Phase-Asymptotic Stability of Transition Front Solutions in Cahn-Hilliard Equations and Systems

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Outline of the Talk

- Introduction
- Physical motivation: spinodal decomposition
- Transition fronts and phase-asymptotic stability
- Overview of recent results
- The Cahn-Hilliard equation in one space dimension
  - Spectral analysis and the Evans function
  - The classical semigroup framework
  - Contour analysis
  - The pointwise semigroup framework
  - Local tracking
  - Closing the argument
- (Time-permitting) Multiple space dimensions
- (Time-permitting) Cahn-Hilliard systems
- (Time-permitting) Further work

Cahn-Hilliard Equations and Systems

For $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, we consider systems of the form

$$u_{j_t} = \nabla \cdot \Big\{ \sum_{i=1}^{m} M_{ji}(u) \nabla \Big((-\Gamma \Delta u)_i + F_{u_i}(u) \Big) \Big\}.$$

This is a standard model of certain phase separation processes such as spinodal decomposition, where the components of u characterize m components of a mixture that contains $m+1$ components in all.

Here, $F \in \mathbb{R}$ is a measure of bulk free energy density, $M \in \mathbb{R}^{m \times m}$ is a measure of molecular mobility, and $\Gamma \in \mathbb{R}^{m \times m}$ characterizes interfacial energy. Based on physical considerations, we assume M and Γ are symmetric and positive definite, M uniformly so.

We are interested in the phase-asymptotic stability of transition fronts $\bar{u}(x)$ (for $n = 1$) and $\bar{u}(x_1)$ (for $n \geq 2$).

Example Cases

For $x \in \mathbb{R}$ and $u \in \mathbb{R}$, we have the Cahn-Hilliard equation

$$u_t = \Big(M(u)(-\gamma u_{xx} + F'(u))_x \Big)_x .$$

For analysis, we often take $M(u) \equiv 1$, giving

$$u_t = (-\gamma u_{xx} + F'(u))_{xx}.$$

For $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ we have respectively

$$u_t = \nabla \cdot \Big\{ M(u) \nabla (-\gamma \Delta u + F'(u)) \Big\},$$

and

$$u_t = \Delta(-\gamma \Delta u + F'(u)).$$

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Spinodal Decomposition

Spinodal decomposition is a phenomenon in which the rapid cooling (quenching) of a homogeneously mixed binary alloy (e.g., iron and chromium) causes separation to occur, resolving the mixture into regions of different crystalline structure, separated by steep transition layers.

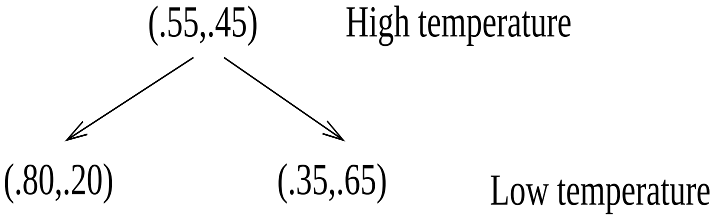

Figure: Possible concentrations.

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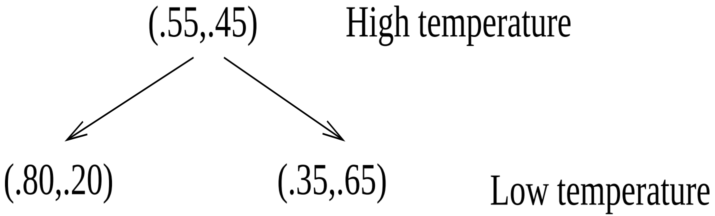

Figure: Possible concentrations.

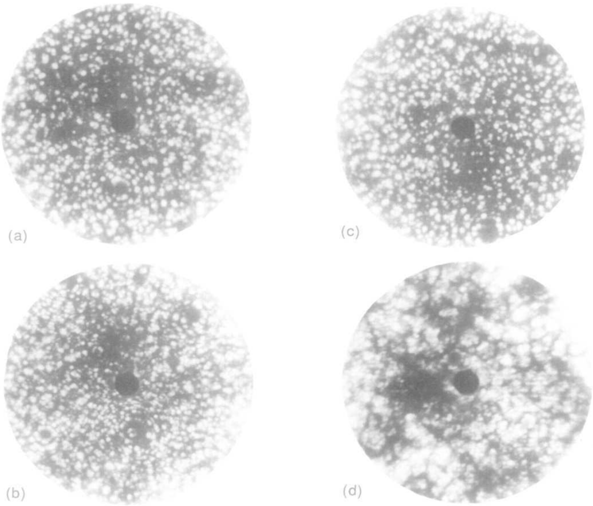

Fig. 2. Field ion micrographs from Fe-45% Cr samples aged for (a) 4, (b) 24, (c) 100 and (d) 500 h. The brightly imaging regions are Cr-enriched and the dark regions Cr-depleted.

Numerical Simulation, T. Sullivan, Kenyon College

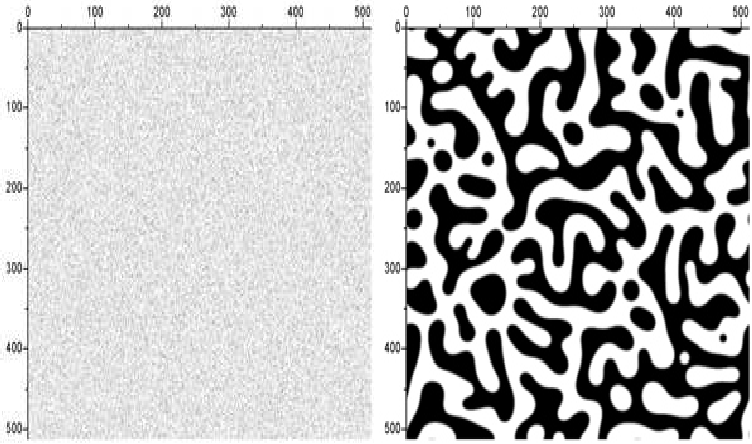

Figure: Numerical simulation for $n = 2$.

Cahn-Hilliard Energy

In 1958, John W. Cahn and John E. Hilliard suggested a reasonable form for the energy associated with this process,

$$E(u) = \int_{\Omega} F(u) + \frac{\gamma}{2}|\nabla u|^2 dx,$$

The bulk free energy density $F(u)$ is a measure of energy density, assuming the alloy is homogeneously mixed.

