Zero-range processes

with applications to transport in random media

Otto Pulkkinen
University of Jyväskylä

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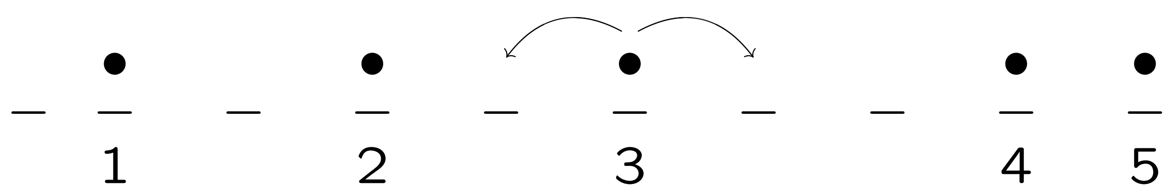

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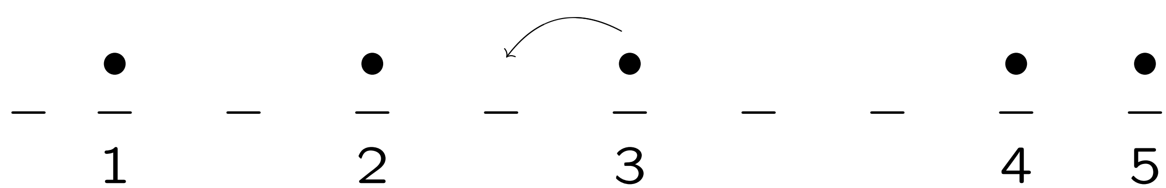

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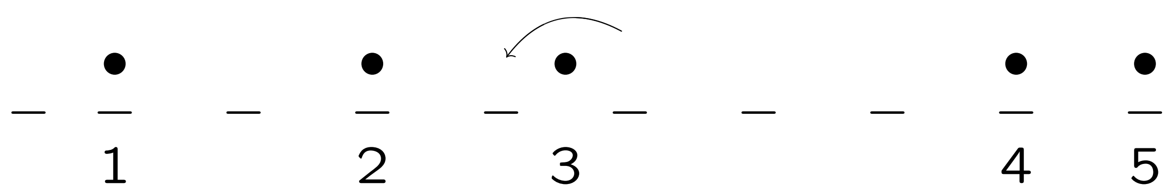

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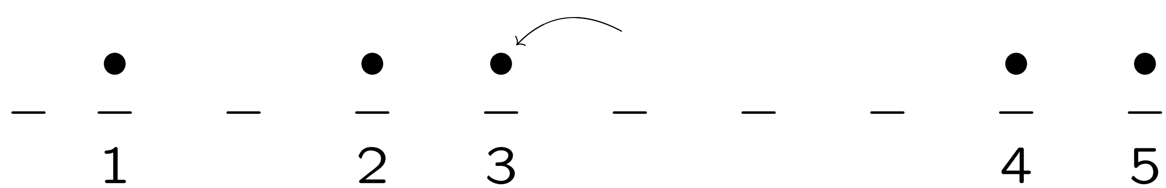

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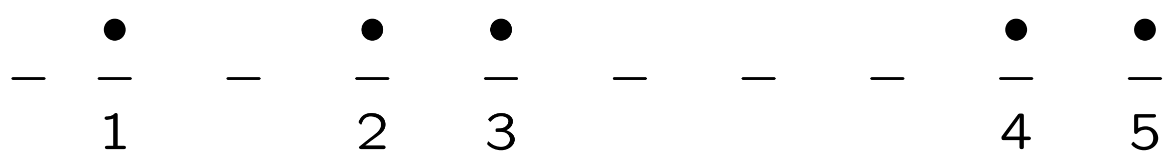

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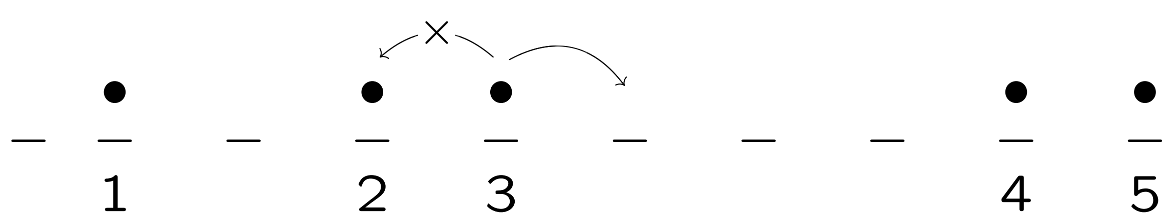

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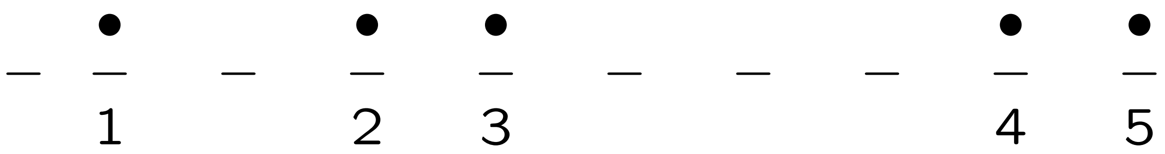

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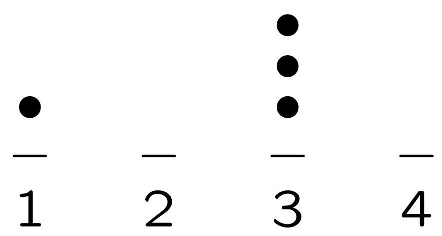

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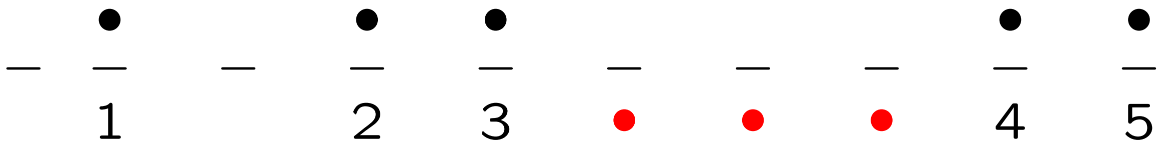

