Spectral and Eigenvector Statistics of Random Matrices

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Gaussian random matrices

What do the eigenvalues and eigenvectors of "typical" large matrices look like?

Simple example: Gaussian Unitary Ensemble (GUE). Let $H = (h_{ij})$ be an $N \times N$ Hermitian matrix with entries

$$h_{ij} \;=\; \bar{h}_{ji} \;=\; \frac{1}{\sqrt{N}}\,(x_{ij} + \mathrm{i} y_{ij})\,, \qquad h_{jj} \;=\; \frac{\sqrt{2}}{\sqrt{N}}\,x_{ii}\,,$$

where $(x_{ij} : i \leqslant j)$ and $(y_{ij} : i < j)$ are independent families of independent standard Gaussian random variables.

Denote by $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ the eigenvalues of H. The scaling of the entries of H was chosen so that λ_i is of order one:

$$\mathbb{E}\frac{1}{N}\sum_i \lambda_i^2 \;=\; \mathbb{E}\frac{1}{N}\operatorname{Tr} H^2 \;=\; \frac{1}{N}\sum_{i,j}\mathbb{E}|h_{ij}|^2 \;=\; 2\,.$$

Therefore the eigenvalue spacing is typically of order N^{-1}.

Eigenvalue density: macroscopic distribution

Let $\rho_N(\lambda_1, \dots, \lambda_N)$ be the symmetric probability density of the eigenvalues of H. Define the k-point correlation function

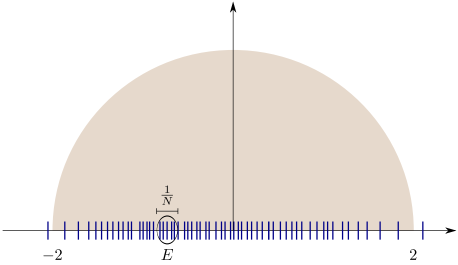
$$\rho_N^{(k)}(\lambda_1, \dots, \lambda_k) \;:=\; \int \rho_N(\lambda_1, \dots, \lambda_N)\, \mathrm{d}\lambda_{k+1} \cdots \mathrm{d}\lambda_N \,.$$

Wigner (1955): The macroscopic statistics follow the semicircle law:

$$\lim_{N \to \infty} \int_{E-\ell}^{E+\ell} \bigl(\rho_N^{(1)}(x) - \rho_{sc}(x)\bigr)\, \mathrm{d}x \;=\; 0\,, \qquad \rho_{sc}(x) \;:=\; \frac{1}{2\pi} \sqrt{[4 - x^2]_+}\,,$$

for any fixed $\ell > 0$.

Eigenvalue correlations: microscopic distribution



Gaudin, Mehta, Dyson (1960s): The local statistics of GUE are given by the sine kernel:

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}(E)^k} \rho_N^{(k)}\left(E + \frac{\alpha_1}{N\rho_{sc}(E)}, \dots, E + \frac{\alpha_k}{N\rho_{sc}(E)}\right) = \det\big[S(\alpha_i - \alpha_j)\big]_{i,j=1}^k$$

for any fixed $E \in (-2, 2)$, where $S(\alpha) := \frac{\sin \pi\alpha}{\pi\alpha}$.

Distribution at the edge

Order the eigenvalues: $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$.

What is the typical eigenvalue spacing near the spectral edges ± 2?

If we take the semicircle law seriously, we expect

$$\int_{\lambda_{N-100}}^{2} N\rho_{sc}(x)\,\mathrm{d}x \;=\; 100 \qquad \Longrightarrow \qquad \lambda_{N-100} \;\approx\; 2 - CN^{-2/3}\,.$$

Therefore the eigenvalue spacing near the edges should be $N^{-2/3}$.

Tracy, Widom (1994): The extreme eigenvalues of GUE converge in distribution,

$$\lim_{N\to\infty} \mathbb{P}(N^{2/3}(\lambda_N - 2) \leqslant s) \;=\; F_2(s)\,,$$

where F_2 can be explicitly computed.

