}_t)$$

where the subscript t betokens the time evolution

$$\imath\partial_t\boldsymbol{\rho}_t = [\mathbb{H}_t\,,\boldsymbol{\rho}_t] \tag{1a}$$

$$\boldsymbol{\rho}_{t_0} = \boldsymbol{\rho} \tag{1b}$$

The solution of the Liouville-von Neumann equation is determined by the unitary operator fundamental solution on

$$\begin{aligned} \imath\partial_t \mathbb{U}_{t,t_0} &= \mathbb{H}_t \mathbb{U}_{t,t_0} \\ \mathbb{U}_{t_0,t_0} &= \mathbb{1}_{\mathcal{H}} \end{aligned} \tag{2}$$

with $\mathbb{1}_{\mathcal{H}}$ the identity on the Hilbert space. We thus get into

$$\mathrm{E}\,\mathscr{A}_t = \mathrm{Tr}\left(\mathbb{A}\mathbb{U}_{t,t_0}\boldsymbol{\rho}\mathbb{U}^{\dagger}_{t,t_0}\right) \tag{3}$$

The cyclic property of the trace affords two dual pictures of the dynamics consistent with (3).

Schrödinger picture

The dynamical variable are paired with the same self-adjoint operator at any time. Only the state operator evolves according to the Liouville von-Neumann equation (1).

Heisenberg picture

The cyclic property yields

$$\mathrm{E}\,\mathscr{A}_t = \mathrm{Tr}\left(\mathbb{U}^{\dagger}_{t,t_0}\mathbb{A}\mathbb{U}_{t,t_0}\boldsymbol{\rho}\right) \tag{4}$$

The relation must hold for any initial state operator $\boldsymbol{\rho}$. Therefore we may equivalently think of state operators being fixed and specifying a linear functional acting on operators over $\mathcal{H}$ [1, 2]:

$$\mathrm{E}\,\mathscr{A}_t = \boldsymbol{\omega}_{\rho}\left(\tau_{t,t_0}(\mathbb{A})\right)$$

where the linear functional $\boldsymbol{\omega}_{\rho}$ is defined by (4) and

$$\tau_{t,t_0}(\mathbb{A}) = \mathbb{U}^{\dagger}_{t,t_0}\mathbb{A}\mathbb{U}_{t,t_0} \;\equiv\; \mathbb{A}_{(H)t}$$

the (H) standing for **Heisenberg picture**. A straightforward calculation yields

$$\imath\,\partial_t \tau_{t,t_0}(\mathbb{A}) = -\left[\mathbb{H}_t\,,\tau_{t,t_0}(\mathbb{A})\right] \tag{5a}$$

$$\tau_{t_0,t_0}(\mathbb{A}) = \mathbb{A} \tag{5b}$$

We refer to τ_{t,t_0} as the **development map** and to (5a) as the Heisenberg equation of motion. The evolution equations (5) hold for any operator on the Hilbert space, not necessarily self-adjoint. We notice that by construction

$$\left(\tau_{t,t_0}(\mathbb{A})\right)^{\dagger} = \tau_{t,t_0}(\mathbb{A}^{\dagger})$$

HARMONIC OSCILLATOR

The classical Hamiltonian in a $d = 1$ configuration space is

$$H(q,p) = \frac{p^2}{2\,m} + \frac{1}{2}m\omega^2 q^2$$

The Hamiltonian admits a particularly simple representation in the non-dimensional coordinates $(\tilde{p},\tilde{q})$ defined by

$$p = \sqrt{m\,\omega}\,\tilde{p} \qquad \& \qquad q = \frac{\tilde{q}}{\sqrt{m\,\omega}} \tag{6}$$

Namely, we thus obtain

$$H(q,p) = \tilde{H}(\tilde{q},\tilde{p}) = \omega\,\frac{\tilde{p}^2 + \tilde{q}^2}{2}$$

