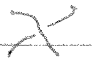
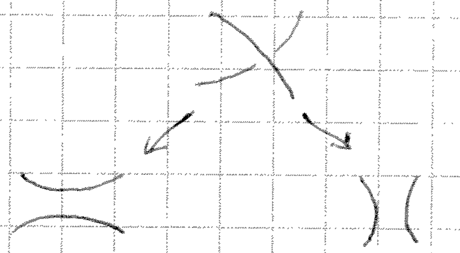
The Kauffman bracket and Temperley-Lieb algebras

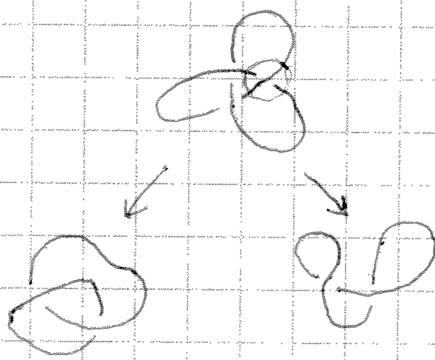
Reference: Kauffman - State models and the Jones polynomial
Topology vol. 26, no. 3, 395-407, 1987

Let us try to directly construct a function on the set of knots that is invariant under the Reidemeister moves. Consider a crossing in an unoriented link diagram:

 Two associated diagrams can be obtained by splicing the crossing:

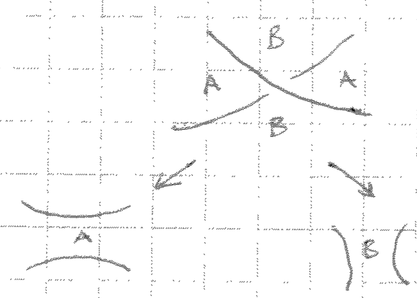


For example:



Repeating this process we end up with a diagram consisting only of trivial knots.

To make the process uniquely defined we add a few ingredients. Let us indicate the type of split as follows:



That is, a given split is said to be of type A or type B according to the convention that an A-split joins the regions labelled A at the crossing and similarly for the B-split. The regions labelled A are those that appear on the left as we approach the crossing along the undercrossing segments. The B-regions appear on the right.

Next we define our function. It is called the Kauffman bracket.

<u>def</u>: Let K be an unoriented knot or link diagram. Let $\langle K \rangle$ be the element of the ring $\mathbb{Z}[A, B, d]$ defined by means of the rules:

i) $\langle \bigcirc \rangle = 1$

ii) $\langle \bigcirc \cup K \rangle = d \langle K \rangle$, K not empty

iii) $\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$

remarks: A formula may involve the bracket and a few small diagrams. These small figures represent larger diagrams that differ only as indicated in the small diagrams. Rule i) says that $\langle K \rangle$ takes the value 1 on a single unknotted circle diagram. Rule ii) says that $\langle K \rangle$ is multiplied by d in the presence of a disjoint circular component. Rule iii) applies to diagrams that differ locally at the site of a single crossing.

We can use rule iii) to keep expanding the formulas until we reach diagrams consisting of disjoint unions of circles. Rules i) and ii) then imply that the value of $\langle K \rangle$ on a disjoint collection of circles is d raised to one less than the cardinality of the collection.

As it stands, the bracket $\langle K \rangle$ is well defined on diagrams but it is not invariant under any of the Reidemeister moves. We can try to find suitable relations among A, B, d to improve the situation.