Universality

Wigner's vision: Any disordered system exhibits one of the following:

A. Poisson statistics, for systems with little or no correlations.

B. Random matrix statistics, for systems with high correlations.

Fundamental belief: universality of random matrices. The macroscopic statistics depend on the particular model, but the microscopic statistics are independent of the details of the model, and depend only on its symmetries.

Generalized Wigner matrices

- Let $H = (h_{ij})$ be an $N \times N$ Hermitian matrix with $h_{ij} = \sigma_{ij} x_{ij}$, where $\sigma_{ij} > 0$ is deterministic, $(x_{ij} : i \leqslant j)$ are independent,

$$\mathbb{E} x_{ij} = 0 \,, \qquad \mathbb{E}|x_{ij}|^2 \;=\; 1 \,, \qquad \text{and} \quad \mathbb{E} x_{ij}^2 \;=\; 0 \text{ for } i \neq j \,,$$

  and the variances satisfy

$$\sum_j \sigma_{ij}^2 \;=\; 1 \,, \qquad \frac{c}{N} \leqslant \sigma_{ij}^2 \leqslant \frac{C}{N} \,.$$

- Similar definition for real symmetric matrices.

- Universality conjecture: The local eigenvalue statistics of H are the same as those of the associated Gaussian ensemble.

  Conjectured by Wigner, Dyson, Gaudin, Mehta for Wigner matrices ($\sigma_{ij}^2 = N^{-1}$).

Bulk universality

Theorem (Erdős, K, Schlein, Yau, Yin; 2009 – 2011)

Bulk universality holds for generalized Wigner matrices provided that

$$\mathbb{E}|x_{ij}|^{4+\varepsilon} \;\leqslant\; C\,,$$

i.e. for $-2 < E < 2$ and $b = N^{-1+\delta}$ we have

$$\operatorname*{w-lim}_{N\to\infty} \int_{E-b}^{E+b} \frac{\mathrm{d}E'}{2b} \big(\rho_N^{(k)} - \rho_{\mu,N}^{(k)}\big)\bigg(E' + \frac{\alpha_1}{N}, \dots, E' + \frac{\alpha_k}{N}\bigg) \;=\; 0\,,$$

where μ stands for GUE or GOE depending on the symmetry of H.

Tao, Vu (2009 – 2010): Bulk universality for Wigner matrices holds if $\mathbb{E}|x_{ij}|^K < C$ for some large enough K and the first four moments of the entries of H match those of GUE/GOE.

Edge universality

Theorem (Erdős, K, Yau, Yin; 2010 – 2011)

Suppose that $H^{\mathbf{v}}$ and $H^{\mathbf{w}}$ are generalized Wigner matrices. Assume that two moments match, i.e. $\mathbb{E}(x_{ij}^{\mathbf{v}})^2 = \mathbb{E}(x_{ij}^{\mathbf{w}})^2$, and that for both ensembles we have

$$\mathbb{E}|x_{ij}|^{12} \leqslant C\,.$$

Then for all $s \in \mathbb{R}$ we have

$$\mathbb{P}^{\mathbf{v}}\big(N^{2/3}(\lambda_N - 2) \leqslant s\big) - \mathbb{P}^{\mathbf{w}}\big(N^{2/3}(\lambda_N - 2) \leqslant s\big) \to 0\,.$$

Similarly: convergence of correlation functions of eigenvalues near edge.

Corollary: The Tracy-Widom law holds for real symmetric Wigner matrices and Hermitian generalized Wigner matrices (using Johansson (2009)).

- Auffinger, Ben Arous, Péché (2010): If $\mathbb{E}|x_{ij}|^{4-\varepsilon} = \infty$, edge universality does not hold.
- Sinai, Soshnikov, Ruzmaikina, Sodin (1992 - 2009): Edge universality if the law of x_{ij} is symmetric and has a finite high moment.
- Tao, Vu (2010): Edge universality assuming third moments vanish and subexponential decay.

Distribution of eigenvectors

Let $\mathbf{u}_1, \dots, \mathbf{u}_N$ be the ℓ^2-normalized eigenvectors associated with the eigenvalues $\lambda_1 \leqslant \cdots \leqslant \lambda_N$.