Classical complexification

We obtain an even simpler representation if we introduce complex coordinates

$$q = \frac{z + \bar{z}}{\sqrt{2}} \qquad \& \qquad p = \frac{z - \bar{z}}{\sqrt{2}\,\imath} \tag{7}$$

Correspondingly, the Hamiltonian transforms as

$$H(q,p) = \tilde{H}(\tilde{q},\tilde{p}) = \tilde{H}\left(\frac{z+\bar{z}}{\sqrt{2}}\,,\frac{z-\bar{z}}{\sqrt{2}\,\imath}\right) = H^{(C)}(z,\bar{z}) = \omega\,\bar{z}\,z$$

A straightforward calculation shows that under (7) Hamilton's equation become Hamilton equations of motion in complex coordinates are

$$\dot{\varkappa}_t = -\imath\,\partial_{\bar{\varkappa}_t} H^{(C)}(\varkappa_t,\bar{\varkappa}_t) \qquad \& \qquad \varkappa_0 = z \tag{8a}$$

$$\dot{\bar{\varkappa}}_t = \imath\,\partial_{\varkappa_t} H^{(C)}(\varkappa_t,\bar{\varkappa}_t) \qquad \& \qquad \bar{\varkappa}_0 = \bar{z} \tag{8b}$$

In particular for the harmonic oscillator we get into

$$\dot{\varkappa}_t = -\imath\,\omega\,\varkappa_t$$
$$\dot{\bar{\varkappa}}_t = \imath\,\omega\,\bar{\varkappa}_t$$

and can be readily integrated

$$\varkappa_t = e^{-\imath\,\omega\,t}\varkappa_0 \qquad \& \qquad \bar{\varkappa}_t = e^{\imath\,\omega\,t}\bar{\varkappa}_0$$

The all the above steps can be repeated in phase spaces of any dimension.

Quantization

The quantum counterpart in the representation where the position operator $\mathbb{Q}$ is diagonal corresponds to the identification

$$\mathbb{P}_q = -\imath\,\partial_q$$

The Hamilton operator in the position representation acts as

$$\mathbb{H}_q f = -\frac{1}{2\,m}\partial_q^2 f + \frac{1}{2} m\omega^2 q^2 f$$

of the space of square integrable functions

$$\operatorname{dom}\mathbb{H}_q = \left\{ f \colon \mathbb{R} \mapsto \mathbb{C} \;\middle|\; \|f\|^2_{\mathbb{L}^2} = \int_{\mathbb{R}} \mathrm{d}q\,|f(q)|^2 \,<\, \infty \right\}$$

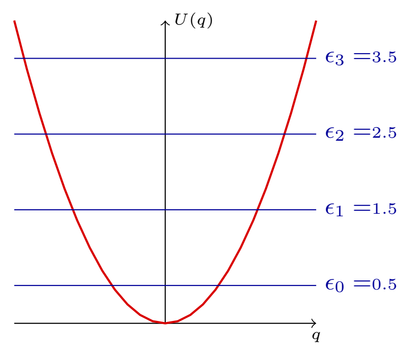
The main features of the quantum model are

1. Discrete spectrum.
2. Equispaced non-degenerate energy levels.
3. Ground state with finite energy ϵ.

To derive these property we introduce **ladder operators**

$$\mathrm{a} = \frac{1}{\sqrt{2\,m\,\omega}}\left(\mathbb{P} - \imath\,m\,\omega\,\mathbb{Q}\right) \tag{9a}$$

$$\mathrm{a}^{\dagger} = \frac{1}{\sqrt{2\,m\,\omega}}\left(\mathbb{P} + \imath\,m\,\omega\,\mathbb{Q}\right) \tag{9b}$$



Harmonic oscllator potential for $m = 2$ and $\omega = 1$.

