Random conformally invariant curves

Kalle Kytölä

Iceland Summer School in Random Geometry, August 8-12 2011

2 Basics of Schramm-Loewner Evolutions

Here we give the argument, due to Oded Schramm [5], which identifies SLEs as the only reasonable candidates of scaling limits of interfaces in critical statistical mechanics models. When the setup is such that all the domains we need are conformally equivalent, this Schramm's principle classifies all possible random curves which can be described by Loewner evolutions, which are conformally invariant, and which satisfy a natural Markovian type property. Having found the classification, we then take that as a definition of SLE.

2.1 Schramm's classification principles

Conformally invariant random curves

Let us now start considering conformally invariant random curves. We thus seek to associate to each domain Λ (with a number of marked points) a probability measure on curves in that domain, such that the push-forward of the probability measure in Λ by a conformal map f from Λ to another domain Λ' coincides with the probability measure associated to Λ'.

In such a consideration, we naturally restrict attention to some class of domains (with marked points) that are conformally equivalent. For concreteness we first discuss one of the simplest conformal types, simply connected domains $\Lambda \subset \mathbb{C}$ with two marked boundary points $a, b \in \partial\Lambda$, and look for (oriented but unparametrized) random curves from a to b in $\overline{\Lambda}$. By the Riemann mapping theorem, if $(\Lambda_1; a_1, b_1)$ and $(\Lambda_2; a_2, b_2)$ are two such domains, there exists a conformal map $f : \Lambda_1 \to \Lambda_2$ such that $f(a_1) = a_2$ and $f(b_1) = b_2$, so this collection of domains indeed forms a conformal equivalence class. Note further that such conformal map is not unique, but there is a one parameter family of conformal self maps of any given domain of this type. The setup of simply connected domains with a random curve connecting two marked boundary points is ofter referred to as the *chordal* case.

In this setup, we seek a collection $(\mathsf{P}_{(\Lambda;a,b)})$ of probability measures associated to domains Λ with marked boundary points $a, b \in \partial\Lambda$, such that for any conformal $f : \Lambda \to f(\Lambda)$ we have

$$f_* \, \mathsf{P}_{(\Lambda;a,b)} \;=\; \mathsf{P}_{(f(\Lambda);f(a),f(b))}.$$

In other words, if a random curve γ (in Λ from a to b) has the law $\mathsf{P}_{(\Lambda;a,b)}$, then its image $f \circ \gamma$ has the law $\mathsf{P}_{(f(\Lambda);f(a),f(b))}$.

Conformal invariance alone is not a very restrictive requirement. Indeed, if we were given *any* probability measure $\mathsf{P}_{\mathrm{ref}}$ on curves in one reference domain $(\Lambda_{\mathrm{ref}}; a_{\mathrm{ref}}, b_{\mathrm{ref}})$, subject only to the condition that $\mathsf{P}_{\mathrm{ref}}$ is invariant under the conformal self maps of the reference domain, then we could *define* the probability measures in $(\Lambda; a, b)$ as $f_* \, \mathsf{P}_{\mathrm{ref}}$, where f is any conformal map from the reference domain to $(\Lambda; a, b)$, and thus we would obtain a conformally invariant collection of probability measures. To find interesting conformally invariant random curves which can also be classified, we impose a further condition of *domain Markov property*, which is motivated by interfaces in statistical mechanics.

The domain Markov property (chordal case)

We still first consider the chordal setup: random curves from $a \in \partial\Lambda$ to $b \in \partial\Lambda$ in the closure of a simply connected domain Λ. We consider the curves as oriented but unparametrized: two parametrized curves $\gamma_1 : [T_1^-, T_1^+] \to \mathbb{C}$ and $\gamma_2 : [T_2^-, T_2^+] \to \mathbb{C}$ are identified if $\gamma_1 = \gamma_2 \circ \theta$ for some increasing bijection $\theta : [T_1^-, T_1^+] \to [T_2^-, T_2^+]$. An *initial segment* of $\gamma : [T^-, T^+] \to \mathbb{C}$ is a restriction of γ to a subinterval containing the beginning, i.e. $\gamma|_{[T^-,s]}$ with $T^- \leq s \leq T^+$. The *tip* of an initial segment $\gamma|_{[T^-,s]}$ is the point $\gamma(s)$.

The crucial assumption which adds significant content to our considerations is the following

- *Domain Markov property*: We assume that given any initial segment $\gamma|_{[0,s]}$ of the random curve $\gamma : [0,T] \to \overline{\Lambda}$ in $(\Lambda; a, b)$, the conditional law of the remaining part $\gamma|_{[s,T]}$ is the probability measure associated to the domain $(\tilde{\Lambda}; \tilde{a}, b)$, where $\tilde{\Lambda}$ is component containing b of the complement $\Lambda \setminus \gamma[0,s]$ of the initial segment and $\tilde{a} = \gamma(s)$ is the tip of the initial segment. Put in another way,

$$\mathsf{P}_{(\Lambda;a,b)}\Big[\ \cdot\ \Big|\gamma|_{[0,s]} = \eta\Big] \ = \ \mathsf{P}_{(\Lambda\setminus\eta[0,s];\eta(s),b)}[\,\cdot\,] \boxplus \eta,$$

  where $\boxplus$ denotes concatenation of curves and $\Lambda \setminus \eta[0,s]$ is understood to stand for the relevant connected component only.