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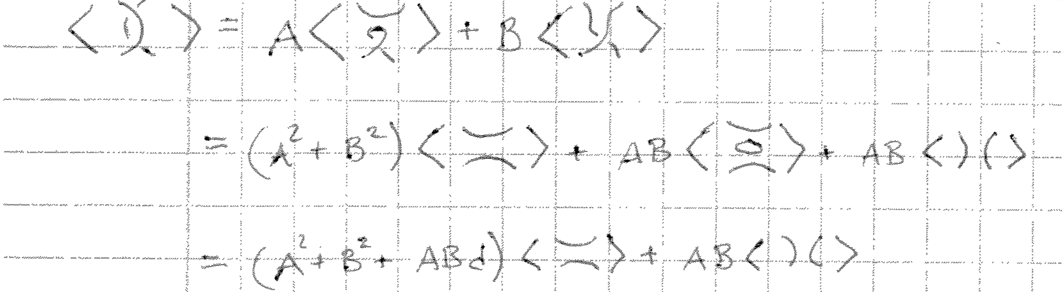
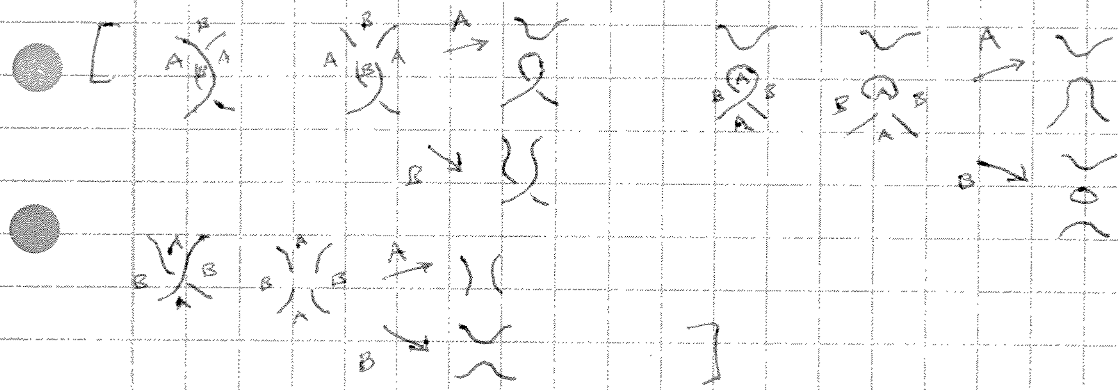
Let us start with invariance under RII



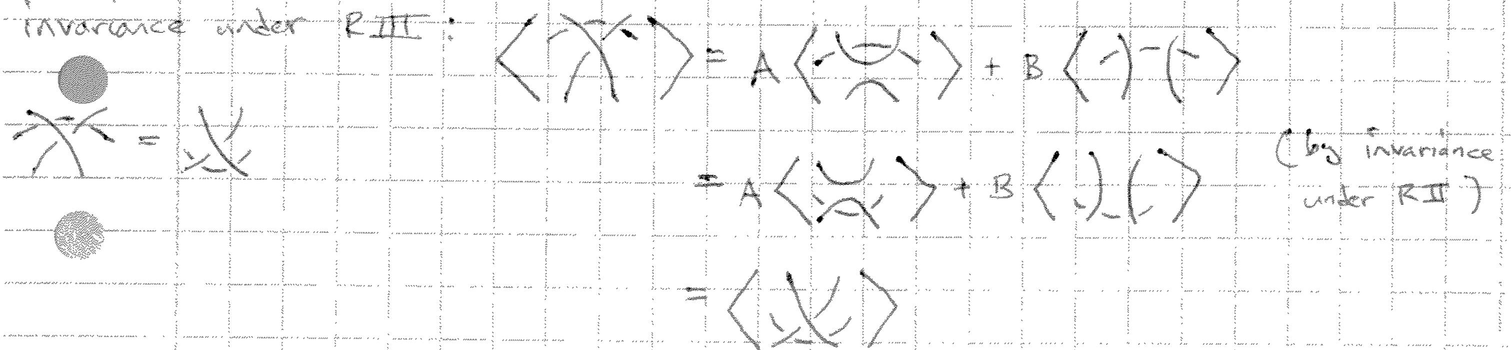


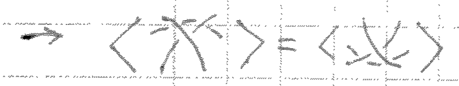
We want - $AB=1$

$A^2+B^2+ABd=0$

let us choose $B=A^{-1}$ and $d=-A^2-A^{-2}$

invariance under RIII:

