The turbulent (homogeneous and isotropic) velocity field is simulated by the gaussian, mean zero Kraichnan ensemble

\[ \langle v_i(t, \bar{x} + \bar{r})v_j(t', \bar{x}) \rangle = \delta(t - t')D_{ij}(\bar{r}, L_v, l_\nu) \]

where the tensor D can be defined e.g. as

\[ D_{ij}(\bar{r}, L_v, l_\nu) = D_0 \int d^d p \frac{\delta_{ij} - p_ip_j/p^2}{(p^2 + 1/L_v^2)^{d/2+\xi/2}} e^{i\bar{r} \cdot \bar{p}} e^{-l_\nu^2 p^2} \]

- \( l_\nu \) is the viscous dissipation scale and \( L_v \) is the integral scale
- The tensor structure ensures incompressibility and isotropy
- In the inertial range \( l_\nu \ll r \ll L_v \) the velocity structure function \( d_{ij} \) scales as \( r^\xi \) and for \( r \ll l_\nu \) it scales as \( r^2 \)
- The expression is finite in the limit \( l_\nu \to 0 \), but it is divergent as \( L_v \to \infty \):

\[ D_{ij}(\bar{r}, L_v, 0) = C_1 \delta_{ij}L_v^\xi + C_2 [(2 + \xi)\delta_{ij} - \xi \hat{r}_i \hat{r}_j] r^\xi + o(L_v) \]

\[ = D_{ij}(0, L_v, 0) - d_{ij}(\bar{r}) \]
### WHY KRAICHNAN MODEL?

- $\delta$-correlation a decent approximation at long time scales, $\lim_{\lambda \to \infty} \lambda^{1/2} v(\bar{x}, \lambda t)$
- Closed Hopf equations. E.g. For the **passive scalar** we have

$$\partial_t \theta - \kappa \Delta \theta + v \cdot \nabla \theta = 0$$

$$\Rightarrow \partial_t \langle \theta_1 \theta_2 \rangle = 2\kappa \Delta \langle \theta_1 \theta_2 \rangle - \langle v_1 \cdot \nabla_1 \theta_1 \theta_2 \rangle - \langle \theta_1 v_2 \cdot \nabla_2 \theta_2 \rangle$$

$$= 2\kappa \Delta \langle \theta_1 \theta_2 \rangle + d_{ij} \partial_i \partial_j \langle \theta_1 \theta_2 \rangle$$

$$= \mathcal{M}_2 \langle \theta_1 \theta_2 \rangle$$

( $\theta_1 = \theta(t, \bar{x}_1)$ etc. and $d_{ij} = D_{ij}(0) - D_{ij}(\bar{x}_1 - \bar{x}_2)$ is the structure function )

For Nth order correlation functions we have similarly

$$\partial_t \langle \prod_{i=1}^N \theta_i \rangle = \mathcal{M}_N \langle \prod_{i=1}^N \theta_i \rangle$$

with a linear multibody operator $\mathcal{M}_N$. 
WHY KRAICHNAN MODEL?

- Add a forcing with covariance \( \langle f(t, \bar{x}) f(t', \bar{x}') \rangle = \delta(t - t') C(|\bar{x} - \bar{x}'|/L) \)

  where \( L \) is the forcing scale and try to find a steady state:

  \[-\mathcal{M}_N \langle \prod_{i=1}^N \theta_i \rangle = \sum \langle \prod_{i<j} \prod_{k \neq i,j}^N \theta_k \rangle C(|\bar{x}_i - \bar{x}_j|/L) \]

  i.e. there are \( N \)-point functions on the left hand side and \( N-2 \) -point functions on the right hand side. E.g. For the two point function:

  \[\mathcal{M}_2 \langle \theta_1 \theta_2 \rangle = C(|\bar{x}_1 - \bar{x}_2|/L)\]

- For a large scale forcing \(|\bar{x}_1 - \bar{x}_2| \ll L\) and \( C(0) = \text{const.} \) we get for the two point function:

  \[\langle \theta_1 \theta_2 \rangle = A_1 L^{2-\xi} - A_2 C(0) r^{2-\xi} + o(L)\]

- The divergent term vanishes for the structure function while the second term is what one would expect from standard scaling arguments (with the Kolmogorov exponent \( \xi = 4/3 \))
However, for $N>2$ we have instead for the structure function

$$S_N(r) = \langle [\theta(t, \bar{r}) - \theta(t, 0)]^N \rangle$$

$$= a_N L^{\Delta_N} r^{\frac{N}{2}(2-\xi) - \Delta_N} + b_N r^{\frac{N}{2}(2-\xi) + o(L)}$$

with $\Delta_N = \frac{N(N-2)}{2(d+2)} \xi + O(\xi^2)$ for small $\xi$

- The first term with amplitude $a_N$ is a zero mode of the operator $\widetilde{\mathcal{M}}_N$ (which is like $\mathcal{M}_N$ but for structure functions). It is the dominant term in the inertial range.
- Therefore, even though we started with a non-intermittent velocity field, the passive scalar structure functions exhibit strong intermittency.
- The anomalous scaling exponents are universal, although the amplitudes are not.
Strongly stratified turbulence