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Double-well Bulk Free Energy

The Helmholtz free energy is

$$\mathcal{H} = U - TS \Rightarrow \frac{\partial \mathcal{H}}{\partial T} = -S.$$

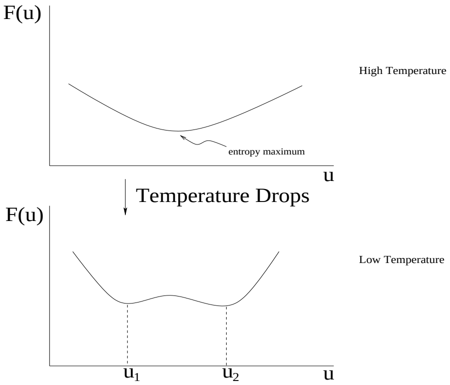

Figure: Double-well form through quenching.

Mass Conservation

For binary alloys this process can be described by a single equation for the concentration of a chosen component u. Since u is conserved, we have a conservation law

$$u_t + \nabla \cdot J = 0,$$

where J denotes flux.

A standard phenomenological assumption, akin to Fourier's law of heat conduction, is

$$J = -M(u)\nabla \frac{\delta E}{\delta u}.$$

This says: the system tends to move from configurations in which small changes in u lead to large changes in E to configurations in which small changes in u lead to small changes in E.

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HEAT EQ $\quad J = -M(u)\nabla u \quad$ HOT $\rightarrow$ COLD

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The Cahn-Hilliard Equation

For the Cahn-Hilliard energy

$$E(u) = \int_\Omega F(u) + \frac{\gamma}{2}|\nabla u|^2 dx,$$

we have

$$\frac{\delta E}{\delta u} = F'(u) - \gamma \Delta u.$$

Thus

$$u_t + \nabla \cdot \{-M(u)\nabla \frac{\delta E}{\delta u}\} = 0$$

is the Cahn-Hilliard equation

$$u_t = \nabla \cdot \left\{ M(u)\nabla(-\gamma \Delta u + F'(u)) \right\}.$$

Historical Remark on Terminology

The (single) Cahn-Hilliard equation first appeared in John W. Cahn's paper from 1961, "On spinodal decomposition."

Cahn-Hilliard systems were first studied by Didier de Fontaine in his 1967 Northwestern thesis "A computer simulation of the evolution of coherent composition variations in solid solutions," carried out under the direction of John Hilliard.

de Fontaine explains that Hilliard was never comfortable having his name on the equation and referred to it himself as "the last unnumbered equation after equation (18) in Cahn's 1961 paper."

Minimizing Energy

Recall that the energy is

$$E(u) := \int_\Omega F(u) + \frac{\gamma}{2}|\nabla u|^2 dx.$$

We expect solutions to evolve toward minimizers of E. More precisely, it's easy to verify that any solution $u(x,t)$ in an appropriate function class will satisfy

$$\frac{d}{dt}E(u) = -\int_\Omega M(u)|\nabla \frac{\delta E}{\delta u}|^2 dx \leq 0.$$

One way the system can minimize this is to approach a minimizer of F, but it's constrained by conservation of mass.

The system can compromise by making transitions from one minimizer of F to another, but these transitions increase the second term in E. The resulting *transition fronts* are a balance between these effects.

The Case $x \in \mathbb{R}$

We focus on the equation

$$u_t = \Big(M(u)(-\gamma u_{xx} + F'(u))_x \Big)_x .$$

First, since F appears as $F'(u)_x$, we can replace F with

$$\tilde{F}(u) = F(u) - au - b,$$

for any constants a and b. It's convenient to subtract off a supporting line.

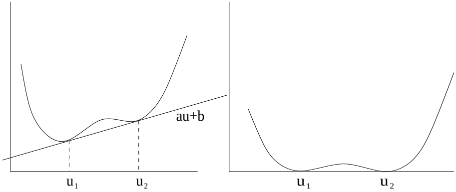

Figure: Subtraction of the supporting line.

Transition Fronts

We say a stationary solution $\bar{u}(x)$ is a transition front if

$$0 = \Big(M(\bar{u})(-\gamma \bar{u}_{xx} + F'(\bar{u}))_x \Big)_x,$$

and

$$\begin{aligned} \lim_{x \to -\infty} \bar{u}(x) &= u_- = u_1 \text{ (respectively } u_2) \\ \lim_{x \to +\infty} \bar{u}(x) &= u_+ = u_2 \text{ (respectively } u_1) \\ \lim_{x \to \pm\infty} \bar{u}'(x) &= 0. \end{aligned}$$

Integrating twice, we find

$$-\gamma \bar{u}_{xx} + F'(\bar{u}) = 0,$$

which is

$$\frac{\delta E}{\delta \bar{u}} = 0.$$

Example Case

For $M(u) \equiv 1$, $\gamma = 1$, and $F(u) = u^2(1-u)^2$.

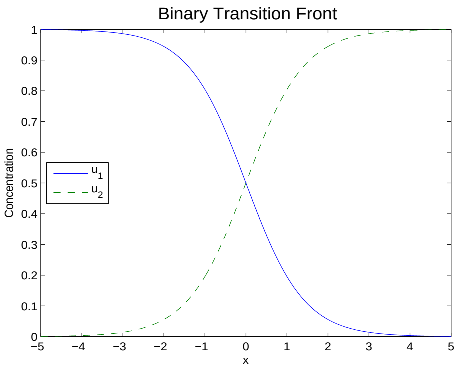

Figure: Binary Transition Front.

Asymptotic Behavior

Intuitively, we expect that for fairly general initial conditions $u(x,0) = u_0(x)$ we will find

$$\lim_{t\to\infty} u(x,t) = \bar{u}(x)$$

for some transition front $\bar{u}(x)$. For systems, we expect $u(x,t)$ to approach a combination of transition fronts consistent with conservation of mass.

Asymptotic Stability

We say $\bar{u}(x)$ is $X \to Y$ stable (for some Banach spaces X and Y) if given any $\epsilon > 0$ there exists $\eta > 0$ so that

$$\|u(x,0) - \bar{u}(x)\|_X < \eta \Rightarrow \|u(x,t) - \bar{u}(x)\|_Y < \epsilon$$

for all $t \geq 0$.

We say $\bar{u}(x)$ is $X \to Y$ asymptotically stable if it is stable and there exists $\eta_0 > 0$ sufficiently small so that

$$\|u(x,0) - \bar{u}(x)\|_X < \eta_0 \Rightarrow \lim_{t \to \infty} u(x,t) = \bar{u}(x)$$

in Y.

The Shift

Since concentration is conserved, perturbations of a transition front $\bar{u}(x)$ will not generally approach the front itself, but rather (in the case of stability) a shift of the front.

