Viscosity solutions for reaction-diffusion systems: heuristic derivation of the G-equation and multi-scale methods

Contents

1 Introduction

The scope of these notes is to illustrate based on heuristic path integral manipulations the relation between *reaction-advection-diffusion* (RAD) equations of Kolmogorov-Petrovskii-Piskunov (KPP) type and "viscosity solutions". The same heuristics provide a justification for the *G-equation* and its geometrical interpretation. Finally the relation with multiscale perturbation theory is discussed. Sources for these notes are the papers [4, 5, 3] and the book [2] where the results discussed here are rigorously derived.

2 Background material from the theory of stochastic differential equations

2.1 Forward dynamics

The stochastic differential equation

$$\begin{aligned} d_t r &= v(r,t)\,dt + \sqrt{2\,\kappa}\,dw_t \\ d_t r &:= r(t+dt) - r(t) \end{aligned} \tag{2.1}$$

provides the Lagrangian picture of

$$\partial_t P + \partial_x (vP) - \kappa\,\partial^2 P = 0 \tag{2.2}$$

with

$$P = P(x,t|x_0,t_0) = \prec \delta^{(d)}(x - \Phi(t,t_0)\circ x_0) \succ \tag{2.3}$$

The flow Φ is the fundamental solution of the SDE such that for all $t\,,t_0$

$$\begin{aligned} r(t) &= \Phi(t,t_0)\circ x_0 \\ r(t_0) &= \Phi(t_0,t_0)\circ x_0 = x_0 \end{aligned} \tag{2.4}$$

The tracer equation is

$$\partial_s \theta + v\cdot\partial_y \theta + \kappa \partial_y^2 \theta = 0 \tag{2.5}$$

with

$$\theta(y,s) = \int d^d x\, \theta(x,t)\, P(x,t|y,s) \tag{2.6}$$

2.2 Backward dynamics

A dual pair of Kolmogorov equations are obtained by studying the dynamics associated with backward increments. Let $T\ \geq\ t\ \geq\ s$ and consider

$$\begin{aligned} d_s^{\star}\tilde{r} &= -v(\tilde{r},s)\,ds + \sqrt{2\,\kappa}\,dw_{T-s} \\ d_s^{\star}\tilde{r} &:= \tilde{r}(s) - \tilde{r}(s-ds)\,, \qquad ds\ >\ 0 \end{aligned} \tag{2.7}$$

The equation is now defined for all values of s decreasing from T. The flow solution of the SDE yields

$$\tilde{r}(s) = \Phi^{\star}(s,t)\circ y \tag{2.8}$$

In order to derive the Kolmogorov pair associated with the dynamics one may introduce $T \geq t \geq s$ which plays the role of the origin in the forward equation. Namely defining the variable

$$\sigma = T - s \tag{2.9}$$

which *increases* for decreasing s one can write

$$\tilde{r}(s) = r(\sigma) \tag{2.10}$$

so that

$$d_s^{\star}\tilde{r} = \tilde{r}(s) - \tilde{r}(s - ds) = \tilde{r}(T - \sigma) - \tilde{r}(T - \sigma - ds) = r(\sigma) - r(\sigma + d\sigma) = -d_\sigma r \tag{2.11}$$

The backward dynamics can be described in terms of a forward one

$$d_\sigma r = v(r, T - \sigma)\, d\sigma + \sqrt{2\,\kappa}\, dw_\sigma \tag{2.12}$$

The relation between the forward and backward flows is then

$$\Phi(\sigma, \tau) = \Phi^{\star}(s, t) \tag{2.13}$$

so that

$$\bar{P}(x, s|y, t) = \prec \delta^{(d)}(x - \Phi^{\star}(s, t) \circ y) \succ = P(x, \sigma|y, \tau) \tag{2.14}$$

Since P is solution of the forward Kolmogorov equation in the variables σ, τ, it follows immediately that

$$\partial_s \bar{P} + \partial_x \left[v(x, s)\bar{P} \right] + \frac{\kappa}{2}\partial_x^2 \bar{P} = 0 \tag{2.15}$$

whilst the evolution of averages according to the backward dynamics is

$$\partial_t \theta + v \cdot \partial_x \theta - \frac{\kappa}{2}\partial^2 \theta = 0 \tag{2.16}$$

and has the solution

$$\theta(x, t) = \int d^d y\, \theta(y, s)\bar{P}(y, s|x, t) \tag{2.17}$$

The path integral representation of the transition probability density can be readily derived from the equivalent forward SDE. In the Ito discretisation one finds

$$\bar{P}(y, s|x, t) = \int \mathcal{D}[q]\delta^{(d)}(q(T - s) - y)\delta^{(d)}(q(T - t) - x)\, e^{-\int_{T-t}^{T-s} d\sigma\, \mathcal{L}(\dot{q}(\sigma), q(\sigma), T - \sigma)} \tag{2.18}$$

with

$$\mathcal{L}(\dot{q}(\sigma), q(\sigma), T - \sigma) = \frac{|\dot{q}(\sigma) + v(q(\sigma), T - \sigma)|^2}{2\,\kappa} \tag{2.19}$$