The annihilation (9a) and creation (9b) operators satisfy the **canonical commutation relation** (CCR)

$$\left[\mathrm{a}\,,\mathrm{a}^{\dagger}\right] = \mathbb{1}_{\mathcal{H}} \tag{10}$$

where 1 is the identity on the Hilbert space $\mathcal{H} = \mathbb{L}^2(\mathbb{R})$. The Hamilton operator takes the form

$$\mathbb{H} = \frac{\omega}{2} \left(\mathrm{a}^\dagger \, \mathrm{a} + \mathrm{a} \, \mathrm{a}^\dagger \right) \tag{11}$$

We use the canonical commutation relation (10) to recast the Hamilton operator in **normal order** form

$$\mathbb{H} = \omega \, \mathrm{a}^\dagger \, \mathrm{a} + \frac{\omega}{2} \mathbb{1}_{\mathcal{H}} \tag{12}$$

Remark. *In (9),(11), (12) we omitted the subscript q as these expressions hold in any basis we choose to represent the harmonic oscillator.*

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Heisenberg algebra

The ladder operators satisfy the Heisenberg algebra with the Hamilton operator

$$[\mathbb{H}, \mathrm{a}] = -\omega \; \mathrm{a} \tag{13a}$$

$$\left[\mathbb{H}, \mathrm{a}^\dagger\right] = \omega \; \mathrm{a}^\dagger \tag{13b}$$

To prove (13) we observe that upon writing the commutator

$$[\mathbb{H}, \mathrm{a}] = -\omega \left[\mathrm{a} \, , \mathrm{a}^\dagger \, \mathrm{a}\right]$$

we notice that the commutator operation satisfies a "Leibniz rule"

$$\left[\mathrm{a} \, , \mathrm{a}^\dagger \, \mathrm{a}\right] = \left[\mathrm{a} \, , \mathrm{a}^\dagger\right] \mathrm{a} + \mathrm{a}^\dagger \left[\mathrm{a} \, , \mathrm{a}\right] = \mathrm{a}$$

We, thus readily, recover (13a). In order to derive (13b) we observe that

$$-\omega \; \mathrm{a}^\dagger = [\mathbb{H}, \mathrm{a}]^\dagger = \left[\mathrm{a}^\dagger \, , \mathbb{H}, \right] = -\left[\mathbb{H}, \mathrm{a}^\dagger\right]$$

Diagonalization in an abstract Hilbert space

First we postulate that ψ_ϵ is an eigenstate corresponding to the a putative eigenvalue ϵ i.e.

$$\mathbb{H}\psi_\epsilon = \epsilon \, \psi_\epsilon$$

The eigenvector is normalized with respect to the inner product in $\mathcal{H}$

$$\|\psi_\epsilon\|^2_{\mathcal{H}} = \langle \, \psi_\epsilon \, , \psi_\epsilon \, \rangle_{\mathcal{H}} = 1$$

We observe that

$$\mathbb{H} \, \mathrm{a} \, \psi_\epsilon = (\epsilon - \omega) \, \mathrm{a} \, \psi_\epsilon \tag{14}$$

namely

$$\mathbb{H} \, \mathrm{a} \, \psi_\epsilon = [\mathbb{H} \, , \mathrm{a}] \, \psi_\epsilon + \mathrm{a} \, \mathbb{H}\psi_\epsilon = -\omega \; \mathrm{a} \; \psi_\epsilon + \epsilon \; \mathrm{a} \; \psi_\epsilon$$

The identity (14) implies an alternative

1. **either**

   $$\mathrm{a} \, \psi_\epsilon \neq 0$$

   whence it follows that

   $$\psi_{\epsilon - \omega} = \mathrm{a} \, \psi_\epsilon$$

   is an eigenvector corresponding to the eigenvalue $\epsilon - \omega$

2. **or**

$$\mathrm{a}\,\psi_{\epsilon} = 0$$

In such a case ψ_{ϵ} specifies the **ground state** or **vacuum** state of the theory.