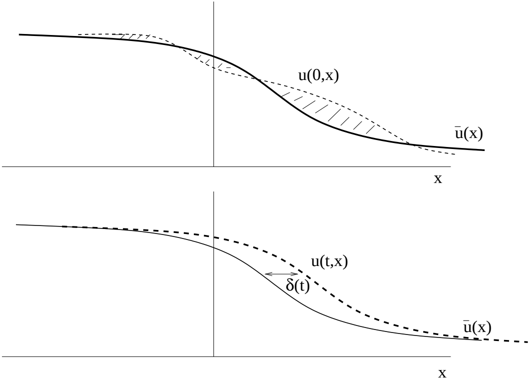

Figure: The shifted wave.

Phase-Asymptotic Stability

We define our perturbation as

$$v(x,t) := u(x+\delta(t),t) - \bar{u}(x).$$

We say $\bar{u}(x)$ is $X \to Y$ phase-stable if there exists a shift function $\delta(t)$ so that for any $\epsilon > 0$ there exists $\eta > 0$ so that

$$\|v(x,0)\|_X < \eta \Rightarrow \|v(x,t)\|_Y < \epsilon$$

for all $t \geq 0$.

We say $\bar{u}(x)$ is $X \to Y$ phase-asymptotically stable if $\bar{u}(x)$ is $X \to Y$ phase-stable and there exists a shift function $\delta(t)$ and a value $\eta_0 > 0$ so that

$$\|v(x,0)\|_X \leq \eta_0 \Rightarrow \lim_{t\to\infty} \|v(x,t)\|_Y = 0.$$

Goal

We establish $L^1 \cap L^\infty \to L^p$ phase-asymptotic stability for all $p > 1$.

We obtain $L^1 \cap L^\infty \to L^1$ phase-stability.

Overview of Recent Results, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

1. For $m = 1$, $n = 1$
    - J. Bricmont, A. Kupiainen, and J. Taskinen, Comm. Pure Appl. Math. 1999
    - E. Carlen, M. Carvalho, and E. Orlandi, Comm. Math. Phys. 2001
    - H., Comm. Math. Phys. 2007
2. For $m = 1$, $n \geq 2$
    - T. Korvola, Doctoral Thesis, University of Helsinki 2003
    - T. Korvola, A. Kupiainen, and J. Taskinen, Comm. Pure Appl. Math. 2005, $n \geq 3$
    - H., Physica D 2007, $n \geq 2$
3. For $m \geq 2$, $n = 1$
    - H. and B. Kwon, Discrete and Continuous Dynamical Systems A (spectral analysis) and 2011 Preprints

http://www.math.tamu.edu/~phoward/mathpubs.html

Existence Theorem for $m = 1$, $n = 1$

For

$$u_t = (M(u)(-\gamma u_{xx} + F'(u))_x)_x$$

suppose $\gamma > 0$, $M(u) \geq M_0 > 0$, F has a double-well form and $M \in C^2(\mathbb{R})$, $F \in C^4(\mathbb{R})$.

Then there exist two transition fronts $\bar{u}(x)$ and $\bar{u}(-x)$, both of which are strictly monotonic. Aside from tranlations of these, there are no other transition fronts.

Stability Theorem for $m = 1$, $n = 1$

Suppose $\bar{u}(x)$ is a transition front as described in the existence theorem. Then for Hölder continuous initial conditions $u_0(x) \in C^\gamma(\mathbb{R})$, $0 < \gamma < 1$, with

$$\|u(0,x) - \bar{u}(x)\|_{L^1} + \|u(0,x) - \bar{u}(x)\|_{L^\infty} \leq \epsilon,$$

for $\epsilon > 0$ sufficiently small, there exists a unique solution of (CH)

$$u \in C^{4+\gamma, 1+\frac{\gamma}{4}}(\mathbb{R} \times (0,\infty)) \cap C^{\gamma, \frac{\gamma}{4}}(\mathbb{R} \times [0,\infty))$$

and a shift $\delta \in C^{1+\gamma}[0,\infty)$ so that

$$\begin{aligned} \|u(x+\delta(t),t) - \bar{u}(x)\|_{L^p} &\leq C\epsilon(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \\ |\delta(t) - \delta_\infty| &\leq C\epsilon(1+t)^{-1/4}. \end{aligned}$$

Pointwise Stability Theorem

Suppose $\bar{u}(x)$ is a transition front as described in the existence theorem. Then for Hölder continuous initial conditions $u_0(x) \in C^\gamma(\mathbb{R})$, $\gamma > 0$, with

$$|u(0,x) - \bar{u}(x)| \le \epsilon(1+|x|)^{-3/2},$$

for $\epsilon > 0$ sufficiently small, there exists a unique solution of (CH)

$$u \in C^{4+\gamma, 1+\frac{\gamma}{4}}(\mathbb{R}\times(0,\infty)) \cap C^{\gamma,\frac{\gamma}{4}}(\mathbb{R}\times[0,\infty))$$

and a shift $\delta \in C^{1+\gamma}[0,\infty)$ so that

$$|u(x+\delta(t),t) - \bar{u}(x)| \le C\epsilon\Big[(1+t)^{-1/2}e^{-\frac{x^2}{Lt}} + (1+|x|+\sqrt{t})^{-3/2}\Big]$$
$$|\delta(t) - \delta_\infty| \le C\epsilon(1+t)^{-1/4}.$$

Linearization

We define our perturbation as

$$v(x,t) := u(x+\delta(t),t) - \bar{u}(x),$$

where $\delta(t)$ tracks the shift between $u(x,t)$ and $\bar{u}(x)$. The perturbation equation is

$$v_t = \Big(M(\bar{u})(-\gamma v_{xx} + F''(\bar{u})v)_x\Big)_x + \dot{\delta}(t)(\bar{u}_x + v_x) + Q_x,$$

where

$$|Q| \le C\Big[e^{-\eta|x|}|v|^2 + |v||v_x| + |v||v_{xxx}|\Big].$$

This is advantageous (over $|v|^2$) because we expect v_x and v_{xxx} to decay faster than v as $|x| + t \to \infty$. On the other hand, we expect v_x and v_{xxx} to blow up respectively like $t^{-1/4}$ and $t^{-3/4}$ as $t \to 0$.

The Associated Eigenvalue Problem

The associated linear equation is

$$v_t = Lv := \Big(M(\bar{u})(-\gamma v_{xx} + F''(\bar{u})v)_x\Big)_x.$$

If we look for solutions of the form $v(x,t) = e^{\lambda t}\phi(x)$, we obtain the eigenvalue problem

$$L\phi = \lambda\phi.$$

The resolvent for this problem is

$$R(\lambda; L) := (\lambda I - L)^{-1},$$

and we say $\lambda \in \mathbb{C}$ is in the resolvent set of L if $R(\lambda; L)$ is a bounded linear operator.