The domain Markov property thus related the conditional law of the remaining part after an initial segment to the law in the remaining domain. This property is motivated by interfaces in statistical mechanics models, as the reader will easily understand by considering for example the Ising model on a hexagonal lattice, and an interface which is a boundary between plus and minus spin clusters. In statistical mechanics, this property does not even need the model to be at a critical point. It is remarkable that when we combine the domain Markov property with the conformal invariance anticipated to emerge at the critical point, we obtain a simple classification of the possible random curves. This observation made in [5] is known as the *Schramm's principle* and will be discussed next.

The Schramm's principle (chordal case)

For the techniques of Loewner chains to be applicable, we still have to impose the following regularity assumption on the random curves

- *Loewner regularity*: We assume that the curve $\gamma : [0,T] \to \overline{\Lambda}$ starts from a, i.e. $\gamma(0) = a$, and that the tip of any initial segment $\gamma|_{[0,s]}$ is in the component containing b of the complement $\Lambda \setminus \gamma[0,s]$ of the initial segment, and that the local growth condition is satisfied.

Recall that by conformal invariance, $f_* \mathsf{P}_{(\Lambda;a,b)} = \mathsf{P}_{(f(\Lambda);f(a),f(b))}$, it is enough to describe the random curve in one reference domain, and to push-forward the definition to other domains by conformal maps. For the chordal case it is convenient to choose the reference domain $(\mathbb{H}; 0, \infty)$, and use the chordal Loewner chain in the half-plane to describe the curve.

Assume that our collection of probability measures $(\mathsf{P}_{(\Lambda;a,b)})$ satisfies *conformal invariance*, *domain Markov property* and *Loewner regularity*. Then choose a random curve γ in the half-plane with law $\mathsf{P}_{(\mathbb{H};0,\infty)}$.

As earlier, parametrize γ by the half-plane capacity of the initial segments and encode the growth of these initial segments in the Loewner chain. More precisely, let H_t be the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ — the Loewer chain (g_t) then consists of the hydrodynamically normalized conformal maps

$$g_t : H_t \to \mathbb{H}, \qquad g_t(z) \ = \ z + 2t\,z^{-1} + O(z^{-2}).$$

By Loewner regularity, the conformal maps g_t satisfy the Loewner equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \ = \ \frac{2}{g_t(z) - X_t}$$

for some continuous driving function $t \mapsto X_t \in \mathbb{R}$. The random curve γ is encoded by the driving function $t \mapsto X_t$, and we henceforth use the term *driving process* to emphasize the randomness of (X_t).

Consider an initial segment $\gamma|_{[0,s]}$ of the random curve. The Loewner chain to describe the initial segment corresponds to the restriction of the driving process $(X_t)_{t\in[0,s]}$. By the domain Markov property, the conditional law of the remaining part $\gamma|_{[s,T^+]}$ given the initial segment is $\mathsf{P}_{H_s;\gamma(s),\infty}$. Now note that the map

$$z \;\mapsto\; g_s(z) - X_s$$

is conformal from H_s to $\mathbb{H}$, and such that $\gamma(s) \mapsto 0$ (definition of driving function) and $\infty \mapsto \infty$ (hydrodynamical normalization). Therefore, conditionally on the initial segment, conformal invariance states that the law of the image of $\gamma|_{[s,T^+]}$ by the map $g_s - X_s$ has the law $\mathsf{P}_{(\mathbb{H};0,\infty)}$. The image curve $\tilde{\gamma}$ is defined by

$$\tilde{\gamma}(t) \;=\; g_s(\gamma(s+t)) - X_s, \qquad t \geq 0.$$

Let $(\tilde{g}_t)$ denote the collection of hydrodynamically normalized conformal maps $\tilde{g}_t : \tilde{H}_t \to \mathbb{H}$, where $\tilde{H}_t$ is the unbounded component of $\mathbb{H} \setminus \tilde{\gamma}[0,t]$. In fact,

$$z \;\mapsto\; g_{s+t}\Big(g_s^{-1}(z + X_s)\Big) - X_s$$

is a conformal map $\tilde{H}_t \to \mathbb{H}$, and it is a matter of simple calculation to verify the normalization

$$\begin{aligned} g_{s+t}\Big(\underbrace{g_s^{-1}(z+X_s)}_{\approx z+X_s-\frac{2s}{z+X_s}+\cdots}\Big) - X_s &= \Big(z + X_s - \frac{2s}{z+X_s} + \cdots\Big) + \frac{2(s+t)}{z + X_s - \frac{2s}{z+X_s} + \cdots} + \cdots - X_s \\ &= \Big(z + X_s - \frac{2s}{z}\Big) + \frac{2(s+t)}{z} - X_s + O(z^{-2}) \\ &= z + \frac{2t}{z} + O(z^{-2}). \end{aligned}$$

We see that not only is the above map hydrodynamically normalized, but also the curve $\tilde{\gamma}$ is still parametrized by capacity. In conclusion the Loewner chain for $\tilde{\gamma}$ is given by

$$\tilde{g}_t(z) \;=\; g_{s+t}\Big(g_s^{-1}(z + X_s)\Big) - X_s.$$

This Loewner chain must satisfy a Loewner's equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{g}_t(z) \;=\; \frac{2}{\tilde{g}_t(z) - \tilde{X}_t},$$

and indeed from the expression above we calculate

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{g}_t(z) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(g_{s+t}\Big(g_s^{-1}(z+X_s)\Big) - X_s\Big) \;=\; \frac{2}{g_{s+t}\Big(g_s^{-1}(z+X_s)\Big) - X_{s+t}} \;=\; \frac{2}{\tilde{g}_t(z) + X_s - X_{s+t}}.$$

