Top Lyapunov exponent of inertial particle pair separation

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Inertial particle:

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} \qquad\qquad \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\frac{1}{\tau}[\vec{v} - \vec{u}(\vec{r}, t)]$$

↖

Stokes time

Inertial particle pair separation:

$$\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = \vec{V} \qquad\qquad \frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = -\frac{1}{\tau}[\vec{V} - \sigma(t)\vec{R}]$$

↗

strain matrix of velocity field

$$\sigma_{ij}(t) = \partial_j u_i((\vec{r}(t), t)$$

If the velocity field $\vec{u}$ is a 2D smooth Kraichnan field, then some exact results can be given.
In this model $\sigma(t)$ is a matrix-valued Gaussian white noise:

$$\langle \sigma_{ij}(t)\sigma_{mn}(t')\rangle = \delta(t-t')\, C_{ijmn} = \delta(t-t')\, 2D\, \tilde{C}_{ijmn}$$

$$\tilde{C}_{ijmn} = (d + 1 - 2\wp)\delta_{im}\delta_{jn} + (\wp d - 1)(\delta_{ij}\delta_{mn} + \delta_{in}\delta_{mj})$$

↗

$\wp$: compressibility degree

$\sqrt{\wp}$ potential field + $\sqrt{1-\wp}$ solenoidal field

Notice that $\vec{V}$ is driven by the noise $\frac{1}{\tau}\sigma\vec{R}$.

$\longrightarrow$ Gaussian white noise in $\mathbb{R}^d$: only its covariance matters

$\longrightarrow$ Replace σ with $\tilde{\sigma}$; in $d = 2$:

$$\tilde{\sigma} = \begin{pmatrix} \sqrt{\beta_L}\eta_1 & -\sqrt{\beta_N}\eta_2 \\ \sqrt{\beta_N}\eta_2 & -\sqrt{\beta_L}\eta_1 \end{pmatrix} \qquad \begin{aligned} \beta_L &= \tfrac{2D}{\tau^2}(2\wp + 1) \\ \beta_N &= \tfrac{2D}{\tau^2}(3 - 2\wp) \end{aligned}$$

Noticing that $\tilde{\sigma}$ is also the matrix notation of the complex multiplication by:

$$\tilde{\sigma} = \sqrt{\beta_L}\eta_1 + i\sqrt{\beta_N}\eta_2$$

pass to complex notation also for $\vec{R}$, $\vec{V}$.

Alternative forms: introduce $\mathcal{U} = \frac{\tilde{\sigma}}{\tau}$ and $E = -\frac{1}{4\tau^2}$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -z^2 - E + \mathcal{U} \qquad z = \frac{V}{R} + \frac{1}{2\tau}$$

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} + \mathcal{U}\psi = E\psi \qquad \psi = e^{\frac{t}{2\tau}}R$$

Lyapunov exponent:

$$\lambda = \left\langle \frac{\mathrm{d}\log R}{\mathrm{d}t} \right\rangle = \langle z \rangle - \frac{1}{2\tau} = \left\langle \frac{\mathrm{d}\log\psi}{\mathrm{d}t} \right\rangle - \frac{1}{2\tau}$$

2 special cases with explicit formula for Lyapunov exponent

$\beta_N = 0$ $\quad \wp = \frac{3}{2}$ (or pure dilatation)

$\mathcal{U}$ real → problem reduces to real Anderson equation (not completely trivial, since initially $\vec{R} \nparallel \vec{V}$)

$$\lambda = \frac{1}{2\tau}\left[-1 + c^{-\frac{1}{2}}\frac{Ai'(c)Ai(c) + Bi'(c)Bi(c)}{Ai^2(c) + Bi^2(c)}\right]$$

$$c = \frac{1}{4\tau^2\left(\frac{\beta_L}{2}\right)^{2/3}} \qquad Ai,\ Bi: \text{ Airy functions}$$