Let H be a GUE matrix. The law of H can be written as

$$\mathbb{P}(\mathrm{d}H) \;=\; \frac{1}{Z} \mathrm{e}^{-N \operatorname{Tr} H^2/2} \, \mathrm{d}H \, .$$

Thus, $\mathbb{P}$ is invariant under unitary conjugations $H \mapsto VHV^*$, where $V \in \mathrm{U}(N)$.

Consequences:

- The eigenvector matrix

$$U \;:=\; [\mathbf{u}_1, \dots, \mathbf{u}_N]$$

  is Haar-distributed on $\mathrm{U}(N)$.
- Each eigenvector $\mathbf{u}_\alpha$ is uniformly distributed on $S^{N-1} \subset \mathbb{C}^N$.
- The eigenvectors U are independent of the eigenvalues $\lambda_1, \dots, \lambda_N$.

Universality of extreme eigenvectors

If $\mathbf{u}_\alpha$ is uniformly distributed on S^{N-1}, then the quantity

$$\sqrt{N}\, u_\alpha(i)$$

is of order one, for all $i = 1, \dots, N$.

Theorem (K, Yin; 2011)

The distribution of edge eigenvectors is universal: Let H be a Wigner matrix whose entries have subexponential decay, and V be a GUE/GOE matrix. Then for any test function function θ we have

$$\lim_{N \to \infty} (\mathbb{E}^H - \mathbb{E}^V) \theta\Big(N \overline{u}_{\alpha_1}(i_1) u_{\alpha_1}(j_1), \dots, N \overline{u}_{\alpha_k}(i_k) u_{\alpha_k}(j_k)\Big) \;=\; 0\,,$$

provided that $\alpha_1, \dots, \alpha_k \leqslant N^\varepsilon$ for some small $\varepsilon > 0$.

- This characterizes the distribution of the eigenvector components completely.
- In the bulk, i.e. under no restriction on α_i, the same result holds if four moments of H match those of V. This was also proved by Tao, Vu (2011).

The Erdős-Rényi graph

A random graph on the N vertices $\{1, \ldots, N\}$, where each edge of the complete graph is chosen independently with probability p.

Example:
($N = 100$, $p = 0.01$)

Adjacency matrix $A = (a_{ij})$: real symmetric with

$$a_{ij} = \frac{\gamma}{q} \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p\,, \end{cases}$$

where $q := \sqrt{pN}$ and $\gamma = (1 - p)^{-1/2}$ so that $\operatorname{Var} a_{ij} = N^{-1}$.
Note that $\mathbb{E}\, a_{ij} \neq 0$.

Each column typically has $pN = q^2$ nonvanishing entries.

Heuristics for connectivity

For simplicity, consider the directed graph with (a_{ij}) i.i.d. (no symmetry).

$$\begin{aligned}
\mathbb{P}\big[\text{no vertex is isolated}\big] &= \mathbb{P}\Bigg[\bigcap_{i=1}^N \big\{\text{vertex } i \text{ is isolated}\big\}^c\Bigg] \\
&= \mathbb{P}\Bigg[\bigcap_{i=1}^N \Bigg(\bigcap_{j=1}^N \{a_{ij} = 0\}\Bigg)^c\Bigg] \\
&= \big(1 - (1-p)^N\big)^N \\
&\sim \exp\big(-N \mathrm{e}^{-pN}\big)\,.
\end{aligned}$$

The critical p is $N^{-1} \log N$. For $p = (1+a) N^{-1} \log N$ we have

$$\mathbb{P}\big[\text{no vertex is isolated}\big] \sim \exp\big(-N^{-a}\big)\,.$$

Theorem (Erdős, Rényi; 1960)

If $q \leqslant (1-\varepsilon)\sqrt{\log N}$, then there are a.s. isolated vertices. (Hence not all eigenvectors can be delocalized.)

Conversely, if $q \geqslant (1+\varepsilon)\sqrt{\log N}$, then the graph is a.s. connected.