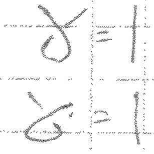


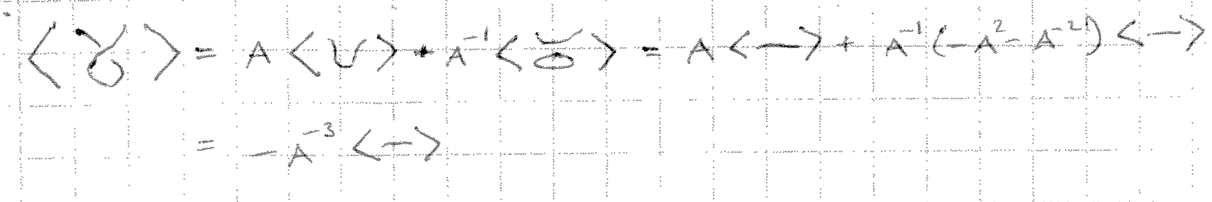


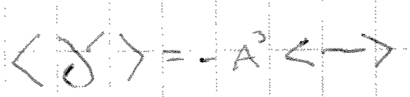
We see that type II invariance of the bracket implies type III invariance.

What about type I ?

RI



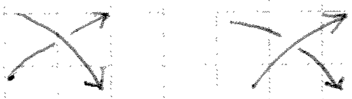


similarly 

so the bracket is not invariant under RI. It is only an invariant of regular isotopy (that is, invariant under RII, RIII but not RI). There are two ways to deal with this problem.

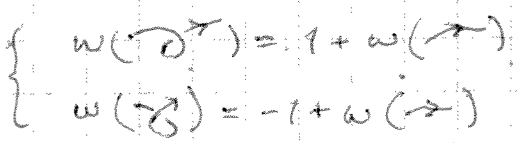
1) Use the twist number to modify the bracket

<u>def</u>: Let K be an oriented link diagram. Define the twist number (or writhe) of K, $w(K)$, by the equation $w(K) = \sum_p \varepsilon(p)$ where p runs over all crossings in K, and $\varepsilon(p)$ is the sign of the crossing:



$\varepsilon = +1$ $\qquad$ $\varepsilon = -1$

$w(K)$ is an invariant of regular isotopy and



Thus, we can define a normalized bracket, α_K, for oriented links K by the formula

$$\alpha_K = (-A^3)^{-w(K)} \langle K \rangle$$

<u>prop</u>: The normalized bracket polynomial α_K is an invariant of ambient isotopy.

proof: Since $w(K)$ is a regular isotopy invariant, and $\langle K \rangle$ is also a regular isotopy invariant, it follows at once that α_K is a regular isotopy invariant. Thus we need only check that α_K is invariant under type I moves. This follows at once. For example

$$\alpha_{\circlearrowright} = (-A^3)^{-w(\circlearrowright)} \langle \circlearrowright \rangle$$

$$= (-A^3)^{-[1+w(\sim)]} (-A^3) \langle \sim \rangle$$

$$= (-A^3)^{-w(\sim)} \langle \sim \rangle = \alpha_{\sim} \qquad \blacksquare$$

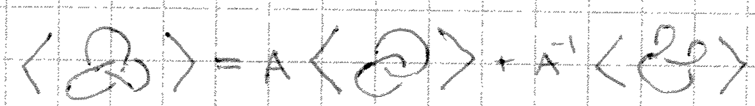
__Prop__: Let K^* denote the mirror image of the (oriented!) link K that is obtained by switching all the crossings of K. Then $\langle K^* \rangle (A) = \langle K \rangle (A^{-1})$ and

$$\alpha_{K^*}(A) = \alpha_K(A^{-1}).$$

proof: Reversing all crossings exchanges the roles of A and A^{-1} in the definition of $\langle K \rangle$ and α_K.