$\beta_L = 0$ $\quad \wp = -\frac{1}{2}$ (or pure rotation)

In the holomorphic writing, equilibrium distribution of z is supported by half-plane $\Re z > \frac{1}{2\tau}$, so that the "complex" Laplace transform $\langle e^{-pz}\rangle$, $p \in \mathbb{R}_+$ is well defined

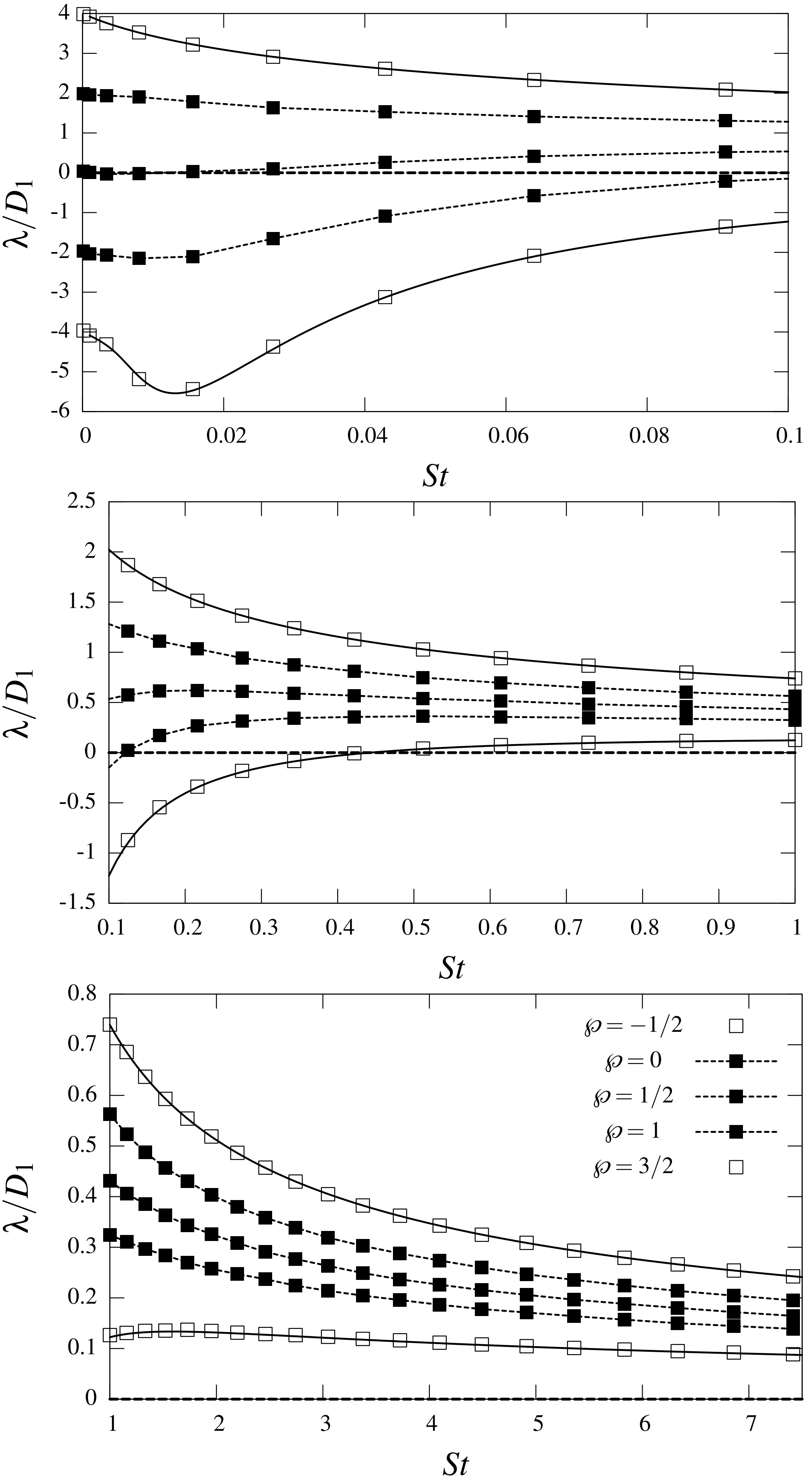
$$\lambda = \frac{1}{2\tau}\left[-1 - c^{-\frac{1}{2}}\frac{Ai'(c)}{Ai(c)}\right]$$