Spectrum of L: $\sigma(L) = \sigma_{pt} \cup \sigma_{ess}$

We consider two (not necessarily disjoint) sets of spectrum. The point spectrum is

$$\sigma_{pt} := \{\lambda \in \mathbb{C} : L\phi = \lambda\phi \text{ for some } \phi \in H^2, \phi \neq 0\}.$$

We refer to elements of σ_{pt} as eigenvalues. By essential spectrum σ_{ess}, we mean any value $\lambda \in \mathbb{C}$ that is not in the resolvent set and is not an isolated eigenvalue with finite multiplicity.

Roughly, eigenvalues characterize local (transitional) behavior of the transition front, while the essential spectrum characterizes endstate behavior.

The Essential Spectrum

The essential spectrum is determined by the asymptotic ($x \to \pm\infty$) eigenvalue equations

$$-M(u_\pm)\gamma\phi_{xxxx} + M(u_\pm)F''(u_\pm)\phi_{xx} = \lambda\phi.$$

It corresponds with solutions

$$\phi(x) = e^{i\xi x},$$

so that

$$\lambda = -M(u_\pm)\gamma\xi^4 - M(u_\pm)F''(u_\pm)\xi^2.$$

By positivity of M and the double-well structure of F,

$$\sigma_{ess} = (-\infty, 0].$$

The Point Spectrum

First, for $\lambda \neq 0$, if $\phi(x;\lambda)$ solves

$$\Big(M(\bar{u})(-\gamma\phi_{xx} + F''(\bar{u})\phi)_x\Big)_x = \lambda\phi,$$

we can show by integrating both sides that $\int_{-\infty}^{+\infty} \phi(x;\lambda)dx = 0$. We set

$$\varphi(x;\lambda) := \int_{-\infty}^{x} \phi(y;\lambda)dy,$$

so that the integrated eigenvalue problem is

$$M(\bar{u})(-\gamma\varphi_{xxx} + F''(\bar{u})\varphi_x)_x = \lambda\varphi.$$

Divide by $M(\bar{u})$, multiply by φ and integrate to find

$$-\langle\varphi_x, -\gamma\varphi_{xxx} + F''(\bar{u})\varphi_x\rangle = \lambda\langle\frac{\varphi}{M(\bar{u})}, \varphi\rangle.$$

Here, $\langle\cdot,\cdot\rangle$ denotes L^2 inner product.

The Eigenvalues $\lambda \neq 0$

We can express this last equation as

$$-\langle \varphi_x, H\varphi_x \rangle = \lambda \langle \frac{\varphi}{M(\bar{u})}, \varphi \rangle,$$

where H is the Schrödinger type operator

$$H := -\gamma \partial_x^2 + F''(\bar{u}).$$

Now (as we'll see in a minute)

$$H\bar{u}' = 0,$$

and $\bar{u}'$ has a fixed sign by monotonicity, so by standard second order theory H is a non-negative operator. Noting also that H is self-adjoint, we conclude

$$\lambda < 0; \quad \text{i.e., } \sigma_{pt} \subset (-\infty, 0].$$

The Eigenvalue $\lambda = 0$

First, by definition

$$-\gamma\bar{u}_{xx} + F'(\bar{u}) = 0 \Rightarrow -\gamma\bar{u}_{xxx} + F''(\bar{u})\bar{u}_x = 0; \quad (H\bar{u}' = 0)$$

so that $\phi = \bar{u}_x$ solves

$$-\gamma\phi_{xx} + F''(\bar{u})\phi = 0.$$

We see that

$$\Big(M(\bar{u})(-\gamma\phi_{xx} + F''(\bar{u})\phi)_x\Big)_x = 0,$$

so that $\lambda = 0$ is an eigenvalue with associated eigenfunction $\bar{u}_x$.

The Evans Function

We characterize the eigenvalue $\lambda = 0$ in terms of the Evans function.

The eigenvalue problem $L\phi = \lambda\phi$ has four linearly independent solutions. For Arg $\lambda \neq \pi$, we can construct these so that two decay as $x \to -\infty$ and two grow as $x \to -\infty$. Alternatively, we can construct these so that two decay as $x \to +\infty$ and two grow as $x \to +\infty$. We denote the associated decaying solutions $\phi_1^+(x;\lambda)$, $\phi_2^+(x;\lambda)$, $\phi_1^-(x;\lambda)$, and $\phi_2^-(x;\lambda)$.

In this context, the Evans function is the Wronskian

$$D(\lambda) := \det \begin{pmatrix} \phi_1^- & \phi_2^- & \phi_1^+ & \phi_2^+ \\ {\phi_1^-}' & {\phi_2^-}' & {\phi_1^+}' & {\phi_2^+}' \\ {\phi_1^-}'' & {\phi_2^-}'' & {\phi_1^+}'' & {\phi_2^+}'' \\ {\phi_1^-}''' & {\phi_2^-}''' & {\phi_1^+}''' & {\phi_2^+}''' \end{pmatrix} \Bigg|_{x=0}.$$

The Evans Function

Any eigenfunction of L must decay at both $-\infty$ and $+\infty$, and so any such eigenfunction must, for some constants α_1, α_2, β_1, and β_2 satisfy

$$\alpha_1\phi_1^-(x;\lambda) + \alpha_2\phi_2^-(x;\lambda) = \phi(x;\lambda) = \beta_1\phi_1^+(x;\lambda) + \beta_2\phi_2^+(x;\lambda).$$

In this case $D(\lambda) = 0$, so D serves as a characteristic function for L. Since $\lambda = 0$ is an eigenvalue, we know $D(0) = 0$.

To further characterize $\lambda = 0$, we would like to compute $D'(0)$, $D''(0)$ etc., but $D(\lambda)$ is not differentiable at $\lambda = 0$.

The Stability Condition

More precisely $\phi_j^{\pm}(x;\lambda) = e^{\mu_j^{\pm}(\lambda)x}(1 + \mathbf{O}(e^{-\eta|x|}))$, where

$$\mu_1^{\pm}(\lambda) = \mp\sqrt{\frac{F''(u_\pm)}{\gamma}} + \mathbf{O}(|\lambda|)$$

$$\mu_2^{\pm}(\lambda) = \mp\sqrt{\frac{\lambda}{M(u_\pm)F''(u_\pm)}} + \mathbf{O}(|\lambda|^{3/2}).$$

We can express D as an analytic function of $\rho = \sqrt{\lambda}$. We let $D_a(\rho)$ denote this analytic function, and we find $D_a(0) = 0$ and $D_a'(0) = 0$. Using the uniqueness of $\bar{u}(x)$ (to the extent specified by our existence theorem), we can verify

$$D_a''(0) \neq 0.$$

Summary of Spectral Analysis

Combining the preceding observations, we have

$$\sigma_{ess} = (-\infty, 0]$$
$$\sigma_{pt} \subset (-\infty, 0]$$
$$D_a''(0) \neq 0.$$

Now, return to the full perturbation equation.