We get that the driving process of $(\tilde{g}_t)$ is given by the increment of the driving process of (g_t)

$$\tilde{X}_t \;=\; X_{s+t} - X_s.$$

Moreover, since $\tilde{\gamma}$ has the same law $\mathsf{P}_{(\mathbb{H};0,\infty)}$ as γ, the driving process $(\tilde{X}_t)$ must have the same law as (X_t). Also recall that the considerations so far were done conditionally on the initial segment $\gamma|_{[0,s]}$ or equivalently conditionally on its driving function $(X_t)_{t\in[0,s]}$, so by we see that $(\tilde{X}_t)$ is independent of $(X_t)_{t\in[0,s]}$. The continuous process (X_t) therefore has independent and identically distributed increments, so its law is necessarily that of a multiple of Brownian motion plus linear drift

$$\Big(X_t\Big)_{t\geq 0} \overset{\text{in law}}{\equiv} \Big(\sqrt{\kappa}\, B_t + \alpha\, t\Big)_{t\geq 0}, \qquad \kappa \geq 0,\ \alpha \in \mathbb{R}.$$

However, we now check that only $\alpha = 0$ is consistent with the requirement that $\mathsf{P}_{(\mathbb{H};0,\infty)}$ is invariant under the one parameter family of conformal self maps of $(\mathbb{H}; 0, \infty)$. These self maps are the scalings of the half-plane, $z \mapsto \lambda z$ for $\lambda > 0$. To map complements of initial segments $\lambda\gamma[0, t]$ of the scaled curve $\lambda\gamma$ hydrodynamically to the half-plane, one uses the map

$$z \;\mapsto\; \lambda\, g_t(z/\lambda).$$

The Laurent expansion

$$\lambda\, g_t(z/\lambda) \;=\; \lambda\left(z/\lambda + \frac{2t}{z/\lambda} + \cdots\right) \;=\; z + \frac{2\lambda^2 t}{z} + \cdots$$

reveals that the correct capacity parametrization of $\lambda\gamma$ is

$$\gamma^{(\lambda)}(t) \;=\; \lambda\,\gamma(\lambda^{-2} t).$$

The capacity parametrized Loewner chain for $\lambda\gamma$ is $(g_t^{(\lambda)})$ with

$$g_t^{(\lambda)}(z) \;=\; \lambda\, g_{\lambda^{-2}t}(z/\lambda).$$

The Loewner equation satisfied by this Loewner chain is obtained by calculating the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t^{(\lambda)}(z) \;= \lambda\,\lambda^{-2}\, \frac{2}{g_{\lambda^{-2}t}(z/\lambda) - X_{\lambda^{-2}t}} \;=\; \frac{2}{\lambda\, g_{\lambda^{-2}t}(z/\lambda) - \lambda\, X_{\lambda^{-2}t}} \;=\; \frac{2}{g_t^{(\lambda)}(z) - \lambda\, X_{\lambda^{-2}t}},$$

from which we see that the driving process $(X_t^{(\lambda)})$ of $(g_t^{(\lambda)})$ is

$$X_t^{(\lambda)} \;=\; \lambda\, X_{\lambda^{-2}t}.$$

If $X_t \equiv \sqrt{\kappa}\, B_t + \alpha t$, then

$$X_t^{(\lambda)} \;\equiv\; \lambda\Big(\sqrt{\kappa}\, B_{\lambda^{-2}t} + \alpha\lambda^{-2} t\Big) \;\equiv\; \lambda\,\sqrt{\kappa}\,\sqrt{\lambda^{-2}}\, B_t + \lambda\alpha\lambda^{-2} t\Big) \;\equiv\; \sqrt{\kappa}\, B_t + \alpha\lambda^{-1} t\Big).$$

The scaled curve $\lambda\gamma$ would have a different law if $\alpha \neq 0$, so the conformal invariance under self maps of $(\mathbb{H}; 0, \infty)$ requires $\alpha = 0$ and finally

$$X_t \;\equiv\; \sqrt{\kappa}\, B_t.$$

We have obtained a strong classification: if a random conformally invariant chordal curve satisfies domain Markov property (and Loewner regularity), then the curve is the push forward by conformal maps from the half-plane $(\mathbb{H}; 0, \infty)$ of a curve whose Loewner driving process is a multiple of Brownian motion. The requirements we imposed motivated by interfaces in critical models of statistical mechanics characterized the law of a curve up to one parameter κ.

The chordal SLE$_\kappa$

The conclusion obtained by Schramm's principle is that there can be no other conformally invariant chordal random curves with domain Markov property except the ones whose half-plane Loewner chain has driving process $(\sqrt{\kappa}\, B_t)$ for some $\kappa \geq 0$. We thus call the Loewner chain determined by

$$g_0(z) \;= z, \qquad \frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \;= \frac{2}{g_t(z) - X_t}, \qquad X_t \;=\; \sqrt{\kappa}\, B_t$$

the *chordal Schramm-Loewner evolution with parameter κ in* $(\mathbb{H}; 0, \infty)$, or briefly *chordal SLE$_\kappa$ in* $(\mathbb{H}; 0, \infty)$.

A priori the chordal SLE$_\kappa$ is a collection of conformal maps $(g_t)_{t\geq 0}$, where $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ is a hydrodynamically normalized map from complements of random hulls $K_t \subset \overline{\mathbb{H}}$, and the hulls are growing: $K_t \subset K_s$ for $t < s$. It is however natural to ask whether there is a curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that $K_t = \gamma[0, t]$, or if at least K_t is generated by a curve in the sense that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. The following result answers the latter question in the affirmative.