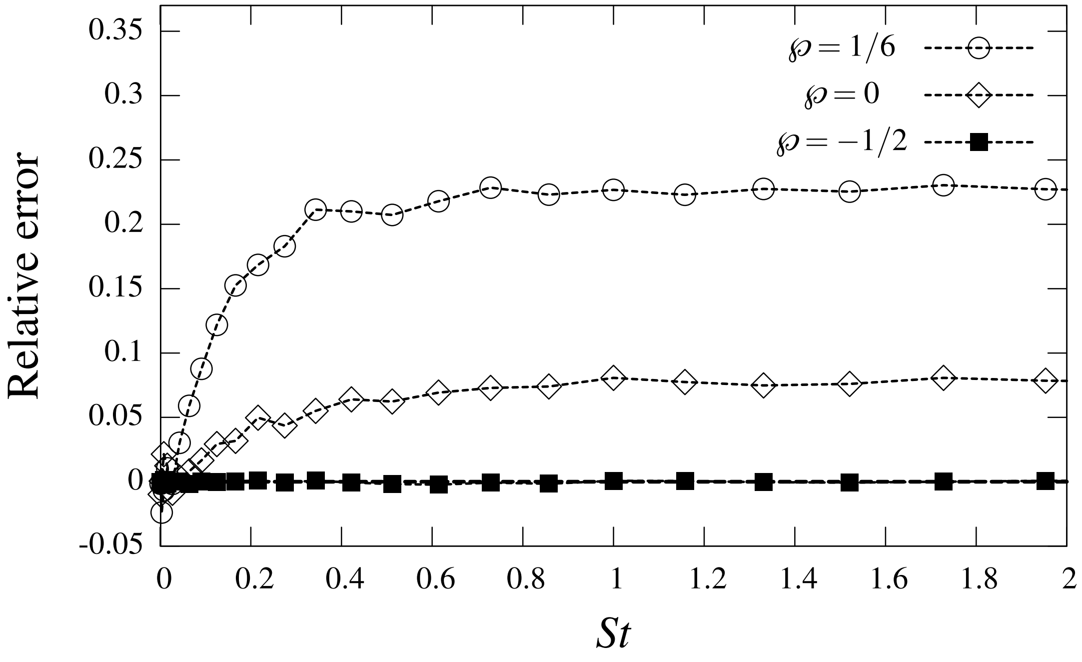
$$c = \frac{1}{4\tau^2\left(\frac{\beta_N}{2}\right)^{2/3}}$$