Based on (14) we identify the ladder operator a as the **annihilation operator** of the theory. Proceeding as in the derivation of (14) we can derive

$$\mathbb{H}\,\mathrm{a}^{\dagger}\,\psi_{\epsilon} = (\epsilon + \omega)\,\mathrm{a}\,\psi_{\epsilon}$$

This latter identity substantiate the interpretation of $\mathrm{a}^{\dagger}$ as **creation operator**.

Existence of the ground state

From (14) we are in the position to reconstruct the spectrum of the harmonic oscillator. Namely, once combined with the observation that the Hamilton operator is **positive definite**

$$\langle\,\psi\,,\mathbb{H}\psi\,\rangle_{\mathcal{H}} = \omega\,\|\mathrm{a}\,\psi\|_{\mathcal{H}}^{2} + \frac{1}{2}\,\|\psi\|_{\mathcal{H}}^{2} = \int_{\mathbb{R}} \mathrm{d}q\,\left(\frac{1}{2\,m}|\partial_{q}\psi(q)|^{2} + \frac{m}{2}\omega^{2}|\psi(q)|^{2}\right) \,\geq\, 0$$

the identity (14) requires elements of the spectrum of $\mathbb{H}$ to be of the form

$$\epsilon_{n} = \epsilon - n\,\omega$$

Hence there must be an n such that

$$\epsilon - n\,\omega\,\geq\,0 \qquad \& \qquad \epsilon - (n+1)\,\omega\,<\,0$$

and correspondingly

$$\mathrm{a}^{n+1}\,\psi_{\epsilon} = 0$$

We conclude that the ground state must satisfy

$$\varphi_{0} \propto \mathrm{a}^{n}\,\psi_{\epsilon}$$

specifies the **ground state** or vacuum of the theory. From the vacuum we reconstruct the full spectrum

$$\varphi_{n} \propto (\mathrm{a}^{\dagger})^{n}\varphi_{0} \qquad \Longrightarrow \qquad \epsilon_{n} = \omega\,\left(n + \frac{1}{2}\right)$$

Remark. *A general result by Leonard Gross [3] guarantees the existence and uniqueness of the ground state for positive definite Hamilton operators. The idea is to consider for $t > 0$ the operator*

$$\mathbb{T} = \exp(-t\,\mathbb{H})$$

which is bounded for $\mathbb{H}$ positive definite. Then the existence and uniqueness of the ground state follows by proving that under minor additional technical condition from the existence and uniqueness of the ground state for bounded operators such as $\mathbb{T}$. This avenue to the proof of the existence of a unique ground stats corresponds to a infinite dimensional extension of ***Perron theorem****. This latter theorem states that a real square matrix with strictly positive elements has a unique largest real eigenvalue and that the corresponding eigenvector can be chosen to have strictly positive components (i.e. eigenstate without nodes).*

An alternative way to proceed is to consider the functional

$$\mathcal{F}[\psi] = \langle\,\psi\,,\mathbb{H}\psi\,\rangle_{\mathcal{H}} + \lambda\,\left(\|\psi\|_{\mathcal{H}}^{2} - 1\right) \tag{15}$$

The second addend is a Lagrange multiplier imposing that the minimizer be unit normalized. Chapter 11 of [5] expounds the proof that a kinetic plus potential Hamilton operator with a potential bounded from ***above*** *and with absolute value integrable on every compact set has a unique ground state. This second approach is adapted to the solution of the important problem of the stability of matter [6]. Finally it is interesting to note that the variational problem (15) was the starting point adopted by Schrödinger to derive the formulation of Quantum Mechanics now going under his name (see [7],* "Zusatz bei der Korrektur am 28.11.1926"*)*