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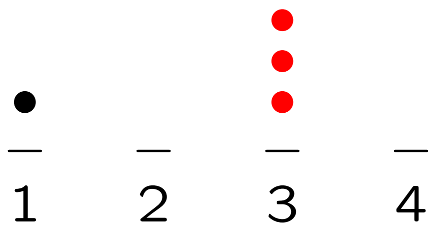

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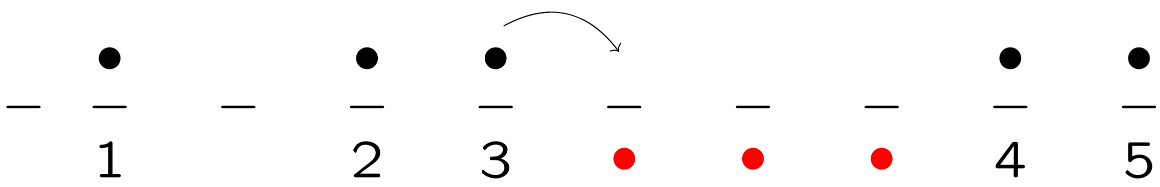

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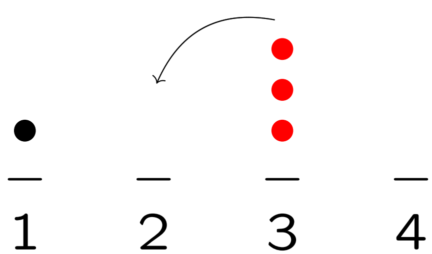

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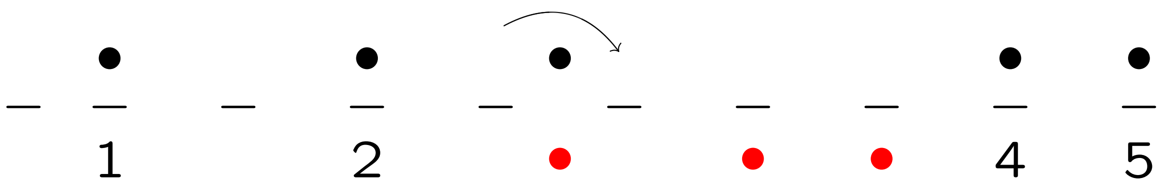

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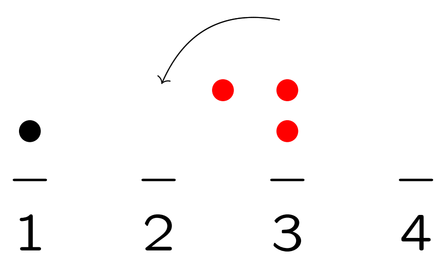

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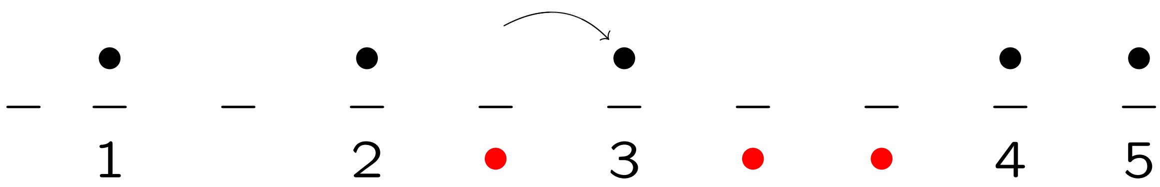

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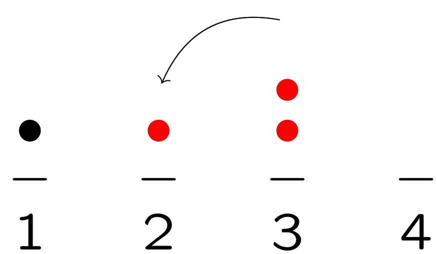

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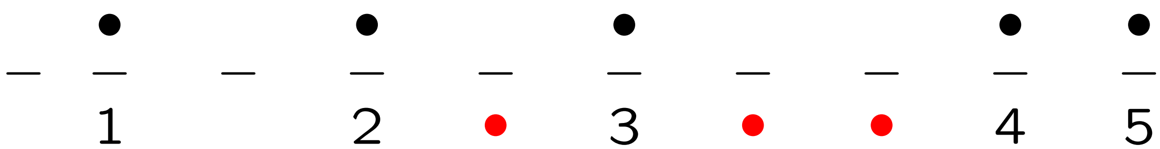

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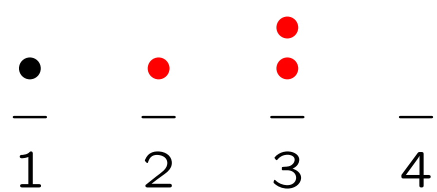

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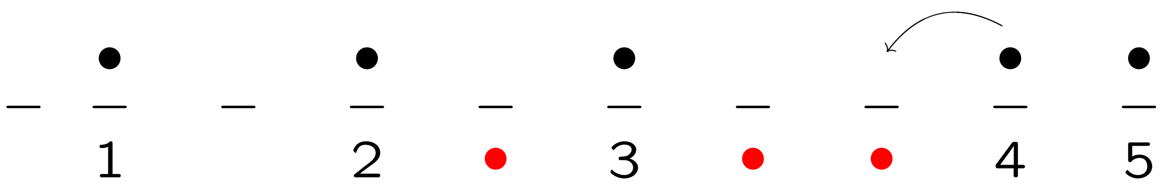