Finally, exploiting the arbitrary choice of T one can set $T = t$ in the above formulae:

$$\bar{P}(y, s|x, t) = \int \mathcal{D}[r]\delta^{(d)}(r(t - s) - y)\delta^{(d)}(r(0) - x)\, e^{-\int_0^{t-s} d\sigma\, \mathcal{L}} \tag{2.20}$$

$$\mathcal{L} = \frac{|\dot{q} + v(q, t - \sigma)|^2}{2\,\kappa} \tag{2.21}$$

3 Reaction-diffusion systems

Consider the RAD system

$$\partial_t \theta + v \cdot \partial \theta = \varepsilon \kappa \, \partial^2 \theta + \frac{f(\theta)}{\varepsilon}$$
$$\theta(x, 0) = \Theta(x) \in [0, 1] \tag{3.1}$$

with $\Theta(x)$ having *compact support* G_0 and the reaction term

$$f(\theta) = \theta \, c(\theta) \tag{3.2}$$

is of KPP type (see appendix A) i.e.

$$f(\theta) \, > \, 0 \qquad \forall \theta \in \, [0, 1[\quad \& \quad f(\theta) \, < \, 0 \qquad \forall \, \theta \, \in] - \infty, 0[\cup]1, \infty[$$
$$c_0 \, \equiv \, c(0) = \max_{0 \geq \theta \geq 1} c(\theta) \tag{3.3}$$

The standard example of KPP reaction is

$$f(\theta) = K \, \theta \, (1 - \theta) \, , \qquad K > 0 \tag{3.4}$$

Finally the velocity field is bounded. The solution can be written in the *implicit* integral form

$$\theta(x, t) = \prec \Theta \circ q(t) \, e^{-\int_0^t du \frac{f \circ \theta(q \circ u, u)}{\varepsilon}} \succ_{q \circ 0 = x} = \int d^d y \, \Theta(y) \prec \delta^{(d)}(y - \Psi(0, t) \circ x) \, e^{\int_0^t du \frac{f \circ \theta(q \circ u, u)}{\varepsilon \, \theta(q \circ u, u)}} \succ \tag{3.5}$$

where

$$\prec \delta^{(d)}(y - \Psi(0, t) \circ x) \, e^{\int_0^t du \frac{f \circ \theta(r \circ u, u)}{\varepsilon \, \theta(q \circ u, u)}} \succ = \int_{q(0)=x}^{q(t)=y} \mathcal{D}q \, e^{-\int_0^t ds \left\{ \frac{|\dot{q} + v(q, t-s)|^2}{4 \, \kappa \, \varepsilon} - \frac{f(\theta)}{\varepsilon \, \theta} \right\}} \tag{3.6}$$

having adopted Ito discretisation. Since $\Theta \, \leq \, 1$ by hypothesis the solution satisfies

$$0 \, \leq \, \theta(x, t) \, \leq \prec e^{\int_0^t du \frac{f \circ \theta(q \circ u, u)}{\varepsilon \, \theta(q \circ u, u)}} \succ_{q(0)=x} \leq 1 \tag{3.7}$$

3.0.1 Qualitative analysis

In the limit ε tending to zero the path integral is dominated by extremals of the action

$$\mathcal{A} = \int_0^t ds \, \mathcal{L}(\dot{q}, q, c) \tag{3.8}$$

with

$$\mathcal{L}(\dot{q}, q, c) = \frac{|\dot{q} + v(q, t-s)|^2}{4 \, \varepsilon \, \kappa} - \frac{f(\theta)}{\varepsilon \, \theta} = \frac{|\dot{q} + v(q, t-s)|^2}{4 \, \varepsilon \, \kappa} - \frac{c(\theta)}{\varepsilon} \tag{3.9}$$

For a reaction of KPP form, the Lagrangian is bounded from below by

$$\mathcal{L}(\dot{q}, q, c) \geq \mathcal{L}_\star(\dot{q}, q) = \frac{|\dot{q} + v(q, t-s)|^2}{4 \, \varepsilon \, \kappa} - \frac{c_0}{\varepsilon} \tag{3.10}$$

consequently

$$\theta(x, t) \leq \theta_\star(x, t) = e^{\frac{c_0 \, t}{\varepsilon}} \int_{q(0)=x} \mathcal{D}q \, \Theta(q(t)) \, e^{-\int_0^t ds \, \frac{|\dot{q} + v(q, t-s)|^2}{4 \, \kappa \, \varepsilon}} \tag{3.11}$$

The bound can be tight only when

$$\mathcal{A}_\star = c_0 \, t - \int_0^t ds \, \frac{|\dot{q} + v(q, t-s)|^2}{4 \, \kappa} \, \leq \, 0 \tag{3.12}$$

i.e. when it self-consistently predicts $\theta \ll 1$

3.1 Exact mathematical result

In Freidlin

$$\lim_{\varepsilon\downarrow 0}\varepsilon\ln\theta(x,t) = \sup\min_{0\geq\tau\geq t}\left\{\int_0^t ds\,\mathcal{L}(\dot{q},q,f)\,;\,q(s)\,\text{absolutely continuous}\,\forall\,s\,\in\,[0,t]\quad\&\quad q(0)=x\,,\;q(t)\,\in\,G_0\right\}\tag{3.13}$$