Theorem 1 (Rohde and Schramm, [4]) *For the chordal SLE$_\kappa$ in* $(\mathbb{H}; 0, \infty)$*, the limits*

$$\gamma(t) \;=\; \lim_{\varepsilon \searrow 0} g_t^{-1}(X_t + \mathrm{i}\varepsilon)$$

exist and depend continuously on $t \geq 0$*. We call the curve* γ *the* chordal SLE$_\kappa$ trace in $(\mathbb{H}; 0, \infty)$*. The hulls* (K_t) *are generated by the trace.*

The proof is somewhat lengthy although not particularly difficult, and in fact the case $\kappa = 8$ needs to be considered separately — it was completed in [3]. The interested reader will find a careful proof in the generic case $\kappa \neq 8$ in [2].

Admitting the above result on the chordal SLE trace, we from here on consider the chordal SLE$_\kappa$ a random curve rather than a Loewner chain. This is certainly closer to the original motivation, and it is worth emphasizing that the curve is the fundamental object, whereas the Loewner chain is merely an artefact resulting from our description of the curve. Figures 1 — 6 portray simulated chordal SLE$_\kappa$ traces for a few different values of κ.

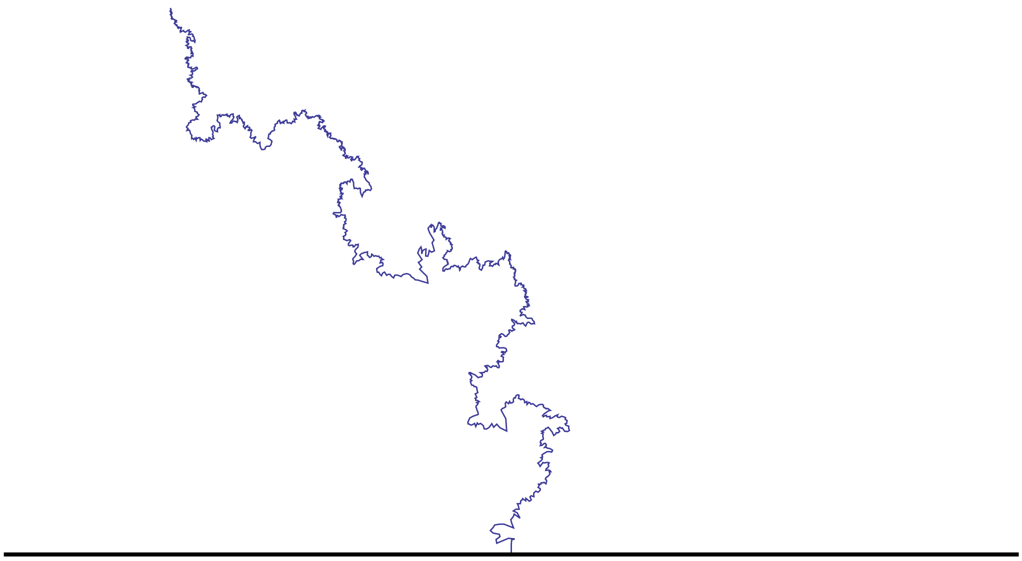

Figure 1: Initial segment of a chordal SLE$_2$ trace in $(\mathbb{H}; 0, \infty)$.

We immediately remark the invariance under conformal self-maps of a domain $(\Lambda; a, b)$. In the following form it directly follows from the scaling calculation we did in the course of establishing the Schramm's principle in the chordal case.

Proposition 1 *The law of chordal SLE$_\kappa$ in* $(\mathbb{H}; 0, \infty)$ *is invariant under the scalings* $z \mapsto \lambda z$*,* $\lambda > 0$*.*

It is also natural to ask for other properties of SLEs. The following result on the qualitative properties divides the parameter regions of κ to three phases.

Theorem 2 (Rohde and Schramm, [4]) *The (trace of the) chordal SLE$_\kappa$ in* $(\mathbb{H}; 0, \infty)$ *is transient,*

$$\lim_{t \nearrow \infty} \gamma(t) \;=\; \infty,$$

and it has the following properties according to the parameter $\kappa \geq 0$

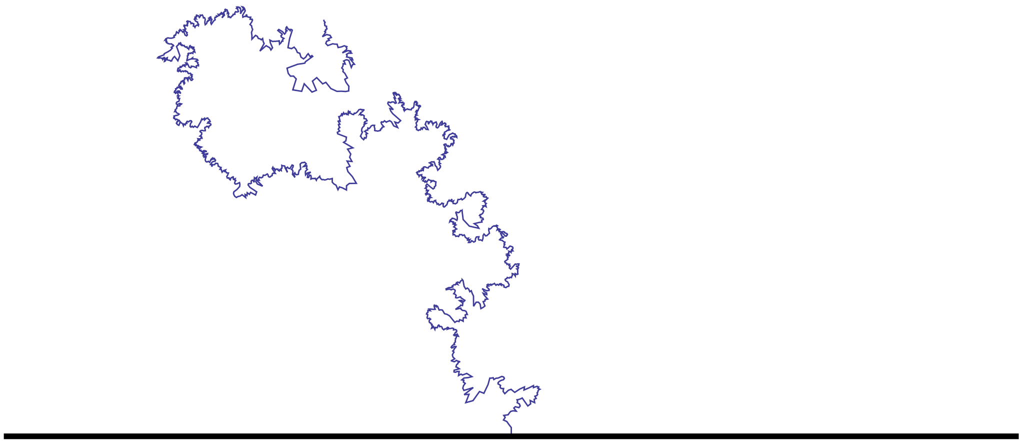

Figure 2: Initial segment of a chordal $\mathrm{SLE}_{8/3}$ trace in $(\mathbb{H}; 0, \infty)$.