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Energy of the vacuum

By hypothesis we must have

$$0 = \|\mathrm{a}\,\varphi_0\|_{\mathcal{H}}^2 = \langle\, \mathrm{a}\,\varphi_0\,, \mathrm{a}\,\varphi_0\,\rangle_{\mathcal{H}} = \left\langle\, \varphi_0\,, \mathrm{a}^\dagger\,\mathrm{a}\,\varphi_0\,\right\rangle_{\mathcal{H}} = \frac{1}{\omega}\left\langle\, \varphi_0\,, \left(\mathbb{H} - \frac{\omega}{2}\mathbb{1}_{\mathcal{H}}\right)\varphi_0\,\right\rangle_{\mathcal{H}}$$

in consequence of

$$\mathrm{a}^\dagger\,\mathrm{a} = \frac{\mathrm{a}^\dagger\,\mathrm{a} + \mathrm{a}\,\mathrm{a}^\dagger}{2} + \frac{\mathrm{a}^\dagger\,\mathrm{a} - \mathrm{a}\,\mathrm{a}^\dagger}{2} = \frac{\mathbb{H}}{\omega} - \frac{\mathbb{1}_{\mathcal{H}}}{2}$$

By definition the norm of a vector vanishes only if its argument vanishes. We conclude that

$$\mathbb{H}\varphi_0 = \frac{\omega}{2}\varphi_0$$

Normalization of the states

We require

$$1 = \|\varphi_{n+1}\|_{\mathcal{H}}^2 = |C_{n+1}|^2\left\|\mathrm{a}^\dagger\,\varphi_n\right\|_{\mathcal{H}}^2 = \left\langle\, \varphi_n\,, \mathrm{a}\,\mathrm{a}^\dagger\,\varphi_n\,\right\rangle_{\mathcal{H}} = |C_{n+1}|^2\left\langle\, \varphi_n\,, \left(\frac{\mathbb{H}}{\omega} + \frac{\mathbb{1}_{\mathcal{H}}}{2}\right)\varphi_n\,\right\rangle_{\mathcal{H}} = |C_{n+1}|^2(n+1)$$

We conclude that

$$\varphi_{n+1} = \frac{\mathrm{a}^\dagger}{\sqrt{n+1}}\varphi_n \tag{16}$$

Furthermore, if we define φ_0 such that

$$\|\varphi_0\|_{\mathcal{H}}^2 = 1$$

we may regard (16) as a recursion equation with unique solution

$$\varphi_n = \frac{(\mathrm{a}^\dagger)^n}{\sqrt{n!}}\varphi_0$$

Ground state in the positon representation

In the position representation the equation

$$\mathrm{a}\,\varphi_0 = \frac{1}{\sqrt{2\,m\,\omega}}\,(\mathbb{P} - \imath\,m\,\mathbb{Q})\,\varphi_0$$

becomes

$$(\partial_q + m\omega q)\phi_0(q) = 0 \qquad \Rightarrow \qquad \phi_0(q) = \left(\frac{m\,\omega}{\pi}\right)^{1/4}\exp\left(-\frac{m\,\omega}{2}q^2\right)$$

the prefactor stemming from the normalization condition.

Summary

The upshot is that the bosonic harmonic oscillator is described by the relations

$$\mathbb{H} = \omega\left(\mathrm{a}^\dagger\,\mathrm{a} + \frac{\mathbb{1}_{\mathcal{H}}}{2}\right) \tag{17a}$$

$$\left[\mathrm{a}\,, \mathrm{a}^\dagger\right] = 1 \qquad \& \qquad \left[\mathbb{H}\,, \mathrm{a}^\dagger\right] = \omega\ \mathrm{a}^\dagger \qquad \& \qquad [\mathbb{H}\,, \mathrm{a}] = -\,\omega\ \mathrm{a} \tag{17b}$$

We emphasize the use of the normal ordering: carrying annihilation operators to the right of the creation operators evinces the action of an operator on the vacuum.

We also notice that the abstract Hilbert formulation allows us to re-define the Hamilton operator as having a zero vacuum energy

$$\tilde{\mathbb{H}} = \omega \, \mathrm{a}^{\dagger} \, \mathrm{a}$$

while leaving unaffected the Heisenberg algebra in (17). From now on we will adopt $\tilde{\mathbb{H}}$ as Hamiltonian of the harmonic oscillator and drop "tilde" to neaten the notation.