3.2 Representation in phase space: viscosity solution

The path integral admits also the phase space representation

$$\prec\delta^{(d)}(y-\Psi(0,t)\circ x)\,e^{\int_0^t du\,\frac{f\circ\theta(r\circ u,u)}{\varepsilon\,\theta(r\circ u,u)}}\succ = \int_{q(0)=y}^{q(t)=x}\mathcal{D}q\,\mathcal{D}p\,e^{-\int_0^t ds\left\{\varepsilon\,\kappa\,p^2+\imath\,p\cdot\,[\dot{q}+v(q,t-s)]-\frac{f(\theta)}{\varepsilon\,\theta}\right\}}\tag{3.14}$$

As above the path integral is constructed using Ito pre-point discretisation.

3.2.1 Viscosity solution by steepest descent

The limit ε tending to zero of the path integral can be studied, using the *steepest descent approximation* [1]. The idea of the steepest descents (see [1] for details) consists in deforming the path of integration in the complex plane so as to make it coincide as far as possible with arcs of steepest paths such that the variation of the integrand is due only to its real part i.e. paths such that

$$\Im\left\{e^{-\mathcal{A}}\right\} = e^{-\Im\{\mathcal{A}\}} = \text{const.}\tag{3.15}$$

is verified. If this is possible the integrand is then expanded around the minimum along the steepest path of its real part. In order to deform the contour of integration one sets

$$p = p_r - \imath p_\imath\tag{3.16}$$

The action functional becomes

$$\mathcal{A} = \int_0^t ds\,\left\{\varepsilon\,\kappa\,(p_r^2-p_\imath^2)+p_\imath\cdot(\dot{q}+v)+\imath p_r\cdot(\dot{q}+v-2\,\varepsilon\,\kappa\,p_\imath)-\frac{c(\theta)}{\varepsilon}\right\}\tag{3.17}$$

Convergence requires

$$\varepsilon\,\kappa\,(p_r^2-p_\imath^2)+p_\imath\cdot(\dot{q}+v) > 0\tag{3.18}$$

The condition is satisfied by the steepest descent path

$$\dot{q}+v-2\,\varepsilon\,\kappa\,p_\imath = 0\tag{3.19}$$

regarded as an equation for $p_\imath$. *Along the steepest descent* the action becomes

$$\mathcal{A}_{s.d.} = \int_0^t ds\,\left\{\varepsilon\,\kappa\,p_r^2+p_\imath\cdot\dot{q}-\mathcal{H}(p_\imath,q,c)\right\}\tag{3.20}$$

with $p_\imath$ given by (3.19) and

$$\mathcal{H}(p,q,f) = \varepsilon\,\kappa\,p^2-p\cdot v+\frac{c(\theta)}{\varepsilon} = \operatorname*{Leg}_{\xi}\left\{\xi\cdot p-\mathcal{L}(\xi,q,c)\right\}\tag{3.21}$$

where $\mathcal{L}$ is the Ventsel-Freidlin Lagrangian (3.9) and Leg denotes the Legendre transform. Obviously, *inserting* (3.19) *into* $\mathcal{A}_{s.d.}$ *recovers* (3.9). The advantage of writing (3.9) in the Hamiltonian form is that it evinces the Hamiltonian structure of the variational problem which is therefore solved by an Hamilton-Jacobi equation. In the present context, the Hamilton-Jacobi equation can also be interpreted as the condition specifying the optimal choice of

$$p_\imath = \frac{\partial_q \psi(q, t-s)}{\varepsilon} \tag{3.22}$$

such to reduce the singular ε dependence to a *local boundary term.* Setting

$$\mathcal{A}_{s.d.} = \int_0^t ds \left\{ -\frac{\kappa\,(\partial\psi)^2 - (\dot{q} + v)\cdot\partial\psi + c(\theta)}{\varepsilon} - \varepsilon\,\kappa\, p_r^2 \right\} \tag{3.23}$$

in the Ito representation

$$\dot{q}\cdot\partial_q\psi(q,t-s) = d_s\psi(q,t-s) + \partial_{t-s}\psi(q,t-s) - \varepsilon\,\kappa\,\partial_q^2\psi(q,t-s) \tag{3.24}$$

yields

$$\begin{aligned} \mathcal{A}_{s.d.} = & -\frac{\psi(x,t) - \psi(x_0,0)}{\varepsilon} \\ & - \int_0^t ds \left\{ \frac{\kappa\,(\partial\psi)^2 - \partial_{t-s}\psi + \varepsilon\,\kappa\,\partial_q^2\psi - v\cdot\partial\psi + c(\theta)}{\varepsilon} - \varepsilon\,\kappa\, p_r^2 \right\} \end{aligned} \tag{3.25}$$