Complete delocalization of eigenvectors

Delocalization of $\mathbf{u}_\alpha$ at scale ℓ means $\|\mathbf{u}_\alpha\|_\infty = O(\ell^{-1/2})$.
($\implies$ $\mathbf{u}_\alpha$ is essentially supported on at least ℓ sites.)

Theorem (Erdős, K, Yau, Yin; 2011)

Let $\mathbf{u}_1, \dots, \mathbf{u}_N$ denote the eigenvectors of the Erdős-Rényi graph. Let $q \geqslant (\log N)^2$. Then

$$\mathbb{P}\left(\max_\alpha \|\mathbf{u}_\alpha\|_\infty \leqslant \frac{(\log N)^8}{\sqrt{N}}\right) \geqslant 1 - \mathrm{e}^{-(\log N)^{1+c}} .$$

- More can be said on $\mathbf{u}_N$: $\|\mathbf{u}_N - \mathbf{e}\|_\infty = o(N^{-1/2})$ where $\mathbf{e} := N^{-1/2}(1, \dots, 1)$.
- Tran, Vu, Wang (2010): If λ_α is away from the edges ± 2 then the weaker estimate $\|\mathbf{u}_\alpha\|_\infty \leqslant q^{-1}$ holds with high probability.

Universality of the Erdős-Rényi graph

The eigenvalues of the ER graph satisfy

$$\lambda_1, \ldots, \lambda_{N-1} \in [-2 - o(1), 2 + o(1)] \,.$$

The largest eigenvalue satisfies $\lambda_N \approx q + q^{-1}$; arises from $\mathbb{E} a_{ij} \neq 0$. Moreover, λ_N has Gaussian fluctuations on scale $N^{-1/2}$.

Theorem (Erdős, K, Yau, Yin; 2011)

If $q \gg N^{1/3}$ (i.e. $p \gg N^{-1/3}$) then the bulk and edge universalities hold. Edge universality means that

$$\lim_{N \to \infty} \Big[\mathbb{P}^A \Big(N^{2/3} (\lambda_{N-1} - 2) \leqslant s \Big) - \mathbb{P}^{\mathrm{GOE}} \Big(N^{2/3} (\lambda_N - 2) \leqslant s \Big) \Big] \;=\; 0 \,.$$

Cornerstone of proof: LSC using resolvents (Green functions).

Let $z = E + \mathrm{i}\eta$ with $\eta > 0$. Define the Stieltjes transform

$$m_{sc}(z) \;:=\; \int \mathrm{d}x \, \frac{\rho_{sc}(x)}{x - z}$$

and the resolvent

$$G(z) \;:=\; (H - z)^{-1}.$$

The Stieltjes transform of the empirical eigenvalue density is

$$m(z) \;:=\; \frac{1}{N} \sum_{\alpha=1}^{N} \frac{1}{\lambda_\alpha - z} \;=\; \frac{1}{N} \operatorname{Tr} G(z)\,.$$

$\eta = \operatorname{Im} z$ describes the spectral resolution: $\operatorname{Im} m(E + \mathrm{i}\eta)$ is the density at E averaged over an interval of size η.

Indeed,

$$\operatorname{Im} m(z) \;=\; \frac{\pi}{N} \sum_{\alpha=1}^{N} \delta_\eta(\lambda_\alpha - E)\,, \qquad \delta_\eta(x) \;:=\; \frac{1}{\pi} \frac{\eta}{x^2 + \eta^2}\,.$$

Main LSC:

Theorem (Erdős, K, Yau, Yin; 2011)

Let $H = (h_{ij})$ be a Hermitian or symmetric matrix, either generalized Wigner or Erdős-Rényi.

Let $z = E + \mathrm{i}\eta$ with $\eta \gtrsim N^{-1}$. With high probability we have

$$\left|G_{ij}(z) - m_{sc}(z)\delta_{ij}\right| \;\lesssim\; \frac{1}{q} + \frac{1}{\sqrt{N\eta}}\,.$$

The averaged quantity $m(z)$ satisfies the stronger bound

$$\left|m(z) - m_{sc}(z)\right| \;\lesssim\; \frac{1}{q^2} + \frac{1}{N\eta}$$

(in the bulk; an analogous formula holds near the edges).