Schrödinger picture

The Liouville-von Neumann equation for the harmonic oscillator admits the solution

$$\boldsymbol{\rho}_t = e^{-\imath \, \omega \, t \, \mathrm{a}^{\dagger} \, \mathrm{a}} \boldsymbol{\rho}_0 \, e^{\imath \, \omega \, t \, \mathrm{a}^{\dagger} \, \mathrm{a}} \tag{18}$$

The infinite dimensional extension of the spectral theorem insures us that the eigenstates of a self-adjoint operator form a complete orthonormal basis of the Hilbert space. In other words dual product (with respect to the inner product in $\mathcal{H}$) must satisfy a completeness relation. The harmonic oscillator has pure point (i.e. discrete) spectrum. The completeness take the form

$$\mathbb{1}_{\mathcal{H}} = \sum_{i=0}^{\infty} \varphi_i \varphi_i^{\dagger}$$

Upon inserting the completeness relation into (18) we get into

$$\boldsymbol{\rho}_t = \sum_{\ell, \hbar=0}^{\infty} e^{-\imath \omega \, t \, \mathrm{a}^{\dagger} \, \mathrm{a}} \varphi_{\ell} \left(\varphi_{\ell}^{\dagger} \boldsymbol{\rho}_0 \varphi_{\hbar} \right) \varphi_{\hbar}^{\dagger} \, e^{\imath \omega \, t \, \mathrm{a}^{\dagger} \, \mathrm{a}} = \sum_{\ell, \hbar=0}^{\infty} e^{-\imath \, \omega \, t \, (\ell - \hbar)} \, R_{\ell \, \hbar} \varphi_{\ell} \varphi_{\hbar}^{\dagger}$$

For any fixed pair $(\ell \, \hbar)$, the **matrix-elements** $R_{\ell \, \hbar}$ are c-numbers specified by the inner product

$$R_{\ell \, \hbar} \; \equiv \; \varphi_{\ell}^{\dagger} \boldsymbol{\rho}_0 \varphi_{\hbar} = \int_{\mathbb{R}} \mathrm{d}q_1 \int_{\mathbb{R}} \mathrm{d}q_2 \bar{\psi}_{\ell}(q_1) \, r(q_1, q_2) \psi_{\hbar}(q)$$

In the foregoing expression we denoted by $\psi_{\hbar}(q)$ the position representation of the $\hbar$-th eigenstate and by $r(q_1, q_2)$ the kernel i.e. the position representation of the initial value of the state operator.

Thermal state

A special initial condition of particular importance describes an harmonic oscillator whose levels are populated according to the canonical distribution at temperature β^{-1}

$$\boldsymbol{\rho}_0 = \frac{e^{-\beta \, \mathbb{H}}}{Z}$$

The normalization factor is specified by the partition function

$$Z = \mathrm{Tr} \, e^{-\beta \, \mathbb{H}} = \sum_{\ell=0}^{\infty} \varphi_{\ell}^{\dagger} e^{-\beta \, \mathbb{H}} \varphi_{\ell} = \sum_{\ell=0}^{\infty} e^{-\beta \, \ell \, \omega} = \frac{1}{1 - e^{-\beta \, \omega}}$$

From the Liouville von-Neumann it follows immediately that

$$\boldsymbol{\rho}_t = \boldsymbol{\rho}_0 = \left(1 - e^{-\beta \, \omega} \right) e^{-\beta \, \mathbb{H}}$$

Heisenberg picture

The Heisenberg time development map for the annihilation operator is

$$\tau_t(\mathrm{a}) = e^{\imath\omega\, t\, \mathrm{a}^\dagger \mathrm{a}}\, \mathrm{a}\, e^{-\imath\omega\, t\, \mathrm{a}^\dagger \mathrm{a}}$$