The (inviscid) Hamilton-Jacobi equation is

$$\partial_{t-s}\psi - \kappa\,(\partial\psi)^2 + v\cdot\partial\psi - c(\theta) = 0 \tag{3.26}$$

The functional dependence on $t-s$ can be then simplified by setting s to zero. If (3.26) holds it follows that for $\varepsilon \downarrow 0$

$$\theta(x,t) = e^{\frac{\psi(x,t)}{\varepsilon}}\,\{1 + \dots\} \tag{3.27}$$

with

$$\theta(x,t) = \begin{cases} 1 & \text{for} \quad \psi(x,t) = 0 \\ 0 & \text{for} \quad \psi(x,t) < 0 \end{cases} \tag{3.28}$$

so that one can self-consistently conclude that to leading order in ε, ψ must satisfy

$$\max_{\psi} \left\{ \partial_t\psi - \kappa\,(\partial\psi)^2 + v\cdot\partial\psi - c_0, \psi \right\} = 0 \tag{3.29}$$

with boundary conditions

$$\psi(x,0) = \begin{cases} 0 & \text{for all } x \text{ such that} \quad \Theta(x) > 0 \\ -\infty & \text{for all } x \text{ such that} \quad \Theta(x) = 0 \end{cases} \tag{3.30}$$

In the derivation of (3.29) one exploits the fact that $f(\theta)$ is of KPP type. Namely

$$\begin{aligned} \psi = 0 \quad &\Rightarrow \quad \theta = 1 \quad \Rightarrow \quad c(\theta) \to 0 \\ \psi < 0 \quad &\Rightarrow \quad \theta \downarrow 0 \quad \Rightarrow \quad c(\theta) \to c_0 \end{aligned} \tag{3.31}$$

3.3 Example of variational equation

A special solvable case is represented by the advection by a constant velocity field

$$v(x,t) = \bar{v} \tag{3.32}$$

One has

$$\max_{\psi} \left\{ \partial_t \psi - \kappa\, (\partial \psi)^2 - \bar{v} \cdot \partial \psi - c_0, \psi \right\} = 0 \tag{3.33}$$

with b constant and

$$\psi(x,0) = \begin{cases} 0 & \text{if} \quad x = x_0 \\ -\infty & \text{if} \quad x \neq x_0 \end{cases} \tag{3.34}$$

The solution is

$$\psi(x,t) = \min \left\{ \left[c_0\, t - \frac{(|x - x_0| + t\, \bar{v})^2}{4\, \kappa\, t} \right], 0 \right\} \tag{3.35}$$

which is a straightforward application of Hopf-Lax formula stating that

$$\psi(x,t) = \inf \left\{ \int_0^t ds\, \mathcal{L}(q(s)) + g(y) \;\middle|\; q(0) = y \;\; q(t) = x \right\} \tag{3.36}$$

is equal to

$$\psi(x,t) = \min_{y \in \mathbb{R}^d} \left\{ t\, \mathcal{L} \left(\frac{x - y}{t} \right) + g(y) \right\} \tag{3.37}$$

4 Fronts and G-equation

4.1 Definition of a front

The natural definition of front [4] is therefore that of the region of space where θ varies abruptly:

$$\Gamma_t = \partial \left\{ x \in \mathbb{R}^d \,|\, \psi \, < \, 0 \right\} \tag{4.1}$$

An estimate of the front domain can be obtained were one able to define a suitable

$$u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \tag{4.2}$$

such that

$$\left\{ x \in \mathbb{R}^d \,|\, u \, = \, 0 \right\} \subseteq \left\{ x \in \mathbb{R}^d \,|\, \psi \, = \, 0 \right\} \tag{4.3}$$

$$\left\{ x \in \mathbb{R}^d \,|\, u \, < \, 0 \right\} \subseteq \left\{ x \in \mathbb{R}^d \,|\, \psi \, < \, 0 \right\} \tag{4.4}$$

Condition (4.3) is generally satisfied by u solution of the so-called G-equation. On the contrary (4.4) is not generally satisfied. Majda and Souganidis show that additive hypothesis are needed in order (4.4) to hold for a *large scale* description of the front.

4.2 Derivation of the G-equation

The Hamiltonian entering the asymptotic problem is

$$\mathcal{H}_{\star} = \varepsilon\,\kappa\,p^2 - v\cdot p + \frac{c_0}{\varepsilon} \tag{4.5}$$

This expression can be minimised with respect to ε

$$\tilde{\mathcal{H}} = \min_{\varepsilon>0}\{\mathcal{H}_{\star}\} = 2\sqrt{\kappa\,c_0}\;|p| - v\cdot p \leq \mathcal{H}_{\star} \tag{4.6}$$

It is therefore intuitive (proof in [4]) that from

$$\begin{aligned}&\max\left\{\partial_t u - \tilde{\mathcal{H}}(\partial_x u, x, t), u\right\} = 0\\ &u(x,0) = g(x)\;>\;0 \qquad \forall\, x\,\in\,G_0\end{aligned} \tag{4.7}$$

may follows

$$u(x,t) \leq \psi(x,t) \tag{4.8}$$

4.3 Geometrical interpretation of $\tilde{\mathcal{H}}$: Maupertuis principle

Let $\mathcal{C}$ the of path space such that

$$\mathcal{C} = \{q(s)\text{ absolutely continuous}\,\forall\, s\,\in[0,t]\,|\,q(0)=x\;\&\;q(t)\;\in\,G_0\} \tag{4.9}$$