The Classical Semigroup Framework

We can write our full perturbation equation as

$$v_t = Lv + \dot{\delta}(t)(\bar{u}_x + v_x) + Q_x.$$

For an initial perturbation $v(x, 0)$, we can express $v(x, t)$ in semigroup formalism as

$$v(x,t) = e^{Lt} v_0 + \int_0^t e^{L(t-s)} \Big[\dot{\delta}(s)\bar{u}'(y) + \dot{\delta}(s) v_y(y,s) + Q_y \Big] ds,$$

where by Laplace transform

$$e^{Lt} := \frac{1}{2\pi i} \int_\Omega e^{\lambda t} R(\lambda; L) d\lambda.$$

Here, Ω denotes a contour in the resolvent set of L, entirely to the right of $\sigma(L)$, so that $\arg \lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$. In addition we can move Ω as allowed by Cauchy's Theorem.

Spectrum and Contour

Clearly, if $\sigma(L) \subset (-\infty, -\kappa]$ for some $\kappa > 0$ then e^{Lt} will decay at exponential rate in t.

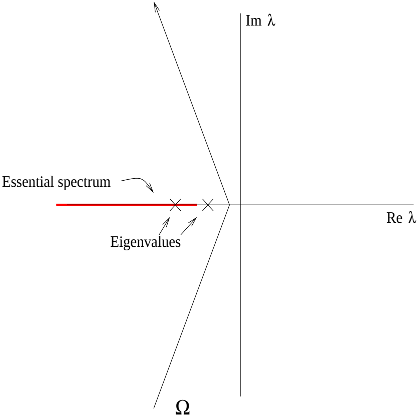

Figure: Example of a clearly stable spectrum.

Spectrum and Contour

By shift invariance, there will be an eigenvalue at $\lambda = 0$.

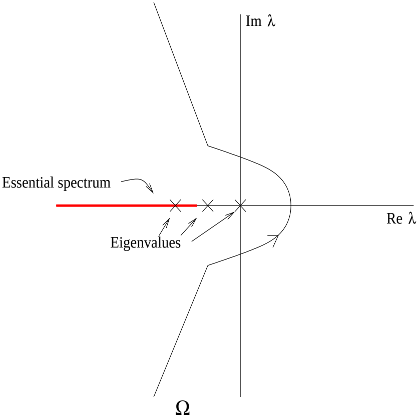

Figure: Neutral eigenvalue at $\lambda = 0$.

Spectrum and Contour

We can accomodate the eigenvalue at $\lambda = 0$ by separating out a term that does not decay in time.

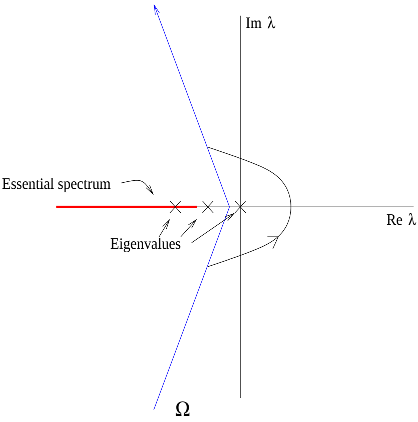

Figure: Accomodating the neutral eigenvalue.

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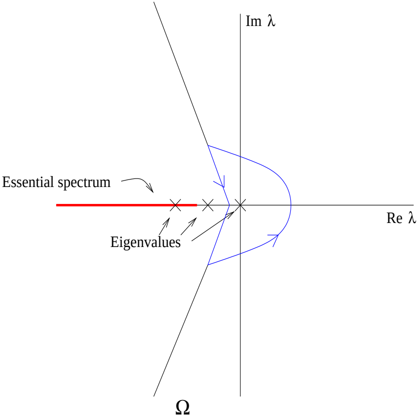

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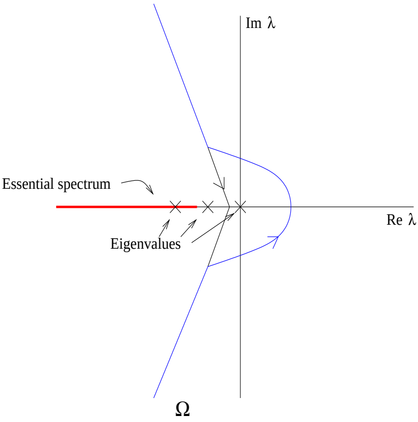

Figure: Accomodating the neutral eigenvalue.

Spectrum and Contour

For Cahn-Hilliard systems $\sigma_{ess} = (-\infty, 0]$.

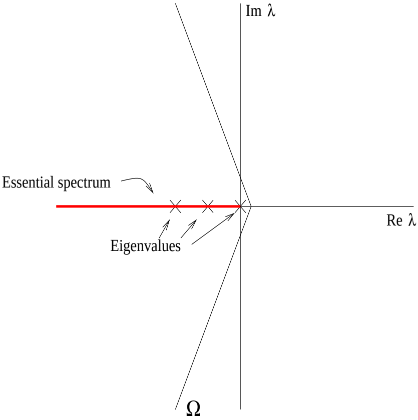

Figure: Essential spectrum for Cahn-Hilliard systems.

The Pointwise Semigroup Framework

Let $G(x,t;y)$ denote a Green's function (distribution) for $v_t = Lv$:

$$G_t = LG$$
$$G(x,0;y) = \delta_y(x).$$

We can express our semigroup operator e^{Lt} as

$$e^{Lt}f = \int_{-\infty}^{+\infty} G(x,t;y)f(y)dy.$$

Our expression for v becomes

$$v(x,t) = \int_{-\infty}^{+\infty} G(x,t;y)v_0(y)dy + \delta(t)\bar{u}'(x)$$
$$+ \int_0^t \int_{-\infty}^{+\infty} G(x,t-s,y)\Big[\dot{\delta}(s)v(y,s) + Q\Big]_y dy.$$

Constructing G

We compute the Laplace transform of G ($t \to \lambda$), writing $\mathcal{L}\{G\} = G_\lambda(x; y)$, so that

$$LG_\lambda - \lambda G_\lambda = -\delta_y(x).$$

Inverting, we find

$$G(x, t; y) = \frac{1}{2\pi i} \int_\Omega e^{\lambda t} G_\lambda(x; y) d\lambda,$$

where Ω denotes the same contour previously described. Note that

$$R(\lambda; L)f = \int_{-\infty}^{+\infty} G_\lambda(x; y) f(y) dy.$$

The Choice of Splitting

The main step in the analysis consists of deriving the splitting

$$G(x,t;y) = \bar{u}'(x)e(t;y) + \tilde{G}(x,t;y),$$

where $\bar{u}'(x)e(t;y)$ is a leading order term associated with $\lambda = 0$ that does not decay as $t \to \infty$ and $\tilde{G}(x,t;y)$ decays roughly like a heat kernel.