We observe that

$$[\mathbb{H}\,,\tau_t(\mathrm{a})] = e^{\imath\omega\, t\, \mathrm{a}^\dagger \mathrm{a}}\,[\mathbb{H}\,,\mathrm{a}]\, e^{-\imath\omega\, t\, \mathrm{a}^\dagger \mathrm{a}} = \tau_t\left([\mathbb{H}\,,\mathrm{a}]\right) = -\omega\,\tau_t(\mathrm{a})$$

Hence, the Heisenberg equation of motion reduces to

$$\begin{aligned} &\imath\partial_t\tau_t(\mathrm{a}) = \omega\,\tau_t(\mathrm{a}) \\ &\tau_0(\mathrm{a}) = \mathrm{a} \end{aligned}$$

and admits the solution

$$\tau_t(\mathrm{a}) = \mathrm{a}\, e^{-\imath\,\omega\, t}$$

We may recover the same result using the completeness relation

$$\tau_t(\mathrm{a}) = \sum_{\ell,\hbar=0}^{\infty} e^{\imath\omega\, t\,(\ell-\hbar)}\varphi_\ell \left(\varphi_\ell^\dagger\, \mathrm{a}\,\varphi_\hbar\right)\varphi_\hbar^\dagger$$

The inner product

$$\varphi_\ell^\dagger\, \mathrm{a}\,\varphi_\hbar = \varphi_\ell^\dagger\, \mathrm{a}\,\varphi_\hbar = \sqrt{\hbar}\varphi_\ell^\dagger\varphi_{\hbar-1} = \begin{cases} \sqrt{\hbar}\,\delta_{\ell\,\hbar-1} & \hbar\ \geq 1 \\ 0 & \hbar = 0 \end{cases}$$

then yields

$$\tau_t(\mathrm{a}) = \sum_{\ell=0}^{\infty}\sum_{\hbar=1}^{\infty} e^{\imath\omega\, t\,(\ell-\hbar)}\varphi_\ell\varphi_\hbar^\dagger\,\sqrt{\hbar}\delta_{\ell\,\hbar-1} = \sum_{\hbar=1}^{\infty} e^{-\imath\omega\, t}\varphi_{\hbar-1}\varphi_\hbar^\dagger\,\sqrt{\hbar} = e^{-\imath\omega\, t}\,\mathrm{a}$$

An immediate consequence is

$$\left(\tau_t(\mathrm{a})\right)^\dagger = \tau_t(\mathrm{a}^\dagger) = e^{\imath\omega\, t}\,\mathrm{a}^\dagger$$

FERMION OSCILLATOR

The fermion oscillator describes a model of Quantum Mechanics where the **canonical commutation relations** is replaced by **canonical anti-commutation relations** (CAR)

$$\left[\mathrm{a}\,,\mathrm{a}^\dagger\right]_+ = \mathrm{a}\,\mathrm{a}^\dagger + \mathrm{a}^\dagger\,\mathrm{a} = \mathbb{1}_{\mathcal{H}} \tag{19a}$$

$$[\mathrm{a}\,,\mathrm{a}]_+ = \left[\mathrm{a}^\dagger\,,\mathrm{a}^\dagger\right]_+ = 0 \tag{19b}$$

We notice that (19b) imply the nilpotency of the ladder operators

$$\mathrm{a}^2 = 2\,(\mathrm{a}^\dagger)^2 = 0 \tag{20}$$

The **normal ordered** Hamilton operator is

$$\mathbb{H} = \omega\;\mathrm{a}^\dagger\,\mathrm{a} \tag{21}$$

We observe that

$$[\mathbb{H}\,,\mathrm{a}] = -\omega\;\mathrm{a} \tag{22a}$$

$$\left[\mathbb{H}\,,\mathrm{a}^\dagger\right] = \omega\;\mathrm{a}^\dagger \tag{22b}$$

continue to hold preserving the interpretation of a and $a^\dagger$ as, respectively, annihilation and creation operators. Namely, standard commutator manipulations combined with nilpotency of the annihilation operator yields

$$[\mathbb{H}\,,\mathrm{a}] = -\omega\left(\left[\mathrm{a}\,,\mathrm{a}^\dagger\right]\mathrm{a} + \mathrm{a}^\dagger\left[\mathrm{a}\,,\mathrm{a}\right]\right) = -\omega\left[\mathrm{a}\,,\mathrm{a}^\dagger\right]\mathrm{a}$$