The path integral representation yields

$$\psi(x,t) = \varepsilon\ln\theta(x,t) \overset{\varepsilon\downarrow 0}{\sim} -\inf_{q\,\in\,\mathcal{C}}\int_0^t ds\left\{\frac{|\dot{q}(s)+v(q(s),t-s)|^2}{4\,\kappa} - c_0\right\} \tag{4.10}$$

A sufficient conditions for ψ to vanish is that

$$c_0 = \frac{|\dot{q}(s)+v(q(s),t-s)|^2}{4\,\kappa} \qquad \Rightarrow \qquad \mathcal{L}_{\star} = 0 \tag{4.11}$$

A general result from complex analysis states that if

$$\mathcal{L}_0(v,q,t) = \sup_p\{p\cdot v - \mathcal{H}_0(p,q,t)\} \tag{4.12}$$

it follows that

$$\max_{\mathcal{L}_0=a}\{p\cdot v\} = \inf_{\lambda>0}\left\{a\,\lambda + \lambda\mathcal{H}_0\left(\frac{p}{\lambda},q,t\right)\right\} \tag{4.13}$$

Setting

$$\begin{aligned}\mathcal{H}_0 = \kappa\,p^2 - p\cdot v \quad &\Leftrightarrow \quad \mathcal{L}_0 = \frac{|\dot{q}+v|^2}{4\,\kappa}\\ \lambda &= \frac{1}{\varepsilon}\end{aligned} \tag{4.14}$$

it is clear that the right hand side of (4.13) defines $\tilde{\mathcal{H}}$ in (4.7). The geometrical meaning of (4.7) comes from Maupertuis principle. Namely

$$\tilde{\mathcal{A}}\Big|_{\mathcal{L}=a} = \int_0^t ds\,\{p\cdot\dot{q} - \mathcal{L}\}_{\mathcal{L}=a} = \int_0^t ds\,\{p\cdot\dot{q}\}_{\mathcal{L}=a} - a\,t \tag{4.15}$$

reaches its extremal on the reduced action

$$\delta\,\tilde{\mathcal{A}}\Big|_{\mathcal{L}=a} = \delta\int_0^t ds\,\{p\cdot\dot{q}\}_{\mathcal{L}=a} = 0 \tag{4.16}$$

4.4 Counter-example to the G-equation

Suppose [4]

$$v(x,t) = \frac{x}{\tau} \tag{4.17}$$

then the variational problem

$$\max_{\psi} \left\{ \partial_t \psi - \kappa\, (\partial \psi)^2 + \frac{x}{\tau} \cdot \partial \psi - c_0, \psi \right\} = 0$$

$$\psi(x,0) = \begin{cases} 0 & x = 0 \\ -\infty & x \neq 0 \end{cases}$$

has the solution

$$\psi(x,t) = \min \left\{ c_0\, t - \frac{x^2}{2\,\kappa\,\tau\,(e^{\frac{2\,t}{\tau}} - 1)}, 0 \right\} \tag{4.18}$$

The front is

$$x(t) = \pm \left[2\,\kappa\,\tau\, c_0\, t\, (e^{2\,t} - 1) \right]^{1/2} \tag{4.19}$$

The G-equation is

$$\partial_t u - 2\,\sqrt{c_0\,\kappa}\,|\partial_x u| + \frac{x}{\tau} \partial u = 0$$

$$u(x,0) \begin{cases} \geq\, 0 & x = 0 \\ <\, 0 & x \neq 0 \end{cases}$$

The solution is

$$u(x,t) \propto 2\sqrt{\kappa\, c_0}\, \left(1 - e^{-\frac{t}{\tau}} \right) - |x|\; e^{-\frac{t}{\tau}} \tag{4.20}$$

The prediction for the front is

$$|x(t)| = 2\sqrt{\kappa\, c_0}\, \left(e^{\frac{t}{\tau}} - 1 \right) \tag{4.21}$$

which differs from the one of the exact variational problem.

5 Cell equation: non-linear multiscale in the presence of "fast" degrees of freedom

Given

$$\begin{aligned} &\partial_t \theta + v \left(x, t, \frac{x}{\varepsilon^{\alpha}}, \frac{t}{\varepsilon^{\alpha}} \right) \cdot \partial \theta = \varepsilon \kappa\, \partial^2 \theta - \frac{f(\theta)}{\varepsilon} \\ &\theta(x,0) = \Theta(x) \\ &0 \leq\; \alpha\; \leq\; 1 \end{aligned} \tag{5.1}$$

for

$$\theta : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \tag{5.2}$$

the average over *space-time periodic* fast scales can be performed using that there exists a *unique* constant $\mathcal{H}(p,x,t)$ such that the equation