$0 \leq \kappa \leq 4$: *The trace* $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ *is a simple curve, and* $\gamma(t) \in \mathbb{H}$ *for all* $t > 0$.

$4 < \kappa < 8$: *For any* $z \in \mathbb{H}$ *almost surely there exists a* $t > 0$ *such that* $z \in K_t$ *but* $z \notin \gamma[0, t]$, *i.e. the trace surrounds (or "swallows") the point* z *without passing through it. Also* $\gamma[0, \infty) \cap \mathbb{R}$ *is unbounded.*

$\kappa \geq 8$: *The trace is space filling curve,* $\gamma[0, \infty) = \overline{\mathbb{H}}$, *i.e. the trace visits every point of the domain.*

We will prove some of the statements below, the others are proven by similar techniques.

The simulated pictures give some hints about the three phases, although due to the necessary discretization of the curves for the simulation, the pictures have no genuinely different phases. It is also somewhat challenging to reduce numerical errors in simulating SLEs, so from the pictures it might not be clear that phase transitions occur at the precise values of the parameter κ.

One of the most notable quantitative properties of chordal SLE_κ is the fractal dimension. Looking at the simulated pictures, one may already guess that the fractal dimension of the curve increases with the parameter κ.

Theorem 3 (Beffara, [1]) *For* $0 \leq \kappa \leq 8$, *the Hausdorff dimension of* γ, *the trace of the chordal* SLE_κ, *is* $1 + \frac{\kappa}{8}$. *For* $\kappa > 4$, *the Hausdorff dimension of* ∂K_t, *the boundary of the SLE hull, is* $1 + \frac{2}{\kappa}$.

It is easy to obtain an upper bound for the Hausdorff dimension, and we will do this below. A reasonably accessible and careful proof of the entire result can be found in [2].

Other SLEs

It turns out that Schramm's principle works rather well in a few of the simplest situtations besides just the chordal case, notably the following:

- SLEs in simply connected domains with three marked boundary points
- SLEs in simply connected domains with a marked boundary point and a marked interior point

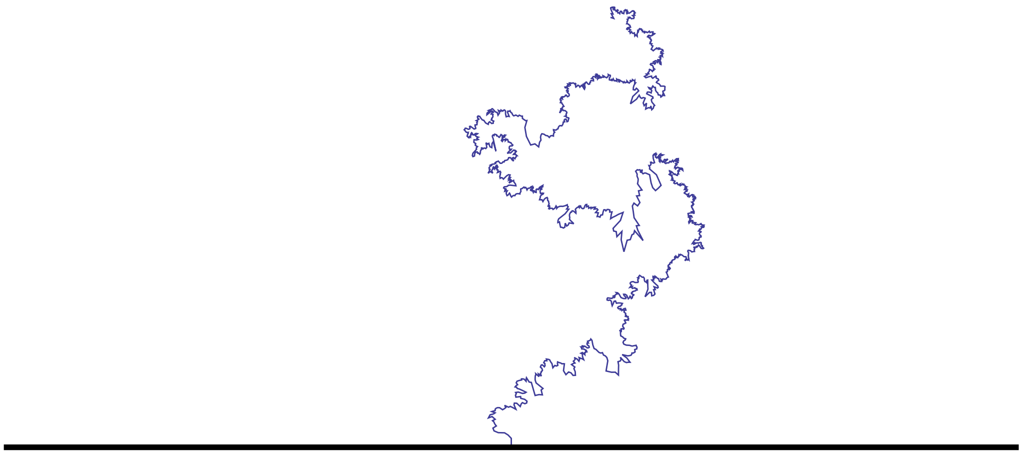

Figure 3: Initial segment of a chordal SLE_3 trace in $(\mathbb{H}; 0, \infty)$.

In each of the two cases, Riemann mapping theorem guarantees that any two domains are conformally equivalent, and in these cases there are no conformal self maps of a domain — all three degrees of freedom are needed to fix the marked points. We may thus expect that the conformal invariance requirement is somewhat less restrictive, and indeed we will find that the classification leaves room for an additional parameter.

Exercise 1 *Find a Schramm's principle for Loewner regular conformally invariant curves with domain Markov property in the following situation: the domain $\Lambda \subsetneq \mathbb{C}$ is simply connected, the curve start from boundary point $a \in \partial\Lambda$, and the law $\mathsf{P}_{(\Lambda;a,b,c)}$ depends also on two other marked boundary points $b, c \in \partial\Lambda$.* Hint: *It is convenient to work in $(\mathbb{S}; 0, +\infty, -\infty)$, and use a Loewner chain corresponding to Loewner vector fields* $\coth(\frac{z-x}{2})\,\partial_z$.

One could also consider other configurations, such as four or more marked boundary points in simply connected domains, or more marked interior points, or multiply connected domains. In each of these cases, however, there are conformal moduli, i.e. two generic such domains are no longer conformally equivalent. The requirement of conformal invariance then has weaker consequences, and an attempt of classification as above becomes less satisfactory — for applications to statistical mechanics models one needs more input from the model itself for identifying the appropriate random curves.

