Random conformally invariant curves

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3 Working with Schramm-Loewner Evolutions

3.1 Coordinate changes of SLEs

SLEs with three marked boundary points

In an earlier exercise, the following version of Schramm's principle was considered. To each simply connected domain $\Lambda \subsetneq \mathbb{C}$ and three boundary points $a, b, c \in \partial\Lambda$ one associates a probability measure $\mathsf{P}_{(\Lambda;a,b,c)}$ on Loewner regular curves starting from a and ending on the arc $\widehat{bc}$, and such that conformally invariance holds in the sense that $f_* \mathsf{P}_{(\Lambda;a,b,c)} = \mathsf{P}_{(f(\Lambda);f(a),f(b),f(c))}$ for f a conformal map. The classification result is that such curve in $(\mathbb{S}; 0, +\infty, -\infty)$ must be described by a Loewner chain

$$h_0(z) = z, \qquad \frac{\mathrm{d}}{\mathrm{d}t} h_t(z) = \coth\Big(\frac{h_t(z) - V_t}{2}\Big), \qquad V_t = \sqrt{\kappa}\, B_t + \alpha\, t,$$

for some $\kappa \geq 0$ and $\alpha \in \mathbb{R}$. In other domains the curve can be defined by conformal transport, as the image of the curve in $(\mathbb{S}; 0, +\infty, -\infty)$ under a conformal map $f : \mathbb{S} \to \Lambda$ such that $f(0) = a$, $f(+\infty) = a$, $f(-\infty) = c$.

For easier comparison with the chordal SLE, let us take the curve in the upper half-plane so that it starts from the origin and one of the marked points is at infinity. To obtain the curve in $(\mathbb{H}; 0, \infty, c)$, where $c < 0$, we use the conformal map f from $\mathbb{S}$ to $\mathbb{H}$ such that $f(0) = 0$, $f(+\infty) = \infty$ and $f(-\infty) = c$.

A formula for that map is

$$f(z) = |c|(e^z - 1).$$

Let $\eta : [0, \infty) \to \overline{\mathbb{S}}$ be the curve in $(\mathbb{S}; 0, +\infty, -\infty)$ and consider the image $\gamma(t) = f(\eta(t))$. The component of $\mathbb{S} \setminus \eta[0, t]$ which contains both $\pm\infty$ is denoted by S_t and the Loewner chain $(h_t)_{t \geq 0}$ consists of conformal maps $h_t : S_t \to \mathbb{S}$ normalized so that $h_t(z) - z \to \pm t$ as $z \to \pm\infty$. The curve γ is Loewner regular, too, and $H_t = f(S_t)$ is the component of $\mathbb{H} \setminus \gamma[0, t]$ which contains infinity (and c). Again, the curve γ as we define it is not parametrized by half-plane capacity, but it takes a C^1 reparametrization to achieve this. Denote again by s_t half the half-plane capacity of the hull generated by $\gamma[0, t]$, so that if $g_t : H_t \to \mathbb{H}$ is the hydrodynamical conformal map, then $g_t(z) = z + 2 s_t z^{-1} + O(z^{-2})$. The maps (g_t) satisfy the Loewner flow equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{2\, \dot{s}_t}{g_t(z) - X_t},$$

where $\dot{s}_t = \frac{\mathrm{d}}{\mathrm{d}t} s_t$ is half the speed of capacity growth and X_t is the image of the position of local growth

$$X_t = g_t(\gamma(t)) = g_t(f(\eta(t))).$$

We have at our disposal one conformal map from H_t to $\mathbb{H}$, namely the composition $f \circ h_t \circ f^{-1}$, but it is not hydrodynamically normalized. The hydrodynamically normalized maps can be obtained by post-composing with the appropriately chosen conformal self map of $\mathbb{H}$,