$$\begin{aligned} &\partial_s w - a(\alpha)\kappa\,\partial_y^2 w - \kappa\,(p+\partial_y w)^2 + v\,(x,t,y,s)\cdot[p+\partial_y w(y,s)] = -\mathcal{H}(p,x,t) \\ &a(\alpha) = \begin{cases} 0 & 0 \leq \alpha < 1 \\ 1 & \alpha = 1 \end{cases} \end{aligned} \tag{5.3}$$

admits a solution for w a function of the *fast variables alone*

$$w = w(y,s) \tag{5.4}$$

By space-time periodic fast scales it is meant that for all (x,t)

$$\begin{aligned} v\,(x,t,y,s) &= v\,(x,t,y+Y,s) \\ v\,(x,t,y,s) &= v\,(x,t,y,s+T) \end{aligned} \tag{5.5}$$

The *cell* is then the hypercube

$$Q \times I \,\subset\, \mathbb{R}^d \times \mathbb{R}_+ \tag{5.6}$$

The Hamiltonian $\mathcal{H}$ featuring (5.3) satisfies the bounds

$$\kappa\,p^2 - |v|_\infty \,\leq\, \mathcal{H}(p,x,t) \,\leq\, \kappa\,p^2 + |v|_\infty \tag{5.7}$$

Observations

1. The uniqueness of $\mathcal{H}(p,x,t)$, a constant w.r.t. the fast variables, stems from the requirement of (5.3) being satisfied by a function fulfilling a determined class of boundary conditions (i.e. periodic in $Q \times I$). Remember that in the linear case, quantisation of the eigenvalues is also a consequence of the *boundary conditions*.

2. Periodicity in Q implies

$$\int_Q d^d y\,\frac{\partial w}{\partial y^i}(y,s) = 0 \tag{5.8}$$

   this provide a *necessary condition* for an element of on a one-parameter $\mathcal{H}$ family of solutions of (5.3) to be periodic i.e. the unique solution of the cell problem.

5.1 Examples

The cell equation can be rewritten in the form

$$\left[\partial_s - 2\,\kappa\,p\cdot\partial_y - a(\alpha)\kappa\,\partial_y^2\right] w - \kappa\,(\partial_y w)^2 + v\,(x,t,y,s)\cdot\partial_y w = -\mathcal{H}(p,x,t) + \kappa\,p^2 - v\,(x,t,y,s)\cdot p \tag{5.9}$$

5.1.1 Fast scales fluctuating only in time

The velocity field is

$$v\,(x,t,y,s) = \bar{v}(x,t) + \delta v(s) \tag{5.10}$$

In such a case it is enough to take w constant so that the cell equation reduces to

$$\partial_s w + \delta v(s)\cdot p = -\mathcal{H}(p,x,t) + \kappa\,p^2 - \bar{v}\,(x,t)\cdot p \tag{5.11}$$

The general solution is

$$w(s) = w(0) + \int_0^s ds'\, \delta v(s') \cdot p - \left[\mathcal{H}(p,x,t) - \kappa\, p^2 + \bar{v}\,(x,t) \cdot p\right]\, s \tag{5.12}$$

Imposing periodicity yields

$$\mathcal{H}(p,x,t) = \kappa\, p^2 - \bar{v}\,(x,t) \cdot p \tag{5.13}$$

5.1.2 Autonomous shear flow in $\mathbb{R}^2$

Suppose fluctuationas around the average velocity are specified by a shear flow

$$v\,(x,t,y,s) = \bar{v}(x,t) + \begin{bmatrix} \delta v_1(y_2) \\ 0 \end{bmatrix} \tag{5.14}$$

with

$$\int_0^Y dy_2\, \delta v_1(y_2) = 0 \tag{5.15}$$

For $\alpha < 1$ the cell equation becomes

$$-\kappa\,(\partial_{y_2} w)^2 + [\bar{v}_2\,(x,t) - 2\,\kappa\, p_2]\, \partial_{y_2} w = -\mathcal{H}(p,x,t) + \kappa\, p^2 - \bar{v}\,(x,t) \cdot p - \delta v_1(y_2)\, p_1 \tag{5.16}$$

The equation can be rewritten as

$$\left\{\partial_{y_2} w + \frac{2\,\kappa\, p_2 - \bar{v}_2\,(x,t)}{2\,\kappa}\right\}^2 = \frac{\mathcal{H}(p,x,t) - \kappa\, p^2 + \bar{v}\,(x,t) \cdot p + \delta v_1(y_2)\, p_1}{\kappa} + \frac{[2\,\kappa\, p_2 - \bar{v}_2\,(x,t)]^2}{4\,\kappa^2} \tag{5.17}$$

The right hand side reduces to

$$\left\{\partial_{y_2} w + \frac{2\,\kappa\, p_2 - \bar{v}_2\,(x,t)}{2\,\kappa}\right\}^2 = \frac{\mathcal{H}(p,x,t) - \kappa\, p_1^2 + \bar{v}_1\,(x,t) \cdot p_1 + \delta v_1(y_2)\, p_1}{\kappa} + \frac{\bar{v}_2^2\,(x,t)}{4\,\kappa^2} \tag{5.18}$$

