Temperley-Lieb Algebras

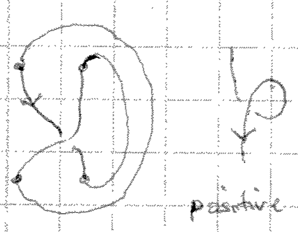
Let us recall how to obtain knot invariants from the braid groups. Suppose that we are given a commutative ring R, and functions $J_n : B_n \to R$ from the n-strand braid group to the ring R, defined for each $n = 2, 3, 4, \ldots$ Then the Markov theorem assures us that the family of functions $\{J_n\}$ can be used to construct link invariants if the following conditions are satisfied:

1. If b and b' are equivalent braid words, then $J_n(b) = J_n(b')$.
2. If $g, b \in B_n$ then $J_n(b) = J_n(gbg^{-1})$.
3. If $b \in B_n$, then there is a constant $\alpha \in R$, independent of n, such that

$$J_{n+1}(b\sigma_n) = \alpha J_n(b)$$
$$J_{n+1}(b\sigma_n^{-1}) = \alpha^{-1} J_n(b)$$

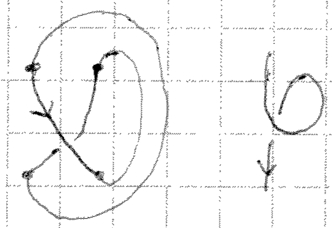
For the closed braid $\overline{b'} = \overline{b\sigma_n}$ the result of the Markov move $b \mapsto b'$ is to perform a type I move on $\overline{b}$. Furthermore, $\overline{b\sigma_n}$ corresponds to a type I move of positive type, while $\overline{b\sigma_n^{-1}}$ corresponds to a type I move of negative type.

$\overline{b\sigma}$



positive twist

$\overline{b\sigma^{-1}}$

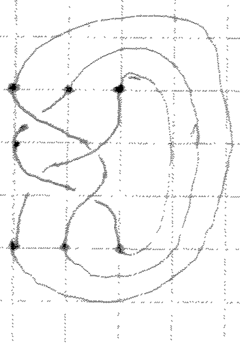


negative twist

The twist number of a braid, $w(b)$, is defined to be its exponent sum. That is, we let $w(b) = \sum_{k=1}^{k} a_k$ in any braid word

$$\sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \cdots \sigma_{i_k}^{a_k} \quad \text{representing } b$$

ex



$b = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$

$w(b) = 0$

It is clear from the example that $w(b) = w(\bar{b})$ where $\bar{b}$ is the oriented link obtained by closing the braid b. Here $w(\bar{b})$ is the twist number of the oriented link $\bar{b}$.

<u>def</u>: Let $\{J_n : B_n \to R\}$ be given with the properties listed above. Call $\{J_n\}$ a Markov trace on $\{B_n\}$. For any link L, let $L \sim \bar{b}$, $b \in B_n$ via Alexander's theorem. Define $J(L) \in R$ via the formula

$$J(L) = \alpha^{-w(b)} J_n(b).$$

Call $J(L)$ the link invariant for the Markov trace $\{J_n\}$.

To see that $J(L)$ really gives a knot invariant, it is enough to observe that the exponent sum is invariant under conjugation, braid moves and it cancels the effect of the type 3 Markov move. Thus the function $\alpha^{-w(b)} J_n(b)$ is invariant under the Markov moves.

Now the Kauffman bracket gives us an example of a Markov trace on $\{B_n\}$. To see this, let us define the bracket on braids $\langle\ \rangle : B_n \to \mathbb{Z}[A, A^{-1}]$ via $\langle b \rangle = \langle \bar{b} \rangle$, the evaluation of the bracket on the closed braid $\bar{b}$. In terms of the Markov trace formalism, we are letting $J_n : B_n \to \mathbb{Z}[A, A^{-1}] = R$ via $J_n(b) = \langle \bar{b} \rangle$. From the properties of the bracket it is then obvious that $\{J_n\}$ is a Markov trace, with $\alpha = -A^3$.