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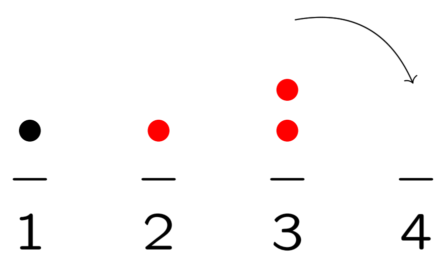

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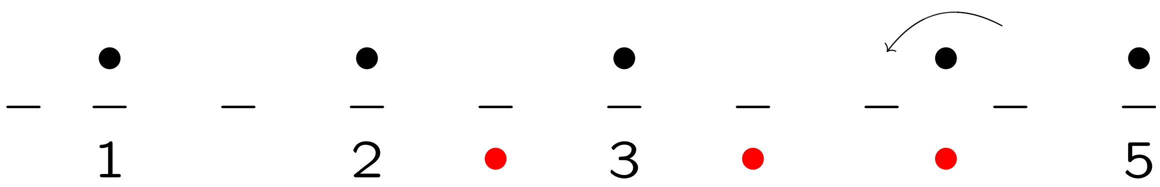

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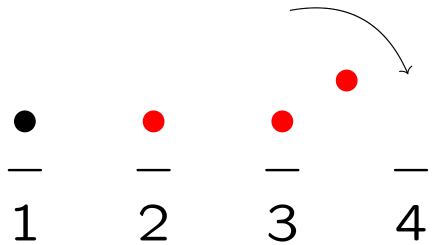

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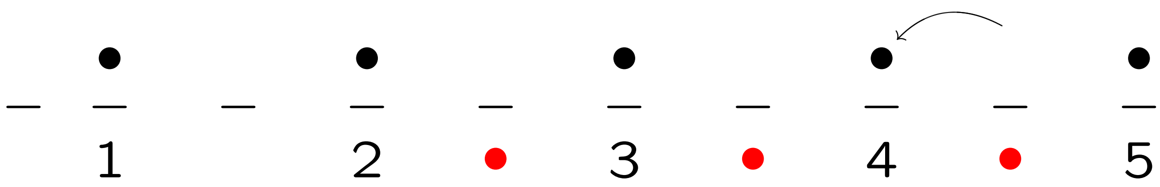

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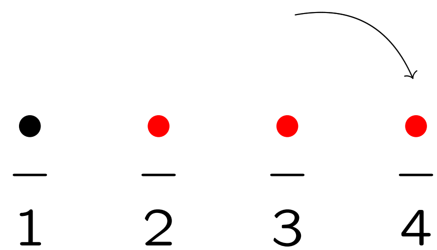

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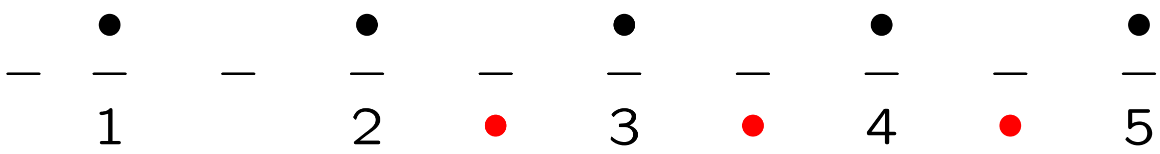

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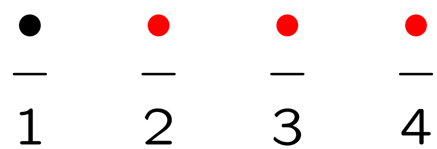
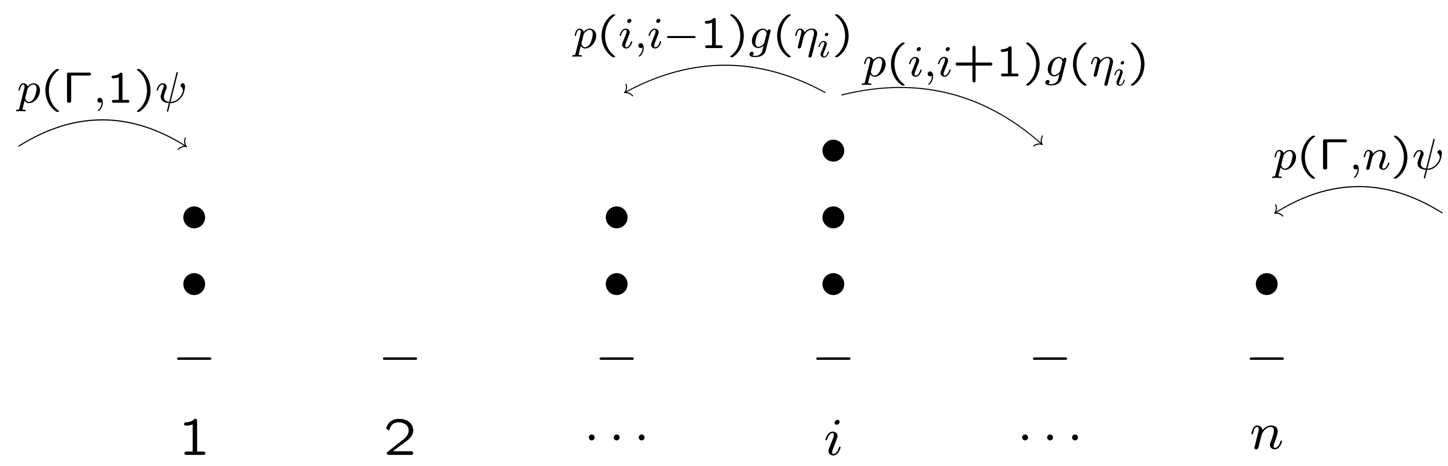


General zero-range process on the set $\{1, \ldots, n\}$



- Interaction function $g : \mathbb{N} \longrightarrow (\epsilon, \infty)$ with bounded increments
- Irreducible matrix of one-particle transition probabilities $p(i, j)$
- Reservoir fugacity $\psi \geq 0$