Consider the auxiliary equation

$$\left[\frac{dw}{dy} + U\right]^2 = F(y) \tag{5.19}$$

with U constant and

$$F(y) > 0 \tag{5.20}$$

for all $y \in [0, Y]$. Then

$$w(y) = w(0) - U\, y \pm \int_0^y d\omega\, \sqrt{F(\omega)} \tag{5.21}$$

A periodic solution must satisfy

$$U = \pm \int_0^Y d\omega\, \frac{\sqrt{F(\omega)}}{Y} \tag{5.22}$$

Therefore $\mathcal{H}$ should be chosen such to let simultaneously

1. the right hand side of (5.18) be positive definite
2. the periodicity condition (5.22) befulfilledd.

Setting

$$M = \max \{\bar{v}_1(x,t) \cdot p_1\} \tag{5.23}$$

then for any $N \geq 0$ independent of y the choice

$$\mathcal{H}(p,x,t) = \kappa p_1^2 - \bar{v}_1(x,t) \cdot p_1 + \frac{\bar{v}_2^2(x,t)}{2} + M + N \tag{5.24}$$

renders the right hand side of (5.18) positive definite

$$\left\{\partial_{y_2} w + \frac{2\kappa p_2 - \bar{v}_2(x,t)}{2\kappa}\right\}^2 = \frac{M + \delta v_1(y_2)\, p_1 + N}{\kappa} \tag{5.25}$$

The periodicity condition becomes

$$\frac{2\kappa p_2 - \bar{v}_2(x,t)}{2\kappa} = \pm \int_0^Y \frac{d\omega}{Y} \sqrt{\frac{M + \delta v_1(\omega)\, p_1 + N}{\kappa}} \tag{5.26}$$

5.2 Comparison with linear theory

The logarithmic transform

$$w = a(\alpha) \ln \phi \tag{5.27}$$

yields

$$\left[\partial_s - a(\alpha)\kappa\, \partial_y^2\right] \phi = -\frac{\mathcal{H}(p,x,t) - \kappa p^2 + v(x,t,y,s) \cdot p}{a(\alpha)} \phi - [2\kappa p + v(x,t,y,s)] \cdot \partial_y \phi \tag{5.28}$$

or equivalently

$$[\partial_s - D_y]\, \phi = -\frac{\mathcal{H}(p,x,t) - \kappa p^2 + \bar{v}(x,t) \cdot p}{a(\alpha)} \phi \tag{5.29}$$

where

$$\bar{v}(x,t) := \int_{Q\times I} d^d y ds\, v(x,t,y,s) \tag{5.30}$$

and

$$D_y := a(\alpha)\kappa\, \partial_y^2 + [2\kappa p + v(x,t,y,s)] \cdot \partial_y + [\bar{v}(x,t) - v(x,t,y,s)] \cdot p \tag{5.31}$$

So if

$$v(x,t,y,s) = \bar{v}(x,t) + \delta v(y,s) \tag{5.32}$$

it follows that

$$\mathcal{H}(p,x,t) = \kappa p^2 - \bar{v}(x,t) \cdot p \tag{5.33}$$

6 Cell equation and front evolution

The main results of [4] is that the limit of fast reaction and slow diffusion for a two scale advection with *periodic fast degrees* of freedom is governed by the variational equation

$$\max\{\partial_t\psi - \mathcal{H}_0(\partial_x\psi, x, t) - c_0, \psi\} = 0 \tag{6.1}$$

$$\psi(x,0) = \begin{cases} 0 & \text{for all } x \text{ such that } \quad \Theta(x) \; > \; 0 \\ -\infty & \text{for all } x \text{ such that } \quad \Theta(x) \; = \; 0 \end{cases} \tag{6.2}$$

with

$$\mathcal{H}_0(\partial_x\psi, x, t) = \text{solution of the cell problem} \tag{6.3}$$

which enjoys the convexity properties of the standard Ventsel-Freidlin Hamiltonian. Furthermore for

$$v(x,t,y,s) = \bar{v}(x,t) + \delta v(y,s) \tag{6.4}$$

with *zero average* fast periodic scales

$$\kappa\, p^2 - p\cdot\bar{v}(x,t) - \max_{(y,s)\in Q\times T}\{p\cdot\delta v(y,s)\} \leq \mathcal{H}_0(p,x,t) \leq \kappa\, p^2 - p\cdot\bar{v}(x,t) - \min_{(y,s)\in Q\times T}\{p\cdot\delta v(y,s)\} \tag{6.5}$$

Furthermore if

$$v(x,t,y,s) = \bar{v} + \delta v(y,s) \tag{6.6}$$

with $\bar{v}$ constant and $\delta v(y,s)$ *incompressible* with zero average the solutions of the G-equation fulfill *both* the conditions (4.3),(4.4).