$$g_t = \mu_t \circ f \circ h_t \circ f^{-1},$$

where $\mu_t : \mathbb{H} \to \mathbb{H}$ is a Möbius transformation. In this situation μ_t preserves infinity, so we can write $\mu_t(z) = a_t z + b_t$, and it turns out not to be necessary to write down a_t and b_t explicitly (although it is not difficult to do so, and the reader should perhaps nevertheless do that as an exercise). For brevity, let us also denote by

$$\varphi_t \;=\; \mu_t \circ f \;=\; g_t \circ f \circ h_t^{-1}$$

the conformal map $\mathbb{S} \to \mathbb{H}$ which is important for us at time t. This map in particular gives us the driving process of γ,

$$X_t \;=\; \varphi_t(V_t).$$

We will next calculate the time derivative of the map φ_t. For this purpose we need the time derivative of h_t^{-1}, and we leave it as an easy exercise for the reader to check that

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t^{-1}(z) \;=\; -(h_t^{-1})'(z)\; \coth\Big(\frac{z - V_t}{2}\Big).$$

Using this and the Loewner flow equation for g_t, we write the time derivative we are interested in as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(g_t(f(h_t^{-1}(z)))\Big) \\
&= \Big(\frac{\mathrm{d}}{\mathrm{d}t} g_t\Big)(f(h_t^{-1}(z))) + (g_t \circ f)'(h_t^{-1}(z)))\, \Big(\frac{\mathrm{d}}{\mathrm{d}t} h_t^{-1}\Big)(z) \\
&= \frac{2\,\dot{s}_t}{g_t(f(h_t^{-1}(z))) - X_t} - (g_t \circ f)'(h_t^{-1}(z)))\; (h_t^{-1})'(z)\; \coth\Big(\frac{z - V_t}{2}\Big) \\
&= \frac{2\,\dot{s}_t}{\varphi_t(z) - X_t} - \varphi_t'(z)\; \coth\Big(\frac{z - V_t}{2}\Big).
\end{aligned}$$

Taylor expansion of the two terms at $z = V_t$ like before gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(z) \;=\; \frac{1}{z - V_t}\Big(\frac{2\,\dot{s}_t}{\varphi_t'(V_t)} - 2\,\varphi_t'(V_t)\Big) - \frac{\dot{s}_t}{\varphi_t'(V_t)^2} - 2\,\varphi_t''(V_t) + O(z - V_t).$$

The maps φ_t as well as their derivatives are regular at the boundary point V_t, so we require the pole to cancel, and obtain the equation

$$\dot{s}_t \;=\; \varphi_t'(V_t)^2,$$

which again is the intuitive property resulting from the change of capacity under the approximate scaling that φ_t does in neighborhoods of V_t. So we have an expression for the (explicit) time derivative of φ_t at the point V_t

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\Big)(z) \;=\; -3\,\varphi_t''(V_t) + O(z - V_t).$$

We are in a position to compute the increment of $X_t = \varphi_t(V_t)$ by Itô's formula, recalling also $\mathrm{d}V_t = \sqrt{\kappa}\,\mathrm{d}B_t + \alpha\,\mathrm{d}t$. The result is

$$\begin{aligned}
\mathrm{d}X_t \;&=\; \mathrm{d}\Big(\varphi_t(V_t)\Big) \\
&= \Big(\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\Big)(V_t)\;\mathrm{d}t + (\sqrt{\kappa}\;\mathrm{d}B_t + \alpha\;\mathrm{d}t)\;\varphi_t'(V_t) + \frac{\kappa}{2}\,\mathrm{d}t\;\varphi_t''(V_t) \\
&= \sqrt{\kappa\,\dot{s}_t}\;\mathrm{d}B_t + \Big(\frac{\kappa - 6}{2}\,\frac{\varphi_t''(V_t)}{\varphi_t'(V_t)^2} + \alpha\,\frac{1}{\varphi_t'(V_t)}\Big)\,\dot{s}_t\;\mathrm{d}t.
\end{aligned}$$