Reminder on the bracket:

<u>def</u>: Let K be an unoriented knot or link diagram. Let $\langle K \rangle$ be the element of the ring $\mathbb{Z}[A, B, d]$ defined by means of the rules:

i) $\langle O \rangle = 1$

ii) $\langle O \cup K \rangle = d \langle K \rangle$, K not empty

iii) $\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$

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The process of computing the bracket for a knot or link diagram K can be expressed as a simple formula in the following way. If U is the underlying planar graph for K, then a state of U is a choice of splitting for every vertex of U. Since splitting all the vertices of a state results in a configuration of disjoint circles, we see that the states are in one-to-one correspondence with final configurations of the expansion of the bracket. Accordingly, we define $\langle K|S\rangle$ for a diagram K and a state S by the formula

$$\langle K|S\rangle = A^i B^j$$

where i is the number of splittings of type A and j is the number of splittings of type B. The total contribution of a given state to the polynomial is then given by the formula

$$\langle K|S\rangle\, d^{|S|-1}$$

where $|S|$ denotes the number of circles in the splitting of S. All in all, these considerations lead to the following proposition

__Prop:__ $\langle K\rangle$ is uniquely determined on diagrams by the rules i), ii) and iii) above. It is given by the formula

$$\langle K\rangle = \sum_S \langle K|S\rangle\, d^{|S|-1}$$

where this summation is over all states of the diagram, and $|S|$ denotes the number of components in the splitting of a state S.

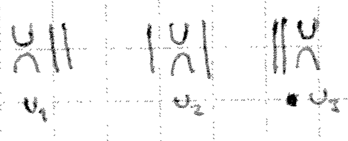
Next we turn to the braid-analog of the state expansion for the bracket. The recursion formula for the bracket applied to braids becomes

$$\langle |\cdots|\times|\cdots|\rangle = A\langle |\cdots||||\cdots|\rangle + A^{-1}\langle |\cdots|{\textstyle\frac{\cup}{\cap}}|\cdots|\rangle$$

$$\langle \sigma_i \rangle = A\langle 1_n \rangle + A^{-1}\langle U_i \rangle$$

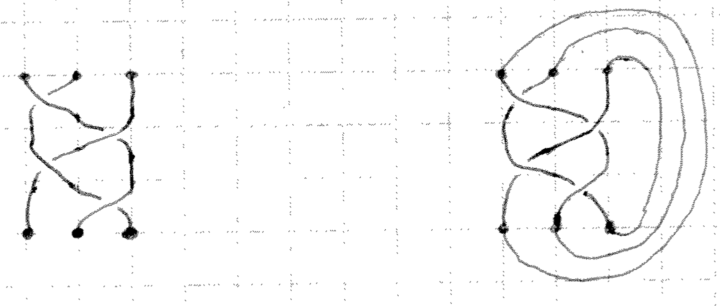
where 1_n denotes the identity element in B_n and U_i is a new kind of element

 at the i-th and $(i+1)$-th strands :

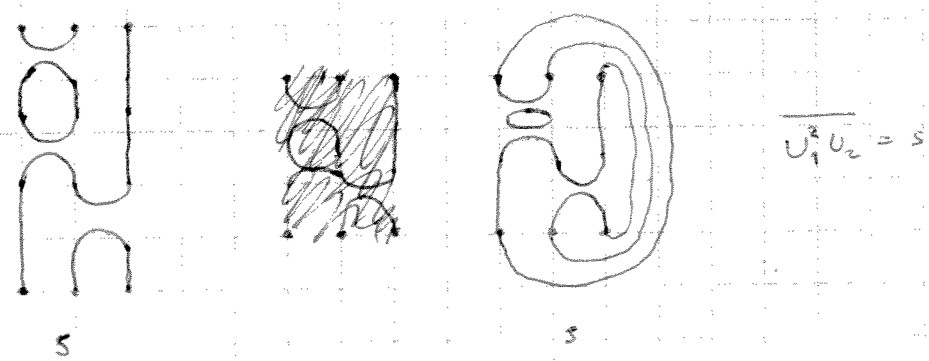


Since a state for $\overline{b}$ is obtained by choosing splice direction for each crossing of b, we see that each state of $\overline{b}$ can be written as the closure of a product of the elements U_i.

<u>ex</u> $L = \overline{b}$ $\quad b = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$



The state s of L below corresponds to the product $U_1^2 U_2$



To compute the bracket for a braid we can use the following formalism : Write

$$\sigma_i = A + A^{-1}U_i \qquad \sigma_i^{-1} = A^{-1} + AU_i$$

Given a braid word b, write $b = \mathcal{U}(b)$ where $\mathcal{U}(b)$ is a sum of products of the U_i's obtained by performing the above substitutions for each σ_i. Each product of U_i's, when closed gives a collection of loops. Thus if U is such