Intuitively, $\bar{u}'(x)e(t;y)$ captures behavior associated with $\lambda = 0$, while $\tilde{G}$ captures behavior associated with essential spectrum.

Choosing the Local Shift

We find (substituting $G = \bar{u}'e + \tilde{G}$ and integrating by parts)

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{+\infty} \tilde{G}(x,t;y)v_0(y)dy + \delta(t)\bar{u}'(x) \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\Big[\dot{\delta}(s)v(y,s) + Q\Big]dyds \\ &+\bar{u}'(x)\Big\{ \int_{-\infty}^{+\infty} e(t;y)v_0(y)dy \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(y,s) + Q\Big]dyds \Big\}. \end{aligned}$$

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$$+\bar{u}'(x)\Big\{ \int_{-\infty}^{+\infty} e(t;y)v_0(y)dy$$
$$- \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(y,s) + Q\Big] dyds \Big\}.$$

Choosing the Local Shift

We take

$$\delta(t) = -\int_{-\infty}^{+\infty} e(t;y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(y,s) + Q\Big]dyds,$$

which leaves

$$v(x,t) = \int_{-\infty}^{+\infty} \tilde{G}(x,t;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\Big[\dot{\delta}(s)v(y,s) + Q\Big]dyds.$$

We obtain similar integral equations for $\dot{\delta}(t)$ and $v_x(x,t)$, making 4 equations (3 since δ only appears as $\dot{\delta}$).

Green's Function Estimates

Using our spectral information and the structure of our equation, we find for $t \geq 1$

$$e(t;y) = c \int_{-\infty}^{\frac{y}{\sqrt{4\gamma M_- t}}} e^{-z^2} dz + R(t;y)$$

$$|R(t;y)| \leq C t^{-1/2} e^{-\frac{y^2}{At}},$$

and for $t \geq 1$ and $|x - y| \leq Kt$

$$|\tilde{G}(x,t;y)| \leq C t^{-1/2} e^{-\frac{(x-y)^2}{At}}.$$

Here, c, C, K, and A are fixed positive constants depending only on the spectrum of L and the structure of the equations.

Integral Equations

We need to recall our system of integral equations:

$$
\begin{aligned}
\dot{\delta}(t) &= -\int_{-\infty}^{+\infty} e_t(t;y)v_0(y)dy \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} e_{ty}(t-s;y)\left[\dot{\delta}(s)v(y,s)+Q\right]dyds, \\
v(x,t) &= \int_{-\infty}^{+\infty} \tilde{G}(x,t;y)v_0(y)dy \\
&\quad - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\left[\dot{\delta}(s)v(y,s)+Q\right]dyds \\
v_x(x,t) &= \int_{-\infty}^{+\infty} \tilde{G}_x(x,t;y)v_0(y)dy \\
&\quad - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_{xy}(x,t-s;y)\left[\dot{\delta}(s)v(y,s)+Q\right]dyds.
\end{aligned}
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\end{aligned}
$$

Linear Stability

Using the estimates we obtain on G, we can verify that

$$\left| \int_{-\infty}^{+\infty} e_t(t;y)v_0(y)dy \right| \leq C(1+t)^{-1}$$

$$\left\| \int_{-\infty}^{+\infty} \tilde{G}(x,t;y)v_0(y)dy \right\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}$$

$$\left\| \int_{-\infty}^{+\infty} \tilde{G}_x(x,t;y)v_0(y)dy \right\|_{L^p} \leq Ct^{-1/4}(1+t)^{-\frac{1}{4}}.$$

Bounding Function $\zeta(t)$

We take these values as preliminary estimates and set

$$\zeta(t) := \sup_{\substack{0 \le s \le t \\ 1 \le p \le \infty}} \Big\{ \|v(\cdot, s)\|_{L^p} (1+s)^{\frac{1}{2}(1-\frac{1}{p})} + \|v_x(\cdot, s)\|_{L^p} s^{1/4} (1+s)^{1/4} + |\dot{\delta}(s)|(1+s) \Big\}.$$

Clearly,

$$\begin{aligned} \|v(\cdot, s)\|_{L^p} &\le \zeta(t)(1+s)^{-\frac{1}{2}(1-\frac{1}{p})} \\ \|v_x(\cdot, s)\|_{L^p} &\le \zeta(t) s^{-1/4} (1+s)^{-1/4} \\ |\dot{\delta}(s)| &\le \zeta(t)(1+s)^{-1}. \end{aligned}$$

Nonlinear Stability

Upon substitution of these bounds into our integral equations, we find

$$\zeta(t) \leq C(\epsilon + \zeta(t)^2).$$

Here C is a new constant and we recall

$$\|v(\cdot, 0)\|_{L^1} + \|v(\cdot, 0)\|_{L^\infty} \leq \epsilon.$$

It's straightforward to verify that this implies

$$\zeta(t) < 2C\epsilon$$

for all $t \geq 0$. This inequality is equivalent with the statement of our theorem.

Multiple Space Dimensions

For $x \in \mathbb{R}^n$ the Cahn-Hilliard equation is

$$u_t = \nabla \cdot \{M(u)\nabla(-\gamma\Delta u + F'(u))\}.$$

In this case, we consider planar transition fronts $\bar{u}(x_1)$. Since $\bar{u}$ depends only on x_1, planar transition fronts have the same structure as transition fronts for $n = 1$.

In this case, our shift δ can depend on the transverse variable $\tilde{x} := (x_2, x_3, \dots, x_n)$, and we define our perturbation variable as

$$v(x, t) := u(x, t) - \bar{u}(x_1 - \delta(\tilde{x}, t)).$$

The Shift For $n = 2$

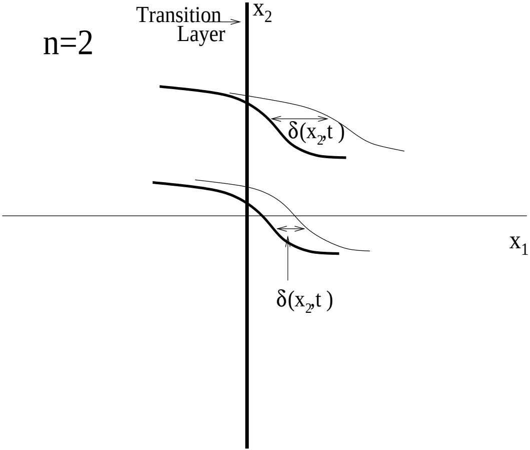

Figure: Shift function $\delta(x_2, t)$ for $n = 2$.