Stationary distributions: Open systems

Grand-canonical distributions

$$\nu(\eta) = \prod_{i=1}^{n} \frac{1}{\mathcal{Z}(\phi_i)} \frac{\phi_i^{\eta_i}}{g!(\eta_i)}, \quad g!(\eta_i) = \prod_{k=1}^{\eta_i} g(k)$$

$$\mathcal{Z}(\phi_i) = \sum_{\eta_i=0}^{\infty} \frac{\phi_i^{\eta_i}}{g!(\eta_i)}$$

- The fugacities ϕ_i satisfy the traffic equations

$$\phi_i = \sum_j p(j,i)\phi_j \quad \text{with} \;\; \phi_\Gamma = \psi.$$

- Ergodicity: At least for $\forall i \; \phi_i < \liminf_{k\to\infty} g(k) =: \Phi$

Stationary distributions: Closed systems

Canonical distributions for total of $[\rho n]$ particles

$$\mu(\eta) = \frac{1}{Z} \prod_{i=1}^{n} \frac{\phi_i^{\eta_i}}{g!(\eta_i)} \quad \text{for } \eta \in \Omega = \{\xi \in \mathbb{Z}_+^n : \sum_{i=1}^{n} \xi_i = [\rho n]\}$$

$$Z = \sum_{\eta \in \Omega} \prod_{i=1}^{n} \frac{\phi_i^{\eta_i}}{g!(\eta_i)}$$

- Fugacities ϕ_i can be chosen from the set of positive solutions to

$$\phi_i = \sum_j p(j,i)\phi_j$$

- The process is ergodic for positive interaction functions.

Idea of the proof:

1. $g(\eta_j + 1)\mu(\eta^{i,j}) = g(\eta_i)\frac{\phi_j}{\phi_i}\mu(\eta),$

2. The fugacities ϕ_i satisfy the traffic equations

$$\phi_i = \sum_j p(j,i)\phi_j$$

Thus the stationary probabilities satisfy the partial balance equations

$$\sum_j [p(j,i)g(\eta_j + 1)\mu(\eta^{i,j}) - p(i,j)g(\eta_i)\mu(\eta)] = 0, \quad i = 1, \ldots, n.$$

- For an open system, there is an equation for the reservoir

$$\sum_j [p(j,\Gamma)g(\eta_j + 1)\nu(\eta^{\Gamma,j}) - p(i,\Gamma)g(\eta_i)\nu(\eta)] = 0.$$

Example: Independent particles, $g(\eta_i) = \eta_i$

- Grand-canonical distributions

$$\nu(\eta) = \prod_{i=1}^{n} \frac{\phi_i^{\eta_i}}{\eta_i!} e^{-\phi_i}$$

$$R(\phi_i) := E^{\nu} X_i = \phi_i \frac{\mathcal{Z}'(\phi_i)}{\mathcal{Z}(\phi_i)} = \phi_i$$

- Canonical distributions with $[\rho n]$ particles

$$\mu(\eta) = \binom{[\rho n]}{\eta_1 \cdots \eta_n} \prod_{i=1}^{n} \phi_i^{\eta_i},$$

where we have chosen $\sum_i \phi_i = 1$.

Example: Independent particles on a torus, p translation invariant

- Canonical marginal distribution on a set A,

$$\mu_A(\eta) = \frac{[\rho n]!(n-|A|)^{[\rho n]-\sum_{i\in A}\eta_i}}{([\rho n]-\sum_{i\in A}\eta_i)!n^{[\rho n]}} \prod_{i\in A}\frac{1}{\eta_i!}$$

$$\stackrel{n\to\infty}{\longrightarrow} \prod_{i\in A}\frac{\rho^{\eta_i}}{\eta_i!}e^{-\rho} = \nu_{A,\rho}(\eta)$$

Equivalence of ensembles!

- Idea: Choose the fugacity according to $\phi(\rho) = R^{-1}(\rho)$, where

$$R(\phi) = E^{\nu}X_i = \phi\frac{\mathcal{Z}'(\phi)}{\mathcal{Z}(\phi)}.$$

is strictly increasing

Equivalence of ensembles

- Product form distributions for both ensembles

- Global constraint $\sum_i X_i = [\rho n]$ in the canonical ensemble

Approximation by independent variables plausible

- But: For strong attractive interactions, $\Phi = \liminf g(k) < \infty$, $\mathcal{Z}(\Phi) < \infty$, and even $R(\Phi) < \infty$.

$$\phi(\rho) = ? \text{ for } \rho > \rho_c := R(\Phi).$$

Theorem

[Großkinsky, Schütz and Spohn, *J. Stat Phys.* **113** (2003)]

Consider a homogeneous zero-range process on an n-site torus with density ρ. Let the grand-canonical distribution have the truncated fugacity

$$\phi(\rho) = \begin{cases} R^{-1}(\rho) & \text{for } \rho < \rho_c, \\ \Phi & \text{for } \rho \geq \rho_c. \end{cases}$$

Then $\mu[f] \stackrel{n\to\infty}{\longrightarrow} \nu[f]$ for all bounded, continuous real functions f depending only on finite number of coordinates.

Sketch of the proof

- Show that the $|A|$-dimensional marginal measures converge weakly. A sufficient condition*: the relative entropy

$$S(\mu_A|\nu_A) = \sum_{\eta\in\Omega(\mathbb{T}_n,[\rho n])} \mu_A(\eta)\,\log\left(\mu_A(\eta)/\nu_A(\eta)\right)$$

vanishes as $n \to \infty$.