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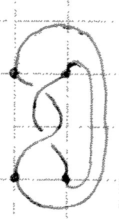
a product, then $\langle U\rangle = \langle \overline{U}\rangle = \delta^{\|U\|}$ where $\|U\| = \#$ of loops in $\overline{U} - 1$ and $\delta = -A^2 - A^{-2}$. Finally if $\mathcal{U}(b)$ is given by

$$\mathcal{U}(b) = \sum_s \langle b|s\rangle U_s$$

where s indexes all the terms in the product, and $\langle b|s\rangle$ is the product of A's and A^{-1}'s multiplying each U-product U_s, then

$$\langle b\rangle = \langle \mathcal{U}(b)\rangle = \sum_s \langle b|s\rangle \langle U_s\rangle$$
$$= \sum_s \langle b|s\rangle \, \delta^{\|s\|}$$

ex $\quad b = \sigma_1^2$

b  $\quad L = \overline{b}$ 

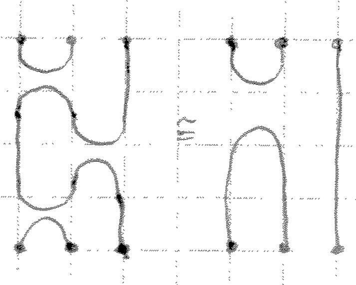
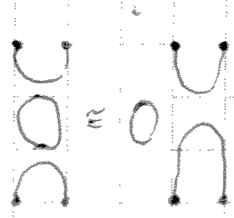
$$\mathcal{U}(b) = (A + A^{-1}U_1)(A + A^{-1}U_1)$$
$$= A^2 + 2U_1 + A^{-2}U_1^2$$

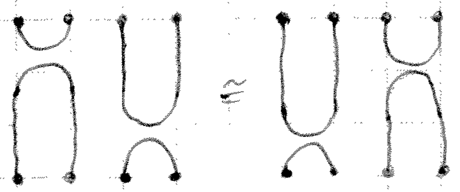
$$\langle b\rangle = \langle \mathcal{U}(b)\rangle = A^2\langle 1_2\rangle + 2\langle U_1\rangle + A^{-2}\langle U_1^2\rangle$$

$|| \rightarrow \langle 1_2\rangle = \delta \qquad \cup\cap \rightarrow \langle U_1\rangle = 1 \qquad \rightarrow \langle U_1^2\rangle = \delta$

$$\rightarrow \langle L\rangle = A^2(-A^2 - A^{-2}) + 2 + A^{-2}(-A^2 - A^{-2})$$
$$= -A^4 - A^{-4}$$

Let us take a closer look at the new diagrams $\cup\cap$ we have obtained. Such diagrams can be multiplied in the same way as we did for braid diagrams so the n-string diagrams $U_1, U_2, \dots, U_{n-1}$ generate a monoid. The relations are easily read off from the following pictures

 $\quad U_1U_2U_1 = U_1$  $\quad U_1^2 = \delta U_1$



$U_1U_3 = U_3U_1$

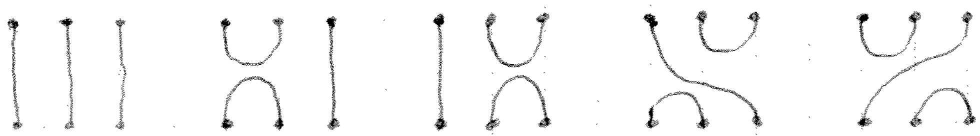
These are precisely the relations for the Temperley-Lieb algebra.

The monoid generated by the U_i has a simple graphical structure. Let D_n denote the collection of all (topological) equivalence classes of diagrams obtained by connecting pairs of points in two parallel rows of n points. The arcs connecting these points must satisfy the following conditions:

1) All arcs are drawn in the space between the two rows of points.
2) No two arcs cross one another.
3) Two elements $a, b \in D_n$ are said to be equivalent if they are topologically equivalent via a planar isotopy through elements of D_n.

We call D_n the diagram monoid on $2n$ points.

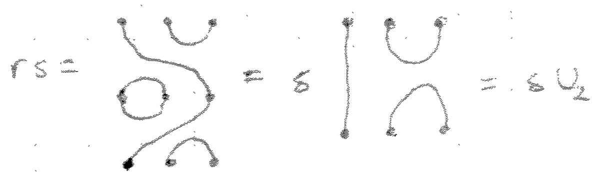
ex D_3 has the following elements



1_3 $\quad U_1$ $\quad U_2$ $\quad r$ $\quad s$

closed loops may appear when multiplying the elements. We write $ab = \delta^k c$ where $c \in D_n$ and k is the number of closed loops in the product.