Linearization

Upon substitution of

$$v(x,t) := u(x,t) - \bar{u}(x_1 - \delta(\tilde{x},t))$$

into the Cahn-Hilliard equation we obtain

$$(\partial_t - L)v = (\partial_t - L)(\delta(\tilde{x},t)\bar{u}'(x_1)) + \nabla \cdot Q,$$

where

$$\begin{aligned} Lv &= \nabla \cdot \{M(\bar{u})\nabla(-\gamma\Delta v + F''(\bar{u})v)\} \\ Q &= Q(\bar{u}, \delta, v). \end{aligned}$$

The Associated Eigenvalue Problem

The relevant eigenvalue problem is $L\phi = \lambda\phi$. We take a Fourier transform in the transverse variable $\tilde{x}$ (transforming $\tilde{x} \to \xi$):

$$L_\xi \hat{\phi} := -D_\xi H_\xi \hat{\phi} = \lambda \hat{\phi}.$$

Here

$$\begin{aligned} D_\xi &:= -\partial_{x_1} M(\bar{u}) \partial_{x_1} + |\xi|^2 M(\bar{u}) \\ H_\xi &:= -\gamma \partial_{x_1 x_1} + (F''(\bar{u}) + \gamma |\xi|^2). \end{aligned}$$

Using asymptotic considerations again, we can verify that the essential spectrum for L_ξ is

$$\sigma_{ess} = \Big(-\infty, -M(u_\pm)F''(u_\pm)|\xi|^2 - \gamma M(u_\pm)|\xi|^4 \Big].$$

The Point Spectrum for $\xi \neq 0$

For $\xi \neq 0$ we set $\varphi := D_\xi^{-1/2} \hat{\phi}$ (akin to integrating for $n = 1$) so that

$$\mathcal{L}_\xi \varphi := D_\xi^{1/2} H_\xi D_\xi^{1/2} \varphi = -\lambda \varphi.$$

We can take advantage of the observation that $\mathcal{L}_\xi$ is self-adjoint to verify that for $M_0 := \min_{x_1 \in \mathbb{R}} M(\bar{u}(x_1))$

$$\sigma_{pt} \subset (-\infty, -\gamma M_0 |\xi|^4].$$

Using Evans function techniques, we can show that for $|\xi|$ sufficiently small, the leading (right-most) eigenvalue of $\mathcal{L}_\xi$ satisfies

$$\lambda_*(\xi) = -\lambda_3 |\xi|^3 + \mathbf{O}(|\xi|^4),$$

where

$$\lambda_3 = \frac{\sqrt{2\gamma}(M(u_-) + M(u_+))}{|u_+ - u_-|^2} \int_{\min\{u_-, u_+\}}^{\max\{u_-, u_+\}} \sqrt{F(x) - F(u_-)} dx.$$

Notes on The Scaling

The cubic scaling for the leading eigenvalue was anticipated by the physical observation that if S denotes the average pattern size during spinodal decomposition then asymptotically

$$S \sim t^{1/3}.$$

The associated behavior of the leading eigenvalue seems first to have been recognized by David Jasnow and R. K. P. Zia in 1987.

Stability Theorem for $n \geq 2$

Suppose $\bar{u}(x_1)$ is a planar transition front as described in the existence theorem. Then for Hölder continuous initial conditions $u_0(x) \in C^\gamma(\mathbb{R}^n)$, $0 < \gamma < 1$, with

$$\|u(0,x) - \bar{u}(x_1)\|_{L^1_x} + \|u(0,x) - \bar{u}(x_1)\|_{L^\infty_x} \leq \epsilon,$$

for $\epsilon > 0$ sufficiently small, there exists a unique solution of (CH)

$$u \in C^{4+\gamma, 1+\frac{\gamma}{4}}(\mathbb{R}^n \times (0,\infty)) \cap C^{\gamma,\frac{\gamma}{4}}(\mathbb{R}^n \times [0,\infty))$$

and a shift $\delta \in C^{3+\gamma,1+\gamma}\left(\mathbb{R}^{n-1} \times [0,\infty)\right)$ so that

$$\|u(x,t) - \bar{u}(x_1 - \delta(\tilde{x},t))\|_{L^p_x} \leq C\epsilon\Big[(1+t)^{-\frac{n}{2}(1-\frac{1}{p})} + (1+t)^{-\frac{n-1}{3}(1-\frac{1}{p})-\frac{2}{3}+\frac{1}{2p}} h_{n,p}(t)\Big]$$

$$\|\delta(\tilde{x},t)\|_{L^p_{\tilde{x}}} \leq C\epsilon(1+t)^{-\frac{1}{3}(1-\frac{1}{p})}.$$

The function $h_{n,p}(t)$

For any $\sigma > 0$, we can take

$$h_{n,p}(t) = \begin{cases} 1 & n = 2, 1 \le p < \infty \\ \log(e + t) & n = 2, p = \infty \\ (1 + t)^\sigma & n \ge 3, 1 \le p \le \infty. \end{cases}$$

Systems for $n = 1$

For $u \in \mathbb{R}^m$ and $x \in \mathbb{R}$, we have systems of the form

$$u_t = \Big(M(u)(-\Gamma u_{xx} + F'(u))_x \Big)_x,$$

where we expect F to have $m + 1$ distinct minimizers $\{\xi_j\}_{j=1}^{m+1}$ associated with energy-preferred phases of the mixture. A common example for $m = 2$ is

$$F(u) = u_1^2 u_2^2 + u_1^2 (1 - u_1 - u_2)^2 + u_2^2 (1 - u_1 - u_2)^2,$$

with minimizers $(0, 0)$, $(1, 0)$, and $(0, 1)$.

In this case, existence of transition fronts is more complicated, because $\bar{u}(x)$ solves the system

$$-\Gamma \bar{u}_{xx} + F'(\bar{u}) = 0.$$

Existence of Transition Fronts

The existence of transition fronts for Cahn-Hilliard systems has been established under quite general conditions by:

- N. D. Alikakos, S. I. Betelu, and X. Chen (2006): for $m = 2$, using complex analysis
- N. D. Alikakos and G. Fusco (2008): $m \geq 2$, for $\Gamma = I$
- V. Stefanopoulos (2008): $m \geq 2$, for Γ positive definite and symmetric

In each of these references, the transition fronts arise as minimizers of the associated energy functional

$$E(u) = \int_{-\infty}^{+\infty} F(u) + \frac{1}{2}\langle u_x, \Gamma u_x \rangle dx.$$

Example Case

For $m = 2$, $M(u) \equiv I$, $\Gamma = I$, and

$$F(u) = u_1^2 u_2^2 + u_1^2(1 - u_1 - u_2)^2 + u_2^2(1 - u_1 - u_2)^2.$$

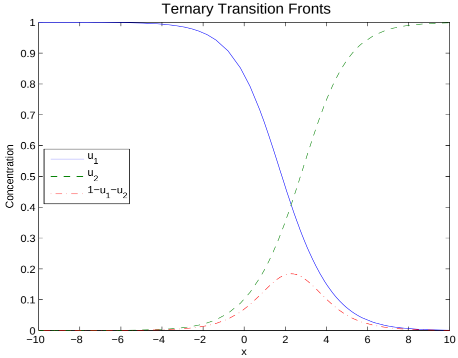

Figure: Ternary Transition Front.