- For Y_i, X_i respectively denoting the canonical and grand-canonical variables, and $\eta \in \Omega(\mathbb{T}_n, [\rho n])$,

$$\frac{\mu(\eta)}{\nu(\eta)} = \frac{P^\mu\left(\cap_{i=0}^{n-1}\{Y_i(t)=\eta_i\}\right)}{P^\nu\left(\cap_{i=0}^{n-1}\{X_i(t)=\eta_i\}\right)} = \frac{\mathcal{Z}(\phi(\rho))^n}{Z}\prod_{i=0}^{n-1}\frac{g!(\eta_i)}{g!(\eta_i)\phi^{\eta_i}}$$

$$= \;=\frac{\mathcal{Z}(\phi(\rho))^n}{Z\phi^{[\rho n]}} = \frac{1}{P^\nu\left(\sum_{i=0}^{n-1}X_i(t)=[\rho n]\right)}.$$

*[Kipnis and Landim, Scaling Limits of Interacting Particle Systems (Springer, 1999)]

- Thus the relative entropy of the full canonical and grand-canonical measures equals $-\log P^{\nu}\left(\sum_{i=0}^{n-1} X_i(t) = [\rho n]\right)$.

- Joint superadditivity of the relative entropy in the decomposition into marginal measures on disjoint sets,

$$S(\mu|\nu) \geq S(\mu_B|\nu_B) + S(\mu_C|\nu_C),$$

implies that

$$S(\mu_A|\nu_A) \leq -\frac{1}{[n/|A|]} \log P^{\nu}\left(\sum_{i=0}^{n-1} X_i(t) = [\rho n]\right).$$

Need to show that the probability decays subexponentially.

1. $\rho < \rho_c$ i.e. $\phi(\rho) < \Phi$:

   The grand-canonical measures have exponential moments, and hence a finite second moment. Thus the the local limit theorem for the convergence to the normal law shows that the probability in question decays as $n^{-1/2}$.

2. $\rho \geq \rho_c$ i.e. $\phi(\rho) = \Phi$:

   The tail of the of the summand distribution vanishes subexponentially and the second moment of the distribution might not exist. These cases are covered by the local limit theorems for attraction to non-normal distributions*:

*Gnedenko and Kolmogorov 1968, Ibragimov 1971

- Exactly at $\rho = \rho_c$,

$$P^\nu\left(\sum_{i=0}^{n-1} X_i(t) = [\rho_c n]\right) \gtrsim n^{-2}.$$

- For $\rho > \rho_c$,

$$\begin{aligned} &P^\nu\left(\sum_{i=0}^{n-1} X_i(t) = [\rho n]\right) \\ &\quad \geq P^\nu\left(X_0 = [\rho n] - [\rho_c(n-1)], \sum_{i=1}^{n-1} X_i(t) = [\rho_c(n-1)]\right) \\ &\quad = P^\nu\Big(X_0 = [\rho n] - [\rho_c(n-1)]\Big) P^\nu\left(\sum_{i=1}^{n-1} X_i(t) = [\rho_c(n-1)]\right). \end{aligned}$$

And we are done!

Comments

- Equivalence on finite sets and of bounded functions only
- Description on small sets in the large n limit – even for $\rho > \rho_c$.
- Extra clump of size $(\rho - R(\Phi))\,n$ of particles on a set of size $o(n)$.

Condensation!

- Example [Evans, *Brazilian J. Phys.* **30** (2000); Großkinsky, Schütz and Spohn, *J. Stat Phys.* **113** (2003)]:

$$g(k) = 1 + \frac{b}{k} + O(k^{-1-\delta}), \qquad \nu_\phi(k) \sim \phi^k k^{-b}, \qquad \Phi = 1$$

Applications

- Spatial separation in shaken sand [Lipowski and Droz, *Phys. Rev. E* **65** (2002)]

- Jamming transitions in asymmetric exclusion processes with particle-wise disorder: inhomogeneous rates g_i [Krug and Ferrari, *J. Phys. A* **29** (1996); Evans, *Europhys. Lett.* **36** (1996)]

- Kafri *et al.* [*Phys. Rev. Lett.* **89** (2002)]: General criterion for phase separation in driven one-dimensional systems

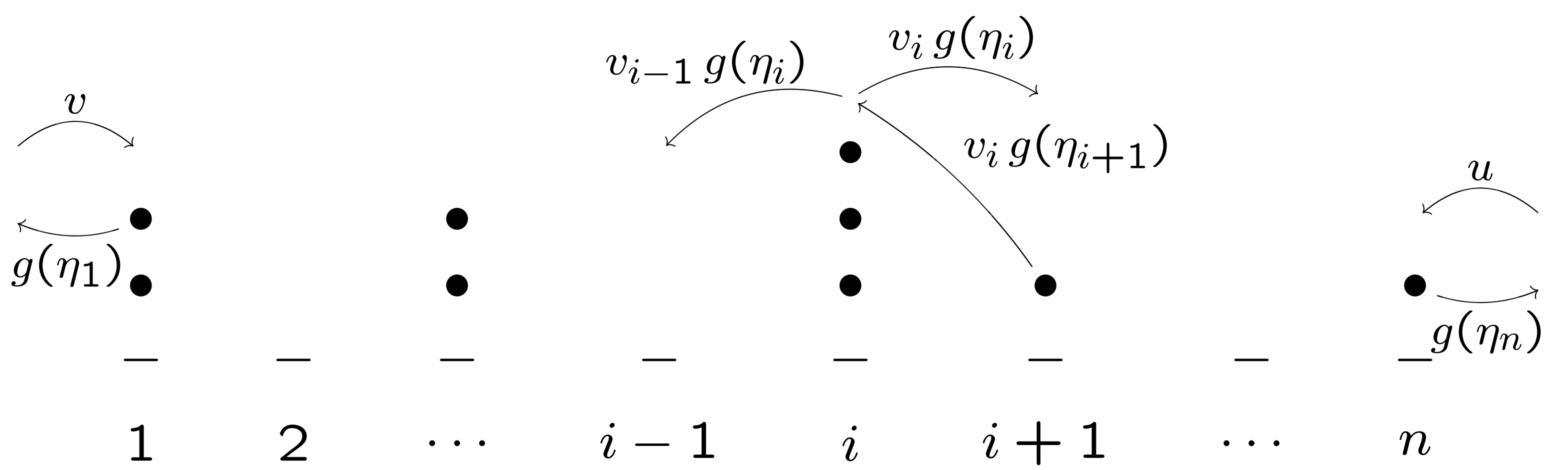
- Phase transitions in evolving networks [Pulkkinen and Merikoski, *J. Stat. Phys.* **119** (2005)]