2.2 Coordinate changes of SLEs

In this section we consider descriptions of the same random curve by different Loewner chains. We emphasize that the random curve is the fundamental object and its parametrization and Loewner chain description are somewhat arbitrary choices, although certain choices are without a doubt more convenient than others.

The article [?] does several coordinate changes systematically. The same idea and almost identical calculations are fundamental for many different SLE problems, so variations on this theme have appeared in the literature ever since SLEs were introduced.

The chordal SLE in half-plane with another endpoint

We have defined the chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ as the curve which generates the Loewner chain with driving process $(\sqrt{\kappa}\, B_t)_{t\geq 0}$. In any other domain $(\Lambda; a, b)$, the chordal SLE_κ is the image

Figure 4: Initial segment of a chordal SLE_4 trace in $(\mathbb{H}; 0, \infty)$.

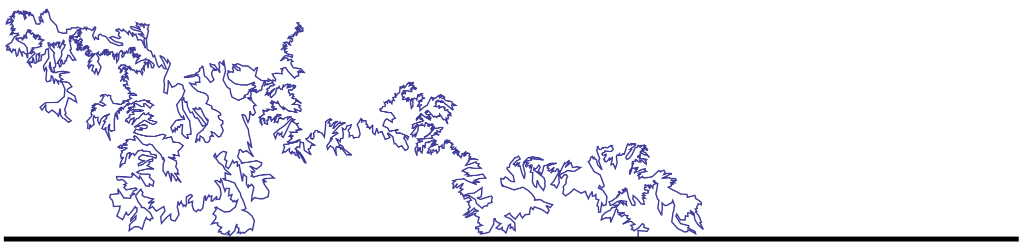

Figure 5: Initial segment of a chordal SLE_6 trace in $(\mathbb{H}; 0, \infty)$.

of this curve by a conformal map from $(\mathbb{H}; 0, \infty)$ to $(\Lambda; a, b)$.

Let us consider the case where the domain is still the half-plane $\mathbb{H}$, the starting point still the origin, but the end point is some point $b \in \mathbb{R} \setminus \{0\}$ at finite distance. The chordal SLE_κ in $(\mathbb{H}; 0, b)$ is clearly a Loewner regular curve (up to the first time it disconnects b from infinity), so we can give a description of it by a chordal Loewner chain.

So, let $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ be the chordal SLE_κ trace in $(\mathbb{H}; 0, \infty)$ parametrized by capacity as above, and let $\hat{\gamma}(t) = \mu(\gamma(t))$, where μ is a conformal map from $(\mathbb{H}; 0, \infty)$ to $(\mathbb{H}; 0, b)$. Such conformal maps are Möbius transformations, and there is a one parameter family of them:

$$\mu(z) \;=\; \frac{b\,z}{z - s}$$

where the parameter $s \in \mathbb{R} \setminus \{0\}$ is the point whose image under μ is infinity. We will use a Loewner chain that fixes infinity to describe the image curve $\hat{\gamma} = \mu \circ \gamma$, so in a sense we are observing the original SLE curve in $(\mathbb{H}; 0, \infty)$ from the point s.

Let $(g_t)_{t \geq 0}$ be the Loewner chain defining the chordal SLE_κ in $(\mathbb{H}; 0, \infty)$,

$$g_0(z) \;=\; z, \qquad \frac{\mathrm{d}}{\mathrm{d}t} g_t(z) \;=\; \frac{2}{g_t(z) - X_t}, \qquad X_t \;=\; \sqrt{\kappa}\, B_t.$$

Figure 6: Initial segment of a chordal SLE_8 trace in $(\mathbb{H}; 0, \infty)$.

We want to find the Loewner chain that describes $\hat{\gamma} = \mu \circ \gamma$, the chordal SLE_κ in $(\mathbb{H}; 0, b)$. To this end, let $\hat{g}_t$ be the hydrodynamically normalized conformal map from the unbounded component $\hat{H}_t$ of $\mathbb{H} \setminus \hat{\gamma}[0, t]$ to $\mathbb{H}$. Note that $\hat{\gamma}$ is not yet parametrized by capacity, but it is not difficult to see that it would only take a differentiable change of parametrization to achieve that. Let us denote by s_t the half-plane capacity of the hull generated by $\hat{\gamma}[0, t]$, so that $\hat{g}_t(z) = z + 2s_t z^{-1} + O(z^{-2})$. Then by Loewner regularity of the image curve $\hat{\gamma}$, until the time that $\hat{\gamma}$ disconnects b from ∞ we have

$$\hat{g}_0(z) = z, \qquad \frac{d}{dt}\hat{g}_t(z) = \frac{2\,\dot{s}_t}{\hat{g}_t(z) - \xi_t}$$

for some driving process $(\xi_t)_{t\geq 0}$, and with $\dot{s}_t = \frac{d}{dt}s_t$ the speed of capacity growth of $\hat{\gamma}$.