Here $q := \sqrt{N}$ for Wigner matrices.

Informally, $G(z) \approx m_{sc}(z)\mathbb{1}$ for η down to optimal scale $\eta \approx N^{-1}$.

Two simple consequences of the LSC

Complete delocalization. Set $\eta \approx N^{-1}$ and estimate

$$C \;\geqslant\; \operatorname{Im} G_{ii}(\lambda_\alpha + \mathrm{i}\eta) \;=\; \sum_\beta \frac{\eta\,|u_\beta(i)|^2}{(\lambda_\beta - \lambda_\alpha)^2 + \eta^2} \;\geqslant\; \frac{|u_\alpha(i)|^2}{\eta}\,.$$

Eigenvalue locations. Use the Helffer-Sjöstrand formula to get eigenvalue rigidity (stated for $\alpha \leqslant N^{1/2}$)

$$|\lambda_\alpha - \gamma_\alpha| \lesssim \big(N^{-2/3}\alpha^{-1/3} + q^{-2}\big)\,,$$

where γ_α is the classical location of the α-th eigenvalue,

$$\int_{-\infty}^{\gamma_\alpha} \rho_{sc}(x)\,\mathrm{d}x = \frac{\alpha}{N}\,.$$

(True for $q \geqslant N^{1/3}$, more complicated formula for $q \leqslant N^{1/3}$.)

Proof of the LSC

Green function method: Use Schur's formula

$$\begin{pmatrix} a & \mathbf{b}^* \\ \mathbf{b} & C \end{pmatrix}^{-1}_{11} = \frac{1}{a - \mathbf{b}^* C^{-1} \mathbf{b}} \quad \Longrightarrow \quad G_{ii} = \frac{1}{h_{ii} - z - \sum_{k,l \neq i} h_{ik} G^{(i)}_{kl} h_{li}} \,,$$

where $G^{(i)} := (H^{(i)} - z)^{-1}$ and $H^{(i)}$ is the minor of H obtained by removing the i-th row and column.

Goal: separate denominator into leading order term + fluctuating errors. (Assume $\sigma_{ij}^2 = N^{-1}$ for simplicity.) We find

$$\mathbb{E}_i \sum_{k,l \neq i} h_{ik} G^{(i)}_{kl} h_{li} = \frac{1}{N} \sum_k G^{(i)}_{kk} =: m^{(i)} \,.$$

The eigenvalues of H and $H^{(i)}$ are interlaced $\Longrightarrow$ $|m - m^{(i)}| \leqslant \frac{1}{N\eta}$.

Therefore

$$G_{ii} \;=\; \frac{1}{-z - m_{sc} - \underbrace{(m - m_{sc}) + \Upsilon_i}_{\text{error}}}\,, \qquad \Upsilon_i \;:=\; h_{ii} - Z_i + O\biggl(\frac{1}{N\eta}\biggr)\,,$$

where

$$Z_i \;:=\; \sum_{k,l \neq i} \Bigl(h_{ik} G^{(i)}_{kl} h_{li} - \mathbb{E}_i h_{ik} G^{(i)}_{kl} h_{li}\Bigr)\,.$$

Expand in error $(m - m_{sc}) - \Upsilon_i$ and average over i:

$$m \;=\; m_{sc} + m_{sc}^2 \biggl(m - m_{sc} - \frac{1}{N} \sum_i \Upsilon_i\biggr) + O\bigl(m - m_{sc} - \Upsilon_i\bigr)^2$$

using the identity $m_{sc} = \frac{1}{-z - m_{sc}}$.

Strategy: Derive bound on $m - m_{sc}$ using large deviation estimates on Υ_i. To control the resolvent expansion, use a bootstrap argument in η from $\eta = C$ down to $\eta \approx N^{-1}$.