We leave it for the reader to verify using the expressions we have found so far, that

$$\frac{\varphi_t''(V_t)}{\varphi_t'(V_t)^2} \;=\; \frac{1}{X_t - g_t(c)} \;=\; \frac{1}{\varphi_t'(V_t)}.$$

Then we may perform a time change to the the half-plane capacity time parameter s, under which we have the ordinary

$$\frac{\mathrm{d}}{\mathrm{d}s} g_{t_s}(z) \;=\; \frac{2}{g_{t_s}(z) - X_{t_s}}.$$

The driving process becomes

$$\mathrm{d}(X_{t_s}) \;=\; \sqrt{\kappa}\,\mathrm{d}\tilde{B}_s + \Big(\frac{\kappa-6}{2} + \alpha\Big)\,\frac{1}{X_{t_s} - g_{t_s}(c)}\,\mathrm{d}s.$$

The driving process of the image curve is the one that defines $\mathrm{SLE}_\kappa(\rho)$ in $(\mathbb{H};0,\infty,c)$, with $\rho = \frac{\kappa-6}{2} + \alpha$. We have thus shown by a Schramm's principle (the exercise with three marked boundary points) and a coordinate change that the most general Loewner regular conformally invariant random curve which satisfies the domain Markov property and depends on three boundary points of a simply connected domain, is $\mathrm{SLE}_\kappa(\rho)$, for $\kappa \geq 0$ and $\rho \in \mathbb{R}$.

As a notable example, the curve η would have been a chordal SLE_κ in $(\mathbb{S};0,+\infty)$ if $\alpha = \frac{6-\kappa}{2}$. Reflecting the sign of the drift α, the chordal SLE_κ in $(\mathbb{S};0,-\infty)$ would have $\alpha = \frac{\kappa-6}{2}$, and coming back to the half-plane picture, the chordal SLE_κ in $(\mathbb{H};0,c)$ corresponds to $\rho = \kappa - 6$ as we already found before.

3.2 SLE martingales constructed by domain Markov property

One of the most important ways of calculating things with SLEs consists in finding a martingale whose end value is the quantity of interest. The domain Markov property provides a natural way of constructing martingales that compute relevant quantities. We will illustrate this technique in two example cases. The first example is a computation of the probability that the chordal SLE_κ touches a given boundary arc. This case explains first of all why $\kappa = 4$ is the point of phase transition from simple curves to self-touching curves, and it also gives a certain crossing probability that is interesting in the statistical mechanics models. The second example concerns the dimension of the SLE trace. Here we in fact only state a property which gives an upper bound for the Hausdorff dimension, and furthermore we only give a heuristic derivation, which could be made rigorous by slightly altering the definitions and putting in a little bit of extra work. The rigorous derivation can be found in the literature, but the heuristic derivation is shorter and at least as enlightening as the proper one.

Boundary visits of chordal SLE

Let $\gamma^{(\Lambda;a,b)}$ denote the chordal SLE_κ trace in the domain Λ from $a \in \partial\Lambda$ to $b \in \partial\Lambda$. Our goal is to find an expression for

$$\mathsf{P}\Big[\gamma^{(\Lambda;a,b)} \cap A \neq \emptyset\Big],$$

where $A \subset \partial\Lambda \setminus \{a,b\}$ is an arc of the boundary of the domain. By conformal invariance, it is sufficient to find an answer in the reference domain $(\mathbb{H};0,\infty)$.

The phase transition at $\kappa = 4$

The first thing we show is that the chordal SLE_κ only touches the boundary when $\kappa > 4$. If the chordal SLE_κ trace γ in $(\mathbb{H};0,\infty)$ touches the boundary at a point $x \in (0,\infty)$, then all the points $x' \in (0,x)$ are disconnected from infinity by the curve, and they belong to the SLE hull after the time s such that $\gamma(s) = x$. Furthermore, by the scale invariance stated in Proposition **??**, either all $x' > 0$ have a positive probability to become part of the hull, or no $x' > 0$ ever becomes a part of the hull. The question of whether the trace can touch the boundary at any other point but the starting point $a = 0$ and the end point $b = \infty$, is equivalent to whether the boundary points can become a part of the hull.