We already have at our disposal the conformal map $g_t \circ \mu^{-1} : \hat{H}_t \to \mathbb{H}$. The hydrodynamically normalized conformal map $\hat{g}_t : \hat{H}_t \to \mathbb{H}$ is obtained by post-composing with an appropriate self map μ_t of the half-plane,

$$\hat{g}_t = \mu_t \circ g_t \circ \mu^{-1}.$$

One could give an explicit expression for the time dependent Möbius transformation μ_t, but it turns out to be not necessary. We note that the driving process (ξ_t) is the image under the Loewner chain $(\hat{g}_t)$ of the tip of $\hat{\gamma}$, or alternatively

$$\xi_t = \hat{g}_t\big(\hat{\gamma}(t)\big) = \big(\mu_t \circ g_t \circ \mu^{-1}\big)\big(\mu(\gamma(t))\big) = \mu_t(X_t).$$

Also since

$$\mu_t = \hat{g}_t \circ \mu \circ g_t^{-1},$$

we can calculate the time derivative of $\mu_t(z)$. We just recall the Loewner equation for $\hat{g}_t$, and observe that the time derivative of g_t^{-1} is easily read from

$$0 = \frac{d}{dt}(z) = \frac{d}{dt}\Big(g_t(g_t^{-1}(z))\Big) = \frac{2}{g_t(g_t^{-1}(z)) - X_t} + g_t'(g_t^{-1}(z))\Big(\frac{d}{dt}g_t^{-1}(z)\Big),$$

with the result

$$\frac{d}{dt}g_t^{-1}(z) = \frac{-2\,(g_t^{-1})'(z)}{z - X_t}.$$

Now we calculate the time derivative of μ_t as follows

$$\begin{aligned}\frac{d}{dt}\mu_t(z) &= \frac{d}{dt}\Big(\hat{g}_t(\mu(g_t^{-1}(z)))\Big) \\ &= \frac{2\,\dot{s}_t}{\hat{g}_t(\mu(g_t^{-1}(z))) - \xi_t} + (\hat{g}_t \circ \mu)'(g_t^{-1}(z))\,\frac{-2\,(g_t^{-1})'(z)}{z - X_t} \\ &= \frac{2\,\dot{s}_t}{\mu_t(z) - \xi_t} - \frac{2\,\mu_t'(z)}{z - X_t}.\end{aligned}$$

The Möbius transformation $\mu_t : \mathbb{H} \to \mathbb{H}$, and its time derivative as well, is regular at the point X_t on the boundary (the only pole of μ_t is at the point $g_t(s)$, so that $\hat{g}_t = \mu_t \circ g_t \circ \mu^{-1}$ fixes infinity). Therefore, the poles at $z \to X_t$ of the two terms in $\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(z)$ must cancel. We do a Laurent expansion for the first term, keeping in mind that $\mu_t(X_t) = \xi_t$,

$$\begin{aligned} \frac{2\,\dot{s}_t}{\mu_t(z) - \xi_t} &= \frac{2\,\dot{s}_t}{\Big(\xi_t + \mu_t'(X_t)\,(z - X_t) + \frac{1}{2}\mu_t''(X_t)\,(z - X_t)^2 + \cdots\Big) - \xi_t} \\ &= \frac{2\,\dot{s}_t}{\mu_t'(X_t)\,(z - X_t)} - \frac{\dot{s}_t\,\mu_t''(X_t)}{\mu_t'(X_t)^2} + O(z - X_t). \end{aligned}$$

The second term is even easier

$$-\frac{2\,\mu_t'(z)}{z - X_t} = \frac{-2\,\mu_t'(X_t)}{z - X_t} - 2\mu_t''(X_t) + O(z - X_t).$$

For the poles to cancel, we must have

$$\dot{s}_t = \mu_t'(X_t)^2.$$

This is of course intuitive. On the one hand, $\dot{s}_t$ is the speed of capacity growth of the curve $\hat{\gamma}$ at time t. On the other hand, a small piece of the curve $\hat{\gamma}[t, t + \Delta t]$ becomes, after mapping to the half-plane by $\hat{g}_t$, the image of $g_t(\gamma[t, t + \Delta t])$ under μ_t. But $g_t(\gamma[t, t + \Delta t])$ is a small piece of curve in $\mathbb{H}$ of capacity Δt, and it is located near X_t. So μ_t essentially scales this piece by the factor $\mu_t'(X_t)$ and the image has capacity approximately $\mu_t'(X_t)^2\,\Delta t$, which is the asserted capacity growth $s_{t+\Delta t} - s_t$.

We made the expansions at $z \to X_t$ of the two terms in $\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(z)$ up to constant terms, so we immediately read the time derivative of μ_t at the point X_t,

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mu_t\Big)(X_t) = -3\,\mu_t''(X_t).$$

This facilitates the determination of the driving process (ξ_t) of the Loewner chain $(\hat{g}_t)$ since $\xi_t = \mu_t(X_t)$, as we observed earlier. Now, recalling that $\mathrm{d}X_t = \sqrt{\kappa}\,\mathrm{d}B_t$, the Itô derivative of ξ_t is

$$\begin{aligned} \mathrm{d}\xi_t &= \mathrm{d}\Big(\mu_t(X_t)\Big) = \mu_t'(X_t)\,\sqrt{\kappa}\,\mathrm{d}B_t + \frac{\kappa}{2}\mu_t''(X_t)\,\mathrm{d}t + \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mu_t\Big)(X_t)\,\mathrm{d}t \\ &= \mu_t'(X_t)\,\sqrt{\kappa}\,\mathrm{d}B_t + \frac{\kappa - 6}{2}\mu_t''(X_t)\,\mathrm{d}t. \end{aligned}$$