The Associated Eigenvalue Problem

We linearize as in the case of a single equation with

$$v(x,t) := u(x+\delta(t),t) - \bar{u}(x).$$

The associated eigenvalue problem is

$$L\phi := \Big(M(\bar{u})(-\Gamma\phi_{xx} + F''(\bar{u})\phi)_x\Big)_x = \lambda\phi.$$

The fact that $\bar{u}(x)$ minimizes E can be used to verify

$$\sigma(L) \subset (-\infty, 0].$$

The primary difficulty involves the Evans function condition at $\lambda = 0$.

The Evans Function

Our eigenvalue problem

$$\Big(M(\bar{u})(-\Gamma\phi_{xx} + F'(\bar{u})\phi)_x\Big)_x = \lambda\phi$$

has $2m$ solutions that decay as $x \to -\infty$, $\{\phi_j^-\}_{j=1}^{2m}$ and $2m$ solutions that decay as $x \to +\infty$, $\{\phi_j^+\}_{j=1}^{2m}$. We will set

$$\Phi_j^\pm := \begin{pmatrix} \phi_j^\pm \\ \phi_j^{\pm\prime} \\ \phi_j^{\pm\prime\prime} \\ \phi_j^{\pm\prime\prime\prime} \end{pmatrix}, \quad \Phi^\pm := (\Phi_1^\pm, \dots, \Phi_{2m}^\pm).$$

The Evans function is

$$D(\lambda) = \det(\Phi^+(0;\lambda), \Phi^-(0;\lambda)).$$

Characteristics of the Evans Function

Recall that any eigenfunction ϕ must be in H^2, and so it must decay at both $\pm\infty$. In this way, there must exist constants $\{\alpha_j\}_{j=1}^{2m}$ and $\{\beta_j\}_{j=1}^{2m}$ so that

$$\sum_{j=1}^{2m} \alpha_j \phi_j^-(x;\lambda) = \phi(x;\lambda) = \sum_{j=1}^{2m} \beta_j \phi_j^+(x;\lambda).$$

By linear dependence $D(\lambda) = 0$. We know $\lambda = 0$ is an eigenvalue, and it follows that $D(0) = 0$.

We would like to characterize this eigenvalue further by computing $D'(0)$, $D''(0)$, etc. However, D is not differentiable at $\lambda = 0$.

Analyticity of the Evans Function

The solutions $\{\phi_j^-\}_{j=1}^{2m}$ and $\{\phi_j^+\}_{j=1}^{2m}$ have the form

$$\phi_j^-(x;\lambda) = e^{\mu_{2m+j}^-(\lambda)x}(r_{2m+j}^- + \mathbf{O}(e^{-\eta|x|}))$$
$$\phi_j^+(x;\lambda) = e^{\mu_j^+(\lambda)x}(r_j^+ + \mathbf{O}(e^{-\eta|x|})),$$

where for $j = 1, \ldots, m$

$$\mu_j^\pm(\lambda) = -\sqrt{\nu_{m+1-j}^\pm} + \mathbf{O}(|\lambda|)$$
$$\mu_{m+j}^\pm(\lambda) = -\sqrt{\frac{\lambda}{\beta_j^\pm}} + \mathbf{O}(|\lambda|^{3/2})$$
$$\mu_{2m+j}^\pm(\lambda) = \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + \mathbf{O}(|\lambda|^{3/2})$$
$$\mu_{3m+j}^\pm(\lambda) = \sqrt{\nu_j^\pm} + \mathbf{O}(|\lambda|).$$

The Stability Condition

We can view the Evans function as an analytic function of $\rho = \sqrt{\lambda}$. We denote this function $D_a(\rho)$. It is straightforward to verify that

$$D_a(0) = D_a'(0) = \cdots = D_a^{(m)}(0) = 0.$$

Our stability condition is

$$\frac{d^{m+1} D_a}{d\rho^{m+1}}(0) \neq 0.$$

For $m = 1$ (the case of a single equation) it's easy to verify that this holds under standard physical assumptions. For $m \geq 2$, we have developed a framework for verifying this condition on a case-by-case basis.

Spectral Stability

It's relatively easy to verify that

$$\sigma_{ess} = (-\infty, 0].$$

If $\bar{u}$ minimizes the Cahn-Hilliard energy, it's easy to show that

$$\sigma_{pt}(L)\backslash\{0\} \subset (-\infty, -\kappa]$$

for some $\kappa > 0$. If these conditions both hold, along with our stability condition

$$\frac{d^{m+1} D_a}{d\rho^{m+1}}(0) \neq 0.$$

we say $\bar{u}(x)$ is spectrally stable.

Stability Theorem

Suppose $\bar{u}(x)$ is a spectrally stable transition front. Then for Hölder continuous initial conditions $u_0(x) \in C^\gamma(\mathbb{R})$, $0 < \gamma < 1$, with

$$\|u(0,x) - \bar{u}(x)\|_{L^1} + \|u(0,x) - \bar{u}(x)\|_{L^\infty} \leq \epsilon,$$

for $\epsilon > 0$ sufficiently small, there exists a unique solution of (CH)

$$u \in C^{4+\gamma, 1+\frac{\gamma}{4}}(\mathbb{R} \times (0,\infty)) \cap C^{\gamma,\frac{\gamma}{4}}(\mathbb{R} \times [0,\infty))$$

and a shift $\delta \in C^{1+\gamma}[0,\infty)$ so that

$$\|u(x+\delta(t),t) - \bar{u}(x)\|_{L^p} \leq C\epsilon(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}$$
$$|\delta(t) - \delta_\infty| \leq C\epsilon(1+t)^{-1/4}.$$

Further Work

- Extension to the case $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $n \geq 2$, $m \geq 2$
- Periodic and pulse-type stationary solutions for $m \geq 2$
- Genuinely multidimensional stationary solutions for $n \geq 2$
- Coarsening dynamics
- Navier-Stokes-Cahn-Hilliard systems
- The functionalized Cahn-Hilliard equation

$$u_t = \Delta\Big\{(\epsilon^2\Delta - F''(u) - \epsilon\eta_1)(\epsilon^2\Delta u - F'(u)) + \epsilon(\eta_2 - \eta_1)F'(u)\Big\}$$