Recall that the hull K_s is defined as the set of points $z \in \overline{\mathbb{H}}$ such that the Loewner equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_t \;=\; \frac{2}{Z_t - X_t}$$

with initial condition

$$Z_0 \;=\; z \in \overline{\mathbb{H}}$$

has no solution up to time s, i.e. that the denominator $D_t = Z_t - X_t$ becomes zero before the times s (or at least its values accumulate at 0). The denominator is governed by the Itô stochastic differential equation

$$\mathrm{d}D_t \;=\; \frac{2}{D_t}\,\mathrm{d}t - \sqrt{\kappa}\,\mathrm{d}B_t.$$

This is in fact just a time change of the familiar Bessel process: if $t(u) = u/\kappa$, then with respect to the time parameter u we have

$$\mathrm{d}D_{t(u)} \;=\; \frac{2/\kappa}{D_{t(u)}}\,\mathrm{d}u + \mathrm{d}\tilde{B}_u,$$

where $(\tilde{B}_u)_{u\geq 0}$ is a standard Brownian motion: $\tilde{B}_u = -\sqrt{\kappa}\,B_{u/\kappa}$. Recall that the Bessel process (β_t) of dimension d is defined by the Itô stochastic differential equation

$$\mathrm{d}\beta_t \;=\; \frac{(d-1)/2}{\beta_t}\,\mathrm{d}t + \mathrm{d}B_t,$$

and for integer d it corresponds to the absolute value of the d-dimensional Brownian motion. The Bessel process hits the origin in finite time (when started away from the origin) if and only if $d < 2$. Comparing the equations we equate $d = 1 + 4/\kappa$, and correspondingly the denominator process D_t hits zero if and only if $\kappa > 4$.

We have shown that the chordal SLE_κ trace touches the boundary if and only if $\kappa > 4$. The reader may judge to which extent the pictures **??** – **??** plausibly illustrate this phenomenon. Let us furthermore remark that SLE's touching the boundary is equivalent to self touching of the curve. Indeed, suppose that the chordal SLE_κ trace γ in $(\mathbb{H}; 0, \infty)$ has a double point $\gamma(t_1) = \gamma(t_2)$ for $0 \leq t_1 < t_2$. Pick $s \in (t_1, t_2)$, and consider conditioning on $\gamma[0, s]$. By the domain Markov property, the conditional law of $\gamma[s, \infty)$ is the law of chordal SLE_κ in $(H_t; \gamma(s); \infty)$. But if $\gamma(t_2) = \gamma(t_1)$, then the point $\gamma(t_2)$ on the curve $\gamma[s, \infty)$ is not in the interior of the domain H_t, which means that the chordal SLE_κ touches the boundary of its domain (by conformal invariance it doesn't matter in which domain this happens). Thus, admitting the existence of the chordal SLE trace, we have shown the first phase transition stated in Theorem **??**: for $\kappa \leq 4$ the trace is a simple curve which doesn't touch the boundary of the domain, and for $\kappa > 4$ the trace has double points and touches the boundary.

The martingale and the probability to touch a boundary interval

Let us then compute the probability for the chordal SLE_κ, $\kappa > 4$, to touch a boundary arc. The equivalent question in the half-plane is to compute

$$P(z^-, z^+) \;=\; \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap [z^-, z^+] \neq \emptyset\Big],$$

where $0 < z^- < z^+$, say (intervals on the negative real axis are handled similarly). The technique to do so relies on finding a martingale whose end value is the indicator of the event that the boundary interval is touched.