Appendices

A Kolmogorov Petrovskii Piskunov problem

The equation

$$\begin{aligned} \partial_t\theta - \frac{\kappa}{2}\partial^2\theta &= f(\theta) \\ \theta(x,0) &= H_0(-x) \end{aligned} \tag{A.1}$$

for H_0 the Heaviside step function and

$$f(\theta) = \theta\, c(\theta) \tag{A.2}$$

with $c(\theta)$ as in (3.3). Kolmogorov Petrovskii Piskunov proved in 1937 that (A.1) admits an *asymptotic running wave solution* for large t. In particular they proved that

1. θ is a strictly monotonic function *decreasing* from $x \downarrow -\infty$ to $x \uparrow \infty$

2. There exists a unique $m(t)$ such that

$$\theta(m(t),t) = \frac{1}{2} \tag{A.3}$$

and

$$\lim_{t\uparrow\infty} \frac{m(t)}{t} = \sqrt{2\,\kappa\,c_0} = v_\star \tag{A.4}$$

3. Asymptotically the solution satisfy

$$\begin{aligned}
&\lim_{t\uparrow\infty} \theta(m(t)+x,t) = \vartheta(x) \\
&\frac{\kappa}{2}\frac{d^2\vartheta}{dx^2} + v_\star\frac{d\vartheta}{dx} + f(\vartheta) = 0 \\
&\lim_{x\downarrow-\infty} \vartheta = 1 \quad \& \quad \lim_{x\uparrow\infty} \vartheta = 0
\end{aligned} \tag{A.5}$$

Simple heuristics permit to recover to some extent the above results. Duhamel's formula evinces that for a positive initial datum the solution remains positive and bounded for all times. The spatial derivative of such solution satisfies the Euclidean quantum mechanics

$$\partial_t\delta\theta - \frac{\kappa}{2}\partial^2\delta\theta = V(x,t)\,\delta\theta \tag{A.6}$$

with time dependent potential

$$V(x,t) = \frac{\partial f(\theta)}{\partial\theta} \tag{A.7}$$

bounded from above and below for all times. The solution can be therefore couched into the form

$$\delta\theta(x,t) = \int d^dy\,K(x,t|y,s)\delta\theta(y,s) \tag{A.8}$$

with positive evolution kernel K (by the Feynman-Kac formula).

Equation (A.5) follows from (A.1) assuming a running wave form

$$\theta(t,x) = \vartheta(x - v_\star\,t) \tag{A.9}$$

Linear analysis consideration fix the asymptotic front speed. Setting

$$\vartheta(x) = \delta\vartheta(x) \tag{A.10}$$

linearising around the asymptotic unity for $x \downarrow \infty$ gives

$$\frac{\kappa}{2}\frac{d^2\delta\vartheta}{dx^2} + v_\star\frac{d\delta\vartheta}{dx} + c_0\,\delta\vartheta = 0 \tag{A.11}$$

as

$$\frac{df}{d\theta} = c(\theta) + \theta\,c'(\theta) \tag{A.12}$$

The associated eigenvalue equation has the solutions

$$\lambda_\pm = -\frac{v_\star}{\kappa} \pm \sqrt{\left(\frac{v_\star}{\kappa}\right)^2 - \frac{2\,c_0}{\kappa}} \tag{A.13}$$

Consistency with monotonic decay imposes

$$v_\star \geq \sqrt{2\,c_0\,\kappa}\,, \qquad \Rightarrow \qquad \vartheta(x - v_\star\,t) \geq \vartheta(x - \sqrt{2\,c_0\,\kappa}\,t) \tag{A.14}$$

B Proof of (4.13) for smooth $\mathcal{H}$

Namely if

$$\mathcal{L} = a \tag{B.1}$$

then by definition for $\lambda > 0$

$$a \geq p \cdot v - \mathcal{H}(p, q, t) = \frac{\tilde{p}}{\lambda} \cdot v - \mathcal{H}\left(\frac{\tilde{p}}{\lambda}, q, t\right) \tag{B.2}$$

whence it follows

$$\tilde{p} \cdot v \leq \lambda\, a + \lambda\, \mathcal{H}\,(\tilde{p}, q, t) \tag{B.3}$$

The opposite inequality in the case of *smooth* $\mathcal{H}$ follows from

$$\frac{d}{d\lambda}\left\{a\,\lambda + \lambda\mathcal{H}\left(\frac{p}{\lambda}, q, t\right)\right\} = a + \mathcal{H}\left(\frac{p}{\lambda}, q, t\right) - \frac{p}{\lambda} \cdot \frac{\partial \mathcal{H}}{\partial \tilde{p}}(\tilde{p}, q, t)\Big|_{\tilde{p}=\frac{p}{\lambda}} \tag{B.4}$$

at stationarity $\lambda = \lambda^\star$ this yields

$$a = \mathcal{L} \tag{B.5}$$

thus

$$\max_{\mathcal{L}=a}\{p \cdot v\} \geq p \cdot \frac{\partial \mathcal{H}}{\partial \tilde{p}}(\tilde{p}, q, t)\Big|_{\tilde{p}=\frac{p}{\lambda}} = \lambda^\star\, a + \lambda^\star\, \mathcal{H}\left(\frac{p}{\lambda^\star}, q, t\right) \tag{B.6}$$

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