Contents

- A brief introduction to zero-range processes
- Transport in random media
  - Stationary states
  - Motion of a tagged particle
- A Bayesian postlude

Transport in random media

- **Random barrier model**
- Single particle rate $i \to i+1$ equals the rate $i+1 \to i$ in the bulk
- Boundary drive by reservoirs with fugacities v and u



Given set of rates $(v_i)_{i=1}^n$

- Stationary state

$$\nu(\eta) = \prod_{i=1}^{n} \frac{1}{\mathcal{Z}(\phi_i)} \frac{\phi_i^{\eta_i}}{g!(\eta_i)},$$

where the fugacities are given by

$$\phi_i = u\frac{s_i}{s_{n+1}} + v\left(1 - \frac{s_i}{s_{n+1}}\right), \qquad s_i = 1 + \sum_{j=1}^{i-1} v_j^{-1}.$$

- The fugacity profile is trivial in equilibrium ($u = v$), and a monotone function bounded by the reservoir fugacities otherwise.

Random rates $(v_i)_{i=1}^n$

1. $\mathbb{E}v_i^{-1} < \infty$

- Continuum version of the sequence s_i

$$S_{n,1}(x) = n^{-1}\Big(1 + \sum_{i=1}^{\lfloor (n+1)x \rfloor - 1} v_i^{-1}\Big), \qquad x \in [0,1]$$

- Continuum analogue of the fugacity

$$\phi_n(x) := u\frac{S_{n,1}(x)}{S_{n,1}(1)} + v\Big(1 - \frac{S_{n,1}(x)}{S_{n,1}(1)}\Big)$$

$$\overset{n\to\infty}{\longrightarrow} ux + v(1-x)$$

uniformly almost surely.

2. $\mathbb{E}v_i^{-1} = \infty$

- Assume that for some $\alpha \in (0,1)$ and slowly varying h,

$$\mathbb{P}(v_i^{-1} > r) = (c + o(1))\, h(r) r^{-\alpha}$$

- New scaling for the auxiliary sequence

$$S_{n,\alpha}(x) = n^{-1/\alpha}\Big(1 + \sum_{i=1}^{\lfloor (n+1)x \rfloor - 1} v_i^{-1}\Big)$$

$$\xrightarrow{n\to\infty} S(x) = \iint_{\mathbb{R}_+ \times [0,x]} r\, J(dr\, dy).$$

α-stable subordinator

- The Poisson random measure $J(dr\,dy)$ counts the jumps within $dr\,dy$. Its intensity equals

$$\nu(dr\,dy) = cr^{-\alpha-1}\,dr\,dy.$$

- For $\alpha \in (0,1)$ and $x > 0$, $\qquad \iint_{\mathbb{R}_+ \times [0,x]} r\nu(dr\,dy) = \infty$

- The fugacity profiles converge in distribution:

$$\phi_n(x) = u\frac{S_{n,\alpha}(x)}{S_{n,\alpha}(1)} + v\left(1 - \frac{S_{n,\alpha}(x)}{S_{n,\alpha}(1)}\right)$$

$$\overset{n\to\infty}{\longrightarrow} u\frac{S(x)}{S(1)} + v\left(1 - \frac{S(x)}{S(1)}\right) =: \phi(x).$$

- $\mathbb{E}\phi(x) = ux + v(1-x).$

Interlude: Skorohod topology

- The topology of uniform convergence is not suited for processes with discontinuities.

- Skorohod's idea:

$$d_S(\phi, \xi) = \inf\{\sup_x |\phi(x) - \xi(\lambda(x))| \vee \sup_x |x - \lambda(x)|\},$$

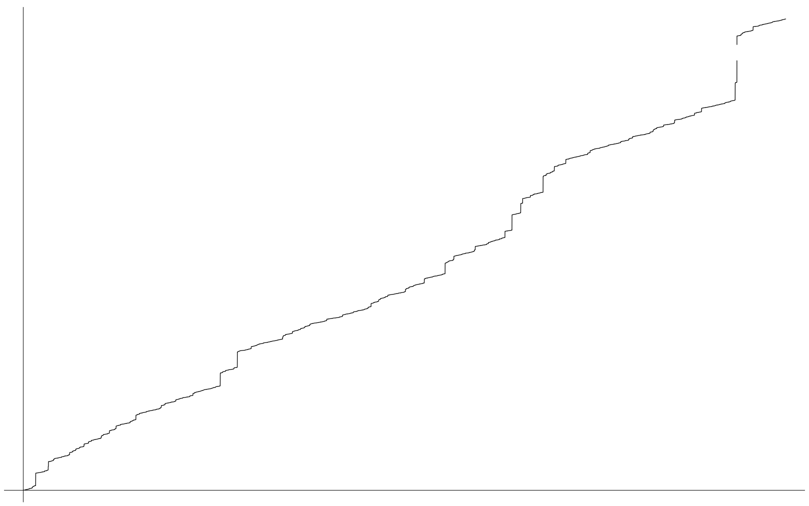
  where the infimum is taken over all continuous, one-to-one functions.