__ex__

Trefoil knot



$$= A(-A^4 - A^{-4}) + A^{-1}(-A^{-3})^2$$

$$\langle T \rangle = -A^5 - A^{-3} + A^{-7}$$

$$w(T) = 3$$

$$\rightarrow \alpha_T = (-A^3)^{-3} \langle T \rangle = -A^{-9}(-A^5 - A^{-3} + A^{-7})$$

$$\alpha_T = A^{-4} + A^{-12} - A^{-16}$$

$$\alpha_{T^*} = A^4 + A^{12} - A^{16}$$

$\rightarrow$ Since $\alpha_{T^*} \neq \alpha_T$ we conclude that the trefoil is not ambient isotopic to its mirror image. This is an example of a result we cannot prove using the Alexander polynomial since it cannot distinguish between knots and their mirror images.

2) Use framed links instead of ordinary ones. We will return to this topic later.

We return now to the braid groups. Recall that the Markov theorem gives us a way to use representations of the braid groups to find invariants of knots and links. Suppose we are given a commutative ring R (polynomials or Laurent polynomials for example) and functions $J_n : B_n \to R$ defined for each $n = 2, 3, \dots$

Then the Markov theorem assures us that the family of functions $\{J_n\}$ can be used to construct link invariants if the following conditions are satisfied:

1. If b and b' are equivalent braid words, then $J_n(b) = J_n(b')$
2. If $g, b \in B_n$ then $J_n(b) = J_n(gbg^{-1})$
3. If $b \in B_n$, then there is a constant $\alpha \in R$, independent of n, such that

$$J_{n+1}(b\sigma_n) = \alpha J_n(b)$$
$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b)$$

For the closed braid $\overline{b'} = \overline{b\sigma_n}$ the result of the Markov move $b \mapsto b'$ is to perform a type I move on $\overline{b}$. Furthermore, $\overline{b\sigma_n}$ corresponds to a type I move of positive type, while $\overline{b\sigma_n^{-1}}$ corresponds to a type I move of negative type. This explains the choice of the constants above.

Next we define the twisting number $w(b)$ of a braid to be

$$w(b) = \sum_{\ell=1}^{k} a_\ell$$

for any braid word $\quad \sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \cdots \sigma_{i_k}^{a_k} \quad$ representing b.

$w(b)$ is defined in this way so that $w(b) = w(\overline{b})$ where $\overline{b}$ is the oriented link obtained by closing the braid b.

<u>def</u>: Let $\{J_n : B_n \to R\}$ be given with properties 1, 2, 3 as given above. Call $\{J_n\}$ a Markov trace on $\{B_n\}$. For any link L, let $L \sim \overline{b}$, $b \in B_n$ via Alexander's theorem. Define $J(L) \in R$ via the formula

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$$J(L) = \alpha^{-w(b)} J_n(b)$$

Call $J(L)$ the link invariant for the Markov trace $\{J_n\}$.

Prop: Let J be the link invariant corresponding to the Markov trace $\{J_n\}$. Then J is an invariant of ambient isotopy for oriented links. That is, if $L \sim L'$ then $J(L) = J(L')$.

Proof: Suppose, by Alexander's theorem, that $L \sim \bar{b}$ and $L' \sim \bar{b}'$ where $b \in B_n$ and $b' \in B_n$ are specific braids. Since L and L' are ambient isotopic, it follows that $\bar{b}$ and $\bar{b}'$ are also ambient isotopic. Hence b' can be obtained from b by a sequence of Markov moves. Each such move leaves the function $\alpha^{-w(b)} J_n(b)$ invariant since the exponent sum is invariant under conjugation, braid moves, and it is used here to cancel the effect of MII move. ∎