The conditional expected values of any random variable, conditioned on the initial segments $\gamma^{(\mathbb{H};0,\infty)}[0, t]$, $t \geq 0$, constitute a martingale by construction. In particular, the conditional probabilities

$$M_t \;=\; \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \cap [z^-, z^+] \neq \emptyset \;\Big|\; \gamma^{(\mathbb{H};0,\infty)}[0, t]\Big]$$

form a martingale (M_t).

By the domain Markov property, given the initial segment $\gamma^{(\mathbb{H};0,\infty)}[0, t]$, the remaining part $\gamma^{(\mathbb{H};0,\infty)}[t, \infty)$ of the curve has the law of the chordal SLE_κ trace in $(H_t; \gamma(t), \infty)$, so we can write

$$M_t \;=\; \mathsf{P}\Big[\gamma^{(H_t;\gamma(t),\infty)} \cap [z^-, z^+] \neq \emptyset\Big].$$

Furthermore, we may use the conformal map $z \mapsto g_t(z) - X_t$ from $(H_t; \gamma(t), \infty)$ back to the reference domain $(\mathbb{H}; 0, \infty)$, and by conformal invariance of SLE the image curve $g_t \circ \gamma|_{[t,\infty)} - X_t$ has the law of a chordal SLE_κ in $(\mathbb{H}; 0, \infty)$. The remaining part $\gamma[t, \infty)$ touches the interval $[x_1, x_2]$ if and only if the image curve touches the interval $[g_t(z^-) - X_t, g_t(z^+) - X_t]$, and thus the martingale reads simply

$$M_t \;=\; \mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \;\cap\; [g_t(z^-) - X_t, g_t(z^+) - X_t] \;\neq\; \emptyset\Big] \;=\; P(g_t(z^-) - X_t, g_t(z^+) - X_t).$$

Thus, the domain Markov property and conformal invariance allowed us to express the martingale as a function of two stochastic processes, essentially the Loewner flows of the points z^- and z^+ (translated by an amount determined by the driving process X_t). In fact Proposition **??**, the scale invariance of the chordal SLE_κ, allows us to simplify further, since the probability of touching an interval $[z^-, z^+]$ can only depend on the ratio z^-/z^+

$$P(z^-, z^+) \;=\; p\Big(\frac{z^-}{z^+}\Big).$$

For the moment, let us suppose that the function $p : (0, 1) \to [0, 1]$ in the expression

$$M_t \;=\; p\Big(\frac{g_t(z^-) - X_t}{g_t(z^+) - X_t}\Big)$$

is nice enough, say twice continuously differentiable, so that we can apply Itô's formula. These types of assumptions become justified in the end of the calculation in an almost automatical manner. Recall that the numerator and denominator in the ratio individually follow time changed Bessel processes, both driven by the same Brownian motion

$$\mathrm{d}Z_t^\pm \;=\; \frac{2}{Z_t^\pm} - \sqrt{\kappa}\,\mathrm{d}B_t.$$

Computing the Itô derivative of $M_t = p(Z_t^-/Z_t^+)$ is routine, and the result is

$$\mathrm{d}M_t \;=\mathrm{d}t\,\Big(\frac{Z_t^- - Z_t^+}{2\,(Z_t^+)^2\,Z_t^-}\,\Big((-4 + (2\kappa - 4)r_t)\,p'(r_t) - \kappa\,r_t\,(1 - r_t)\,p''(r_t)\Big)\Big) + \mathrm{d}B_t\Big(\cdots\Big),$$

where we have denoted the ratio by $r_t = Z_t^-/Z_t^+$. We only care about the $\mathrm{d}t$ term, because if (M_t) is to be a martingale, this term has to vanish. Now requiring the $\mathrm{d}t$ term to vanish for generic values of the ratio r_t amounts to the differential equation

$$(-4 + r(2\kappa - 4))\,p'(r) - \kappa(1 - r)r\,p''(r) \;=\; 0.$$

Integrate to get

$$p'(r) \;=\; \text{const.} \times r^{-\frac{4}{\kappa}}\,(1 - r)^{2\frac{4-\kappa}{\kappa}},$$

and thus

$$p(r) \;=\; c_1 + c_2\int_r^1 u^{-\frac{4}{\kappa}}\,(1 - u)^{2\frac{4-\kappa}{\kappa}}\,\mathrm{d}u.$$