Next we use (19a)

$$\left[\mathrm{a}\,,\mathrm{a}^\dagger\right] = \mathrm{a}\,\mathrm{a}^\dagger - \mathrm{a}^\dagger\,\mathrm{a} = \mathbb{1}_{\mathcal{H}} - 2\,\mathrm{a}^\dagger\,\mathrm{a}$$

whence

$$[\mathbb{H}\,,\mathrm{a}] = -\omega\left(\mathbb{1}_{\mathcal{H}} - 2\,\mathrm{a}^\dagger\,\mathrm{a}\right)\mathrm{a} = -\omega\,\mathrm{a}$$

follows using again nilpotency. Similarly the chain of identities

$$\left[\mathbb{H}\,,\mathrm{a}^\dagger\right]^\dagger = \left[\mathrm{a}^\dagger\,,\mathbb{H}\right] = -\left[\mathbb{H}\,,\mathrm{a}^\dagger\right]$$

yields immediately (22b).

Spectral properties

In view of (21), (22) the only possible eigenstates are

$$\varphi_\ell = (\mathrm{a}^\dagger)^\ell\varphi_0 \qquad \ell = 0, 1$$

and the spectrum consists only of two levels

$$\mathrm{Sp}\,\mathbb{H} = \{0, \omega\}$$

Matrix representation

The eigenstates with the elements of te canonical basis of $\mathbb{C}^2$

$$\varphi_1 \sim \boldsymbol{e}_1 \qquad \& \qquad \varphi_0 \sim \boldsymbol{e}_2$$

whereas

$$\mathbb{H} \sim \frac{\mathbb{1} + \sigma_3}{2}$$

In such a case, the representation of the ladder operators is

$$\mathrm{a} \sim \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{\sigma_1 - \imath\,\sigma_2}{2} \qquad \& \qquad \mathrm{a}^\dagger \sim \sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{\sigma_1 + \imath\,\sigma_2}{2}$$

Upon recalling Pauli matrix algebra

$$\sigma_i\sigma_j = \imath\,\epsilon_{ijk}\sigma_k$$

where ϵ_{ijk} is the totally antisymmetric symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for } i = 1\,, j = 2\,, k = 3 \text{ and cyclic permutation} \\ -1 & \text{for } i = 2\,, j = 1\,, k = 3 \text{ and cyclic permutation} \\ 0 & \text{otherwise} \end{cases}$$

It is then straightforward to verify

$$\left[\frac{\mathbb{1} + \sigma_3}{2}, \frac{\sigma_1 \pm \imath\,\sigma_2}{2}\right] = \pm\,\omega\,\frac{\sigma_1 \pm \imath\,\sigma_2}{2}$$

Thermal state

As for the harmonic oscillator we define the thermal state as

$$\boldsymbol{\rho} = \frac{e^{-\beta\,\mathbb{H}}}{Z}$$

The partition function is simply

$$Z = \operatorname{Tr} e^{-\beta\,\mathbb{H}} = 1 + e^{-\beta\,\omega}$$

The only non vanishing matrix elements are therefore

$$\varphi_0^\dagger \boldsymbol{\rho} \varphi_0 = \frac{1}{1 + e^{-\beta\,\omega}}$$

and

$$\varphi_1^\dagger \boldsymbol{\rho} \varphi_1 = \frac{e^{-\beta\,\omega}}{1 + e^{-\beta\,\omega}}$$

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