For example in D_3



prop: The elements $1, U_1, U_2, \dots, U_{n-1}$ generate D_n. If an element $x \in D_n$ is equivalent to two products, P and Q, of the elements $\{U_i\}$, then Q can be obtained from P by a series of applications of the T-L relations

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$$\text{T-L} \quad \begin{cases} U_i U_{i\pm 1} U_i = U_i \\ U_i^2 = \delta U_i \\ U_i U_j = U_j U_i \quad \text{if} \quad |i-j|>1 \end{cases}$$

The members of the diagram monoid correspond to states of elements of the braid group. Thus the bracket is computed as a sum of evaluations of elements of the diagram monoid. To formalize this, we make the diagram monoid into an algebra.

<u>def</u>: Given a commutative ring R and a monoid M, let RM denote the free additive algebra over R with multiplication generated by M. That is, an element of RM is a formal linear combination $r_1m_1 + \dots + r_nm_n$ with $r_i \in R$ and $m_i \in M$. Multiplication follows the distributive law ($a(b+c) = ab+ac$) and $(r_im_i)(r_jm_j) = (r_ir_j)(m_im_j)$ where the product r_ir_j is in the ring R, and the product m_im_j is in the monoid M.

We choose $R = \mathbb{Z}[A, A^{-1}]$ and $\delta = -A^2 - A^{-2} \in \mathbb{Z}[A, A^{-1}]$ and call the resulting monoid algebra A_n. This algebra is also known as the Temperley-Lieb algebra.

We can now define a mapping

$$\rho : B_n \to A_n$$

by the formulas:

$$\rho(\sigma_i) = A + A^{-1}U_i$$
$$\rho(\sigma_i^{-1}) = A^{-1} + AU_i$$

<u>prop</u>: $\rho : B_n \to A_n$ as defined above, is a representation of the braid group.

proof: It is necessary to verify that $\rho(\sigma_i)\rho(\sigma_i^{-1}) = 1$, $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$ and that $\rho(\sigma_i\sigma_j) = \rho(\sigma_j\sigma_i)$ when $|i-j|>1$. These follow by direct computation

$$\begin{aligned} \rho(\sigma_i)\rho(\sigma_i^{-1}) &= (A + A^{-1}U_i)(A^{-1} + AU_i) = 1 + (A^{-2} + A^2)U_i + U_i^2 \\ &= 1 + (A^{-2} + A^2)U_i + (-A^{-2} - A^2)U_i \\ &= 1 \end{aligned}$$

given $|i-j|>1$ $\quad \rho(\sigma_i\sigma_j) = \rho(\sigma_i)\rho(\sigma_j) = (A + A^{-1}U_i)(A + A^{-1}U_j)$

$$= (A + A^{-1}U_j)(A + A^{-1}U_i)$$
$$= \rho(\sigma_j\sigma_i)$$

$$\rho(\sigma_i\sigma_{i+1}\sigma_i) = (A + A^{-1}U_i)(A + A^{-1}U_{i+1})(A + A^{-1}U_i)$$
$$= (A^2 + U_{i+1} + U_i + A^{-2}U_iU_{i+1})(A + A^{-1}U_i)$$
$$= A^3 + AU_{i+1} + AU_i + A^{-1}U_iU_{i+1} + AU_i + A^{-1}U_{i+1}U_i + A^{-1}U_i^2 + A^{-3}U_iU_{i+1}U_i$$
$$= A^3 + AU_{i+1} + (A^{-1}\delta + 2A)U_i + A^{-1}(U_iU_{i+1} + U_{i+1}U_i) + A^{-3}U_i$$
$$= A^3 + AU_{i+1} + (A^{-1}(-A^2 - A^{-2}) + 2A + A^{-3})U_i + A^{-1}(U_iU_{i+1} + U_{i+1}U_i)$$
$$= A^3 + A(U_{i+1} + U_i) + A^{-1}(U_iU_{i+1} + U_{i+1}U_i)$$

This expression is symmetric in i and $i+1$ $\rightarrow$ $\rho(\sigma_i\sigma_{i+1}\sigma_i) = \rho(\sigma_{i+1}\sigma_i\sigma_{i+1})$ $\blacksquare$

We define a trace on A_n $tr: A_n \rightarrow \mathbb{Z}(A, A^{-1})$ by $tr(U) = \langle \bar{U} \rangle$ for $U \in D_n$ and extending tr linearly to A_n. We then have the formula

$$\langle b \rangle = tr(\rho(b)) \qquad \text{for any } b \in B_n.$$

We have then recovered all the ingredients Jones used to define the Jones polynomial, namely a representation of the braid groups on a suitable algebra (in this case the Temperley-Lieb algebra) having a Markov trace. We will return to other examples later.