- The weak convergence of the fugacity profiles is guaranteed by the almost sure continuity of the map

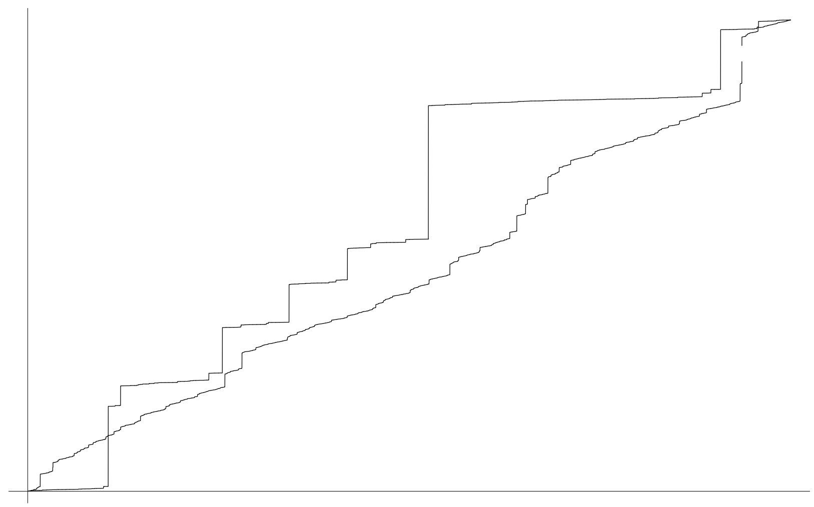
$$\{S_{n,\alpha}(x) : x \in [0,1]\} \mapsto \{\frac{S_{n,\alpha}(x)}{S_{n,\alpha}(1)} : x \in [0,1]\}$$

  in this topology.

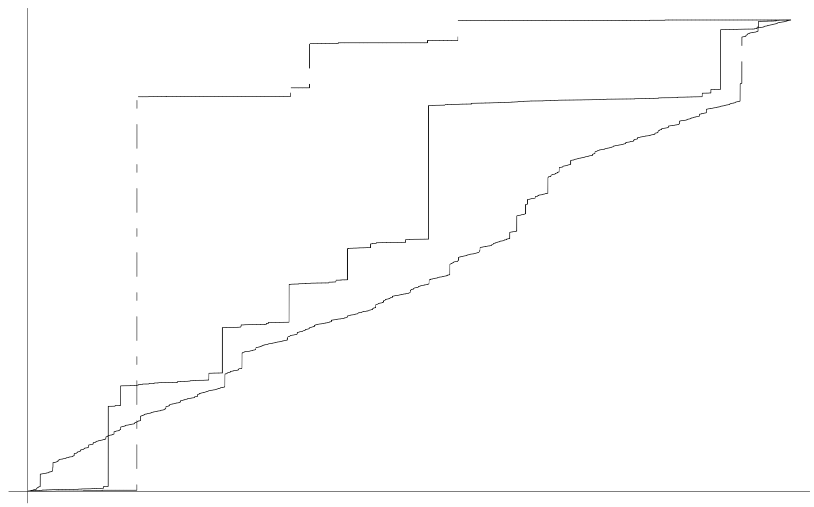
Fugacity $\phi(x)$ for $\alpha = 0.95$



Fugacity $\phi(x)$ for $\alpha = 0.95, 0.7$



Fugacity $\phi(x)$ for $\alpha = 0.95, 0.7, 0.3$



Characterization of the fugacity profiles

Let us evaluate the increments of the fugacity process ($x_i + y_i < x_{i+1}$):

$$
\begin{aligned}
&\mathbb{E} \prod_{i=1}^{k} \frac{S(x_i + y_i) - S(x_i)}{S(1)} \\
&= \mathbb{E} \left(\prod_{i=1}^{k} \left(S(x_i + y_i) - S(x_i) \right) \right) \int_0^\infty \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda S(1)} \, d\lambda \\
&= \int_0^\infty \frac{\lambda^{k-1}}{(k-1)!} e^{-x_1 c \lambda^\alpha} \left(-\frac{d}{d\lambda} e^{-y_1 c \lambda^\alpha} \right) e^{-(x_2 - x_1 - y_1) c \lambda^\alpha} \\
&\qquad \cdots \left(-\frac{d}{d\lambda} e^{-y_k c \lambda^\alpha} \right) e^{-(1 - x_k - y_k) c \lambda^\alpha} \, d\lambda \\
&= \int_0^\infty \frac{\lambda^{k-1}}{(k-1)!} c^k \alpha^k \lambda^{(\alpha-1)k} e^{-c\lambda^\alpha} \, d\lambda = \alpha^{-1} \prod_{i=1}^{k} \alpha y_i
\end{aligned}
$$

Characterization of the fugacity profiles

- Thus $\phi(x)$ has negative correlations

$$\mathbb{E}\prod_{i=1}^{k}[\phi(x_i+y_i)-\phi(x_i)]-\prod_{i=1}^{k}\mathbb{E}[\phi(x_i+y_i)-\phi(x_i)]$$
$$=(u-v)^k\left(\alpha^{k-1}-1\right)\prod_{i=1}^{k}y_i.$$

- Also the pointwise distribution of the fugacity can be calculated*,

$$\mathbb{P}(\phi(x)\le t)=\frac{1}{2}$$
$$-\frac{\operatorname{sgn}(u-v)}{\pi\alpha}\arctan\left(\frac{x|u-t|^\alpha-(1-x)|t-v|^\alpha}{x|u-t|^\alpha+(1-x)|t-v|^\alpha}\tan\frac{\pi\alpha}{2}\right).$$

*By the methods of [Regazzini *et al.*, *Ann. Statist.* **31** (2003)]

Motion of a tagged particle

- Y_t – position of the tagged particle in a stationary zero-range process
- T_J – time of exit from a set J
- Diffusions and random walks can be defined by their Green's function G_J and speed measure $m(dx)$ such that*

$$E^x \int_0^{T_J} f(Y_t)\, dt = \int_{-\infty}^{\infty} f(y) G_J(x,y)\, m(dy),$$

*[Revuz and Yor, Continuous Martingales and Brownian Motion (Springer, 1999)]

Given rates v_i

- Let $s(x) = 1 + \sum_{i=1}^{\lfloor x \rfloor - 1} v_i^{-1}$ denote the continuum version of the sum s_i, and let

$$G_J(x,y) = \frac{[s(x \wedge y) - s(a)][s(b) - s(x \vee y)]}{s(b) - s(a)} I\{x, y \in (a,b)\},$$

- Define the speed measure as $m_n(dx) = \sum_{i=1}^{n} \frac{\mathcal{Z}'(\phi_i)}{\mathcal{Z}(\phi_i)} \delta_i(dx)$, where ϕ_i is the fugacity and δ_i is the Dirac measure at i.