Main difficulty: control

$$Z_i := \sum_{k,l \neq i} \Big(h_{ik} G_{kl}^{(i)} h_{li} - \mathbb{E}_i h_{ik} G_{kl}^{(i)} h_{li} \Big) \,.$$

Use large deviations estimates to get the naive bound

$$|Z_i| \lesssim \frac{1}{\sqrt{N\eta}} + \frac{1}{q} \,.$$

In order to improve the bound for the average

$$m - m_{sc} = \frac{1}{N} \sum_i (G_{ii} - m_{sc})$$

we have to exploit the averaging over i in $\frac{1}{N}\sum_i Z_i$. If the CLT held, then averaging would reduce size by $N^{-1/2}$. In reality we get

$$\bigg| \frac{1}{N} \sum_i Z_i \bigg| \lesssim \frac{1}{N\eta} + \frac{1}{q^2} \,.$$

(Correlation among Z_i's are large for small η and q.)

Proof: high moment expansion

$$\mathbb{E}\left|\frac{1}{N}\sum_i Z_i\right|^k = \frac{1}{N^k}\sum_{i_1,\dots,i_k} \mathbb{E} Z_{i_1}\cdots \overline{Z}_{i_k}\,,$$

decouple the random dependencies of Z_{i_s} by using a systematic size-vs-independence expansion of Z_i. Terms are either small (with high probability) or independent of many rows of H.

Key resolvent identity:

$$G_{ij} = \underbrace{G_{ij}^{(\ell)}}_{\text{independent of row } \ell} + \underbrace{\frac{G_{i\ell}G_{\ell j}}{G_{\ell\ell}}}_{\text{small}}\,.$$

Expansion: Set $\mathbb{S} := \{i_1, \dots, i_k\}$ and write

$$Z_{i_s} = \sum_{\mathbb{U}\subset\mathbb{S}} Z_{i_s}^{\mathbb{S},\mathbb{U}}\,,$$

where $Z_{i_s}^{\mathbb{S},\mathbb{U}}$ is independent of rows indexed by $\mathbb{S}\setminus\mathbb{U}$, and is of size

$$\left|Z_{i_s}^{\mathbb{S},\mathbb{U}}\right| \leqslant \left[|\mathbb{U}|\left(\frac{1}{\sqrt{N\eta}} + \frac{1}{q}\right)\right]^{|\mathbb{U}|+1}.$$

Universality of eigenvectors: sketch of proof

Extract information on eigenvectors from a good control of the resolvent, by writing

$$G_{ij}(z) \;=\; (H-z)^{-1}_{ij} \;=\; \sum_\beta \frac{\overline{u}_\beta(i) u_\beta(j)}{\lambda_\beta - z}\,, \qquad z \;=\; E + \mathrm{i}\eta\,.$$

Define

$$X_{ij}(z) \;:=\; \frac{N}{2\pi \mathrm{i}}\big[G_{ij}(z) - G_{ij}(\overline{z})\big] \;=\; \sum_\beta \frac{\eta/\pi}{(E-\lambda_\beta)^2 + \eta^2} N \overline{u}_\beta(i) u_\beta(j)\,.$$

We want to extract a single eigenvalue λ_α from the sum. Thus we choose $\eta = N^{-2/3-\delta}$. For a typical eigenvalue configuration the eigenvalue separation is of the order $N^{-2/3}$, and hence

$$N\overline{u}_\alpha(i) u_\alpha(j) \;\approx\; \int \mathrm{d}E\, X_{ij}(E + \mathrm{i}\eta)\, \mathbf{1}\Big(-(\log N)^C \eta \leqslant E - \lambda_\alpha \leqslant (\log N)^C \eta\Big)\,.$$

Key tools:

- Precise control of G for small η and level repulsion estimates.
- Green function comparison (resolvent expansion).

Summary

- The local statistics of a random matrix are independent of the laws of its entries (universality). The proof is always by comparison to the explicitly solvable Gaussian case.
- Good control of fluctuations allows heavy tails and sparse matrices such as the Erdős-Rényi graph.
- A precise control of the resolvent is pivotal in all proofs.
- Local semicircle law immediately implies complete eigenvector delocalization.

Open problems:

- Universality of Erdős-Rényi graph for $q \geqslant (\log N)^C$.
- Universality of random band matrices.