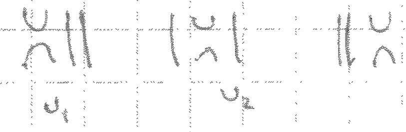
Next we define the bracket $\langle\ \rangle : B_n \to \mathbb{Z}[A, A^{-1}]$ for braids via $\langle b \rangle = \langle \bar{b} \rangle$, the evaluation of the bracket on the closed braid b. In terms of the Markov trace formalism, we are letting $J_n : B_n \to \mathbb{Z}[A, A^{-1}] = R$ via $J_n(b) = \langle \bar{b} \rangle$. Given what we already know about the bracket it is then obvious that $\{J_n\}$ is a Markov trace with $\alpha = -A^3$.

Consider the states of a braid (a state is a given choice for each splicing). They are determined by the recursion formula for the bracket

$$\langle \| \cdots | \times | \cdots \| \rangle = A \langle \| \cdots ||| \cdots \| \rangle + A^{-1} \langle \| \cdots | \overset{\cup}{\cap} | \cdots \| \rangle$$

$$\langle \sigma_i \rangle = A \langle 1_n \rangle + A^{-1} \langle U_i \rangle$$

U_i is a new kind of element $\overset{\cup}{\cap}$ at i, $i+1$



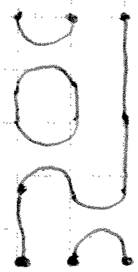
U_1 U_2 U_3

Since a state for $\overline{b}$ is obtained by choosing splice direction for each crossing of b, we see that each state of $\overline{b}$ can be written as the closure of a product of the elements U_i.

<u>ex</u> Let $L = \overline{b}$ be the closure of

$$b = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$$

Then the state s of L corresponds to the closure of $U_1^2 U_2$



$$s = \overline{U_1^2 U_2}$$

(that is, the bracket)

We can use the following formalism to compute the state expansion for a braid. Write

$$\sigma_i \equiv A + A^{-1}U_i \qquad \sigma_i^{-1} \equiv A^{-1} + AU_i$$

Given a braid word b, write $b = \mathcal{U}(b)$ where $\mathcal{U}(b)$ is a sum of products of the U_i's obtained by performing the above substitutions for each σ_i. Each product of U_i's when closed gives a collection of loops. Thus if U is such a product, then $\langle U \rangle = \langle \overline{U} \rangle = \delta^{\|U\|}$ where $\|U\| = \#(\text{of loops in})\,\overline{U} - 1$ and $\delta = -A^2 - A^{-2}$. Finally if $\mathcal{U}(b)$ is given by

$$\mathcal{U}(b) = \sum_s \langle b|s \rangle U_s$$

where s indexes all the terms in the product, and $\langle b|s \rangle$ is the product of A's and A^{-1}'s multiplying each U-product U_s, then

$$\langle b \rangle = \langle \mathcal{U}(b) \rangle = \sum_s \langle b|s \rangle \langle U_s \rangle$$

$$= \sum_s \langle b|s \rangle \, \delta^{\|s\|}$$

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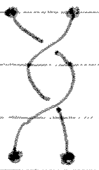
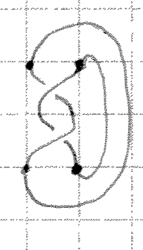
ex $b = \sigma_1^2$

$$U(b) = (A + A^{-1}U_1)(A + A^{-1}U_1)$$
$$= A^2 + 2U_1 + A^{-2}U_1^2$$



$L = \hat{b}$



$$\langle b \rangle = A^2 \langle 1_2 \rangle + 2 \langle U_1 \rangle + A^{-2} \langle U_1^2 \rangle$$

$\langle 1_2 \rangle = \delta$

$\langle U_1 \rangle = 1$

$\langle U_1^2 \rangle = \delta$

$$\to \langle L \rangle = A^2(-A^2 - A^{-2}) + 2 + A^{-2}(-A^2 - A^{-2})$$
$$= -A^4 - A^{-4}$$

This is in agreement with the result we would get by directly computing the bracket $\langle L \rangle$.