We may further remark that any Möbius transformation ν has the property

$$\frac{\nu'(z)^2}{\nu''(z)} = \frac{1}{2}\Big(\nu(z) - \nu(\infty)\Big).$$

Applied to μ_t at X_t, noting $\mu_t(\infty) = \hat{g}_t(b)$, this gives

$$\frac{\mu_t''(X_t)}{\mu_t'(X_t)^2} = \frac{2}{\xi_t - \hat{g}_t(b)},$$

which allows us to simplify the Itô derivative of ξ_t to

$$\mathrm{d}\xi_t = \sqrt{\kappa\,\dot{s}_t}\,\mathrm{d}B_t + \frac{\kappa - 6}{\xi_t - \hat{g}_t(b)}\,\dot{s}_t\,\mathrm{d}t.$$

In order to have a standard Loewner chain description of $\hat{\gamma}$, the chordal SLE_κ in $(\mathbb{H}; 0, b)$, we should use $s = s_t$ as the time parameter. Denote by $s \mapsto t_s$ the inverse function of $t \mapsto s_t$. Then the Loewner equation takes the usual form

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{g}_{t_s}(z) = \frac{2}{\hat{g}_{t_s}(z) - \xi_{t_s}}$$

and the change of time parametrization of the driving process leads to

$$\mathrm{d}\xi_{t_s} = \sqrt{\kappa}\,\mathrm{d}\hat{B}_s + \frac{\kappa - 6}{\xi_t - \hat{g}_t(b)}\,\mathrm{d}s,$$

where $(\hat{B}_s)_{s\geq 0}$ is a standard Brownian motion with respect to the time parameter s. This displays that the change of the chordal SLE endpoint to b excerts a drift on the driving process, whose strength is inversely proportional to the conformal distance of the tip and the endpoint. The sign and strength of the drift depend on κ, and at $\kappa = 6$ the additional drift vanishes.[1]

The Loewner chain $(\hat{g}_{t_s})_{s\geq 0}$ is of the form that is usually taken as definition of the SLE variant $\mathrm{SLE}_\kappa(\rho)$ in the domain $(\mathbb{H}; 0, b, \infty)$, with the particular value $\rho = \kappa - 6$ here. Below we will discuss the Schramm's principle applied to simply connected domains with three marked boundary points, and conclude that the most general (Loewner regular) conformally invariant random curves with domain Markov property are $\mathrm{SLE}_\kappa(\rho)$, for $\kappa \geq 0$ and $\rho \in \mathbb{R}$. In view of this fact, the result of the coordinate change had to be of this form.

The process $\mathrm{SLE}_\kappa(\kappa - 6)$ is also instrumental for the construction of so called conformal loop ensembles via an exploration tree, but for this purpose the process has to be continued in a slightly nontrivial fashion beyond the first time that the curve disconnects the target point b from the observation point ∞. This is beyond the scope of the present minicourse, but the interested reader may consult the article [7] for further information.

SLEs with three marked boundary points

In an earlier exercise, the following version of Schramm's principle was considered. To each simply connected domain $\Lambda \subsetneq \mathbb{C}$ and three boundary points $a, b, c \in \partial\Lambda$ one associates a probability measure $\mathsf{P}_{(\Lambda;a,b,c)}$ on Loewner regular curves starting from a and ending on the arc $\widehat{bc}$, and such that conformally invariance holds in the sense that $f_*\mathsf{P}_{(\Lambda;a,b,c)} = \mathsf{P}_{(f(\Lambda);f(a),f(b),f(c))}$ for f a conformal map. The classification result is that such curve in $(\mathbb{S}; 0, +\infty, -\infty)$ must be described by a Loewner chain

$$h_0(z) = z, \qquad \frac{\mathrm{d}}{\mathrm{d}t} h_t(z) = \coth\Big(\frac{h_t(z) - V_t}{2}\Big), \qquad V_t = \sqrt{\kappa}\,B_t + \alpha\,t,$$

for some $\kappa \geq 0$ and $\alpha \in \mathbb{R}$. In other domains the curve can be defined by conformal transport, as the image of the curve in $(\mathbb{S}; 0, +\infty, -\infty)$ under a conformal map $f : \mathbb{S} \to \Lambda$ such that $f(0) = a$, $f(+\infty) = a$, $f(-\infty) = c$.

For easier comparison with the chordal SLE, let us take the curve in the upper half-plane so that it starts from the origin and one of the marked points is at infinity. To obtain the curve in $(\mathbb{H}; 0, \infty, c)$, where $c < 0$, we use the conformal map f from $\mathbb{S}$ to $\mathbb{H}$ such that $f(0) = 0$, $f(+\infty) = \infty$ and $f(-\infty) = c$.

References

[1] V. Beffara, The dimension of the SLE curves. *Ann. Probab.* **36**(4) 1421–1452 (2008).

[2] G. Lawler, Fractal and multifractal properties of SLE. *Lecture notes for the Clay Mathematics Summer School*, Buzios, Brazil (2010).

[3] G. Lawler, O. Schramm and W. Werner, Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, **32**(1B), 939–995 (2004).

[4] S. Rohde and O. Schramm, Basic properties of SLE. *Ann. Math.*, **161** (2), 883–924 (2005).

[5] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, **118** 221–288 (2000).

[1] This particular phenomenon at $\kappa = 6$ gets a natural interpretation from a percolation result of Smirnov. The chordal SLE_6 is the scaling limit of exploration path of critical percolation — and the exploration path doesn't feel where its declared endpoint is.

[6] O. Schramm and D. Wilson, SLE coordinate changes. *New York J. Math.*, **11**, 659–669 (electronic) (2005).

[7] S. Sheffield, Exploration trees and conformal loop ensembles. *Duke Math. J.*, **147**, 79–129 (2009).

Department of Mathematics and Statistics
University of Helsinki
Finland
E-mail: kalle.kytola@helsinki.fi