- Let $E^{a \to b}$ be the expectation with respect to Palm measure under which the particle travels with certainty from a to b without returning to a. $E^{a \hookleftarrow b}$ is conditioned on return to a without reaching b*.

*See [Kook and Serfozo, *Ann. Appl. Probab.* **3** (1993)] for the construction.

Then the following equalities hold:

$$E^{a\to b}\int_0^{T_J} f(Y_t)\,dt = \int_{-\infty}^{\infty} f(y)G_J(y,y)\,m_n(dy)$$

$$E^{a\leftrightarrow b}\int_0^{T_J} f(Y_t)\,dt = \int_{-\infty}^{\infty} f(y)\frac{v_a(s(b)-s(y))^2}{(s(b)-s(a))(s(b)-s(a+1))}\,m_n(dy).$$

Idea of the proof:

Let $T_J(i)$ denote the time spent at $i \in J$ before the exit time T_J. Then

$$E^{a\to b}\int_0^{T_J} f(Y_t)\,dt = E^{a\to b}\sum_{i=a+1}^{b-1} f(i)T_J(i)$$

$$= \sum_{i=a+1}^{b-1} f(i)E^{a\to b}T_J(i) = \sum_{i=a+1}^{b-1} f(i)p_i(a\to b)\frac{EX_i}{EN_{a\to b}(0,1]}.$$

The last equality is a consequence of a Little law [Kook and Serfozo].

- The probability $p_i(a \to b)$ can be split into a product of two factors:
  1. $\alpha_i(a,b)$, the probability of a particle at i reaching b before a.
  2. $\alpha_i^*(a,b)$, the probability of reaching a before b when it moves as determined by the adjoint generator.

- Simple recursions for these

- The expectations $EX_i = R(\phi_i)$ and $EN_{a\to b} = v_a \phi_a \alpha_{a+1}(a,b)$.

- The second assertion is proved in a similar manner.

Corollary

$$E^i \int_0^{T_J} f(Y_t)\, dt = \int_{-\infty}^{\infty} f(y) G_J(i,y)\, m_n(dy),$$

$$G_J(x,y) = \frac{[s(x \wedge y) - s(a)][s(b) - s(x \vee y)]}{s(b) - s(a)}\, I\{x, y \in (a,b)\},$$

$$m_n(dx) = \sum_{i=1}^{n} \frac{\mathcal{Z}'(\phi_i)}{\mathcal{Z}(\phi_i)}\, \delta_i(dx),$$

- The proof is obtained by first summing up the paths that return to their starting site, and the last journey of the particle to the boundary of J.

Random rates: $\mathbb{E}v_i^{-1} < \infty$

- $J_n = \{\lfloor an \rfloor + 1, \ldots, \lfloor bn \rfloor - 1\}$

- For almost all environments,

$$E^{\lfloor xn \rfloor} \int_0^{T^{(2)}_{J_n}} f\left(\frac{1}{n} Y_{n^2 t}\right) \, dt \to \frac{1}{u-v} \int_a^b f(y) \frac{(y \wedge x - a)(b - y \vee x)}{b-a} \, m(dy)$$

$$m(dy) = \frac{\mathcal{Z}'(uy + v(1-y))}{\mathcal{Z}(uy + v(1-y))} \, dy.$$

Random rates: $\mathbb{E}v_i^{-1} = \infty$, α-stable

- Convergence in distribution

$$E^{\lfloor xn \rfloor} \int_0^{T_{J_n}^{(1+1/\alpha)}} f\left(\frac{1}{n} Y_{n^{1+1/\alpha} t}\right) dt \longrightarrow \int_a^b f(y) \tilde{G}_J(x,y)\, \tilde{m}(dy)$$

$$\tilde{G}_J(x,y) = \frac{[S(x \wedge y) - S(a)][S(b) - S(y \vee x)]}{S(b) - S(a)}.$$

$$\tilde{m}(dy) = \frac{\mathcal{Z}'(\phi(y))}{\mathcal{Z}(\phi(y))} dy,$$

A Bayesian postlude

- Noninteracting particles in equilibrium ($u = v$)
- Dilute one of the reservoirs with a marker, $\rho = c/n$
- Then take the limit $n \to \infty$

$\longrightarrow$ **Cox process** (doubly stochastic Poisson point process)

$$P\left(\cap_{i=1}^{k}\{N(A_i) = m_i\}\right) = \prod_{i=1}^{k} \frac{\Lambda(A_i)^{m_i}}{m_i!} e^{-\Lambda(A_i)}$$

$$\Lambda(A_i) = \int_{A_i \cap [0,1]} \phi(x)\, dx$$

Parametric inference

- What are good estimates for the parameter α if marker particles at $x = (x_1, \ldots, x_k)$ are observed?

Posterior distribution

$$f_{\mathcal{A}|X}(\alpha|x) = \frac{f_{X|\mathcal{A}}(x|\alpha) f_{\mathcal{A}}(\alpha)}{\int f_{X|\mathcal{A}}(x|\alpha) f_{\mathcal{A}}(\alpha)\, d\alpha}$$

$$f_{X|\mathcal{A}}(x|\alpha) = \frac{1}{k!} \mathbb{E}\, e^{-\int_0^1 \phi(x)\, dx} \prod_{i=1}^{k} \phi(x_i)$$

Janossy densities

Non-parametric inference

- What can be deduced about the whole structure of the whole random media on the basis of snapshots with marker particle positions?

- Advanced methods [Karr, Point processes and their statistical inference (Dekker, 1991)]

Thank You