- Boussinesq approximation of Navier-Stokes equations:

\[
\begin{align*}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{1}{\rho_0} \nabla p + \nu \Delta \mathbf{u} - \frac{\rho g}{\rho_0} \mathbf{e}_z \\
\nabla \cdot \mathbf{u} &= 0 \\
\partial_t \rho + \mathbf{u} \cdot \nabla \rho &= \kappa \Delta \rho + \frac{\rho_0}{g} N^2 \bar{\rho} \, w
\end{align*}
\]

- \( N = \sqrt{-\frac{g}{\rho_0} \frac{\partial \bar{\rho}}{\partial z}} \) is the Brunt-Väisälä frequency, \( g \) is gravity, \( \rho_0 \) is a constant reference density and \( \rho \) is a deviation from the linear density profile \( \bar{\rho} \)

- Gravity (and boundary conditions) obviously spoil isotropy

- Some other important numbers (besides \( Re \)) are the (horizontal) Froude number \( F_h = \frac{U}{L_H N} \) and the aspect ratio \( \alpha = L_V / L_H \)
Strongly stratified turbulence

- Brethouwer et. al. 2007: For $Re \gg 1$, $F_h \ll 1$ and $\mathcal{R} \approx Re F_h^2 \gg 1$ (strongly stratified turbulence), $F_h \sim \alpha$ and the vertical and horizontal energy spectra are approx. $k_V^{-3}$ and $k_H^{-5/3}$

- Pancake structures: vertically squashed and horizontally elongated

- There's no density, hence no Brunt-Väisälä frequency or Froude numbers in the Kraichnan ensemble!
Try to generalize the Kraichnan ensemble e.g. by defining the velocity correlations as

\[ D_{\mu\nu}(\mathbf{r}) = D_0 \int d^3 p \frac{P_{\mu\nu}(\mathbf{p}) e^{-lp}}{(p^2 + L^{-2})^{(3+\xi)/2}} e^{\mathbf{i} p \cdot \mathbf{r}} \]

for the horizontal and cross terms \( (\mu \wedge \nu \neq 3) \) and similarly for \( D_{33} \) except that the exponent \( \xi \) is replaced with \( \eta \) and \( 0 < \xi < \eta < 2 \) (+ dimensional adjustment)

- Note that the model is no longer incompressible, except in the limit \( \eta \to \xi \)
- Note the appearance of an UV regulator \( e^{-lp} \) with a viscous length scale \( l \). This will ensure finiteness of the compressibility, \( \propto D_{\mu\nu,\mu\nu}(0) \)
- Due to nonzero compressibility, the density is no longer constant! Therefore we also need to reconsider the conservation of mass equation, \( \partial_t \rho + \nabla \cdot (\mathbf{v} \rho) = 0 \)

**Strongly stratified turbulence**
We must also account for a fluctuating density in the passive scalar equation:

\[ \partial_t (\rho \theta) + \nabla \cdot (\nabla \rho \theta) - \kappa \nabla \cdot (\rho \nabla \theta) = 0 \]

Using conservation of mass and rearranging, we can write it as

\[ \partial_t \theta + (\hat{\mathbf{v}} - \kappa \nabla \chi) \cdot \nabla \theta - \kappa \Delta \theta = 0 \]

where \( \chi = \log(\rho) \)

Note that the passive scalar is now advected by an effective velocity field, \( \hat{\mathbf{v}} - \kappa \nabla \chi \) where \( \chi \) is to be (partially) solved from the (Itô stochastic) equation

\[ \partial_t \chi = \frac{1}{2} D_{\mu \nu}(0) \partial_{\mu} \partial_{\nu} \chi - \mathbf{v} \cdot \nabla \chi - \nabla \cdot \mathbf{v} \]

The fields \( \nabla \chi \) are not decorrelated in time like the velocity field! It is reasonable to expect that this will have an effect on inertial particle clustering.

One strategy could be to approximate the statistics of \( \chi \) as gaussian.
Warhaft (2000): "PASSIVE SCALARS IN TURBULENT FLOWS"

- $S_4 = \langle \delta \theta^4 \rangle \propto r^{\zeta_4}$
- $\zeta_4 = 0.9 \ldots 1.2$

- Determine the anomalous scaling exponents of the passive scalar structure functions
- Clustering of inertial particles
- Fluxes and other quantities, "pancake structures"

Figure 11  The scaling exponent $\zeta_n$ for the scalar structure function $\langle [\Delta \theta(r)]^n \rangle$ within the inertial subrange as a function of $n$. Squares are from the data of Antonia et al (1984) (heated jet), crosses are from the data of Ruiz-Chavarria et al (1996) (heated wake), triangles are from the data of Meneveau et al (1990) (heated wake), circles are from the data of Mydlarski & Warhaft (1998a) (grid turbulence), and plus signs are from the full, three dimensional Navier-Stokes numerical simulations of Chen & Kraichnan (1998). Vertical bars represent uncertainty for the Mydlarski & Warhaft data. The long-dashed line is the white-noise estimate from Kraichnan (1994). The short-dashed line is for the velocity field from Anselmet (1984). The solid line is the KOC prediction.