The differential equation has a two dimensional solution space, but of course there are boundary conditions that the correct solution must satisfy. When the ratio $r = z^-/z^+$ tends to zero, scale invariance tells us that the probability that the SLE curve touches the interval must become one if the curve is ever to touch the boundary. Similarly, when r tends to one, the interval shrinks to a point and the probability to touch the interval should tend to zero (unless the curve visits each point, which is indeed what would happen for $\kappa \geq 8$). The constants are thus determined

$$c_1 \;=\; 0, \qquad c_2 \;=\; \frac{1}{\int_0^1 u^{-\frac{4}{\kappa}}\,(1 - u)^{2\frac{4-\kappa}{\kappa}}\,\mathrm{d}u},$$

although the latter only makes sense if the integral is convergent, which indeed requires $\kappa > 4$ (for convergence at $u = 0$) and $\kappa < 8$ (for convergence at $u = 1$), and we have found

$$p(r) \;=\; \frac{4\,\sqrt{\pi}}{2^{8/\kappa}\,\Gamma(\frac{8-\kappa}{2\kappa})\,\Gamma(\frac{\kappa-4}{\kappa})}\int_r^1 u^{-\frac{4}{\kappa}}\,(1 - u)^{2\frac{4-\kappa}{\kappa}}\,\mathrm{d}u.$$

The standard way to finish the argument, and in particular to get rid of the so far unjustified assumption that p is twice continuously differentiable, is to reverse the logic of the above reasoning. We define p by the formula we just found, and then notice that the process

$$t \mapsto p\Big(\frac{Z_t^-}{Z_t^+}\Big)$$

is a local martingale (by Itô's formula) — not surprisingly as this is what we believe M_t is! The process is furthermore bounded (the values of p are between zero and one), so it is a uniformly integrable martingale up to any stopping time until which both $Z_t^{\pm}$ are well defined. Take τ_{z^-} to be the stopping time at which z^- becomes a part of the hull — this is also the first time at which the SLE trace γ touches the interval $[z^-, \infty)$. There are two possibilities regarding z^+. If the point $\gamma(\tau_{z^-})$ is on the interval $[z^-, z^+]$, then the point z^+ does not become a part of the hull yet, and the ratio Z_t^-/Z_t^+ tends to zero (the denominator remains non-zero, while the numerator tends to zero). In this case the value of our uniformly integrable martingale tends to $p(0) = 1$. If, however, the point $\gamma(\tau_{z^-})$ is on the interval (z^+, ∞), then easy estimates of harmonic measure show that the ratio Z_t^-/Z_t^+ tends to one. In this case the value of our uniformly integrable martingale tends to $p(1) = 0$. We conclude that the endvalue of the uniformly integrable matringale is the indicator of the event we are interested in

$$\lim_{t \nearrow \tau_{z^-}} p\Big(\frac{Z_t^-}{Z_t^+}\Big) = \begin{cases} 1 & \text{if } \gamma \text{ touches } [z^-, z^+] \\ 0 & \text{if } \gamma \text{ doesn't touch } [z^-, z^+] \end{cases},$$

so with the optional stopping theorem we have derived the probability we were interested in

$$\mathsf{P}\Big[\gamma^{(\mathbb{H};0,\infty)} \text{ touches } [z^-, z^+]\Big] = \mathsf{E}\Big[p\Big(\frac{Z_{\tau_{z^-}}^-}{Z_{\tau_{z^+}}^+}\Big)\Big] = \mathsf{E}\Big[p\Big(\frac{Z_0^-}{Z_0^+}\Big)\Big] = p\Big(\frac{z^-}{z^+}\Big).$$

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