Other solvable cases if we allow for

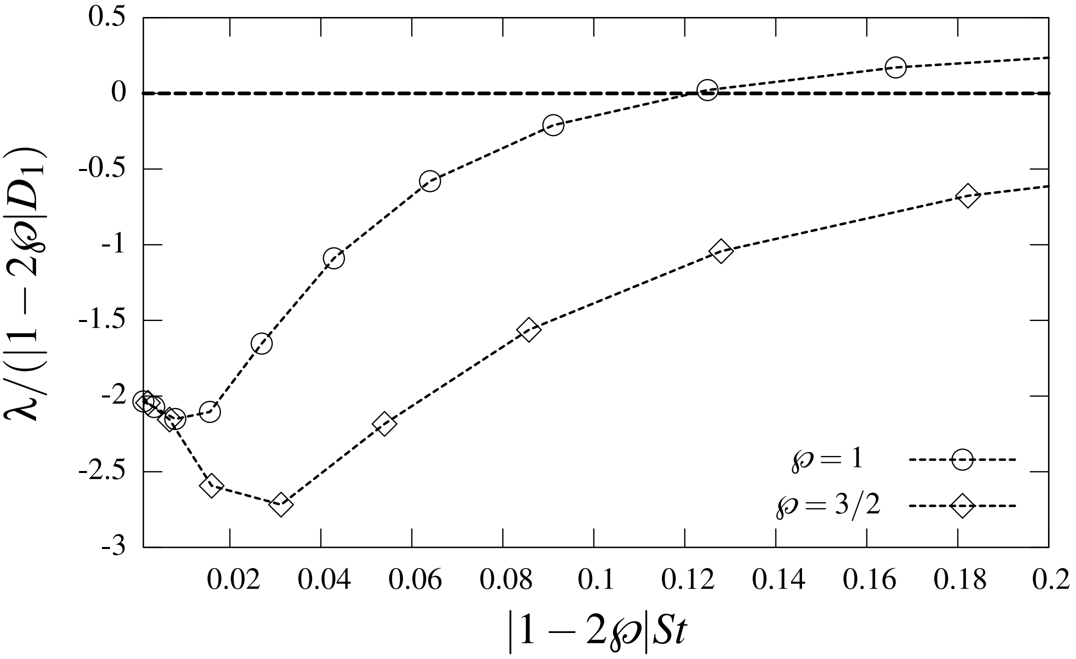
- breaking of spatial homogeneity of velocity field:
  → only homogenous *increments*
- breaking of parity invariant statistics

Adimensionalized Lyapunov exponent in function of Stokes number at different values of compressibility degree $\wp$





Relative error of numerical results with respect to the "analytical" formula , for $\wp < 1/2$. A zoom (not represented) on the curves for very small St is not incompatible with the prediction that all derivatives of the relative error curves should vanish at $St = 0$, but quality of our data in that range is too poor.



Non-collapse of numeric curves for $\wp > 1/2$

Numerical simulations

$\longrightarrow$ λ seems monotone with $\wp$

$\Rightarrow$ for $0 \leq \wp \leq 1$: $\lambda_{\frac{3}{2}} < \lambda_{\wp} < \lambda_{-\frac{1}{2}}$

notice: $\lambda_{\frac{3}{2}}, \lambda_{-\frac{1}{2}} \underset{\tau\to\infty}{\sim} \tau^{-\frac{2}{3}}$ $\longrightarrow$ valid for all $\lambda_{\wp}$, $0 \leq \wp \leq 1$

$\longrightarrow$ λ goes to passive tracer Lyapunov as $\tau \to 0$

$\longrightarrow$ For τ large enough, $\lambda > 0$ always!

Difficulty of simulation: stiffness

$\frac{1}{D\tau}$ is a large parameter when τ is small

To overcome this, write:

$$\mathrm{d}\begin{pmatrix}\vec{R}\\ \vec{V}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & -\frac{1}{\tau}\end{pmatrix}\begin{pmatrix}\vec{R}\\ \vec{V}\end{pmatrix}\mathrm{d}t + \begin{pmatrix}0\\ \sigma(t)\underset{\vec{R}(0)}{\vec{R}(\cancel{t})}\end{pmatrix}$$

$\longrightarrow$ Linear system with constant coefficient

$\Rightarrow$ can be solved exactly over a time-step

Asymptotic analysis

Consider regime of $\lambda\tau \gg 1$, do self-consistent analysis

$$\begin{array}{c} \dfrac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} + \dfrac{1}{\tau}\dfrac{\mathrm{d}\vec{R}}{\mathrm{d}t} = \dfrac{\sigma}{\tau}\vec{R} \\ \uparrow \qquad\qquad \uparrow \\ \lambda^2 R \gg \dfrac{\lambda}{\tau} R \end{array}$$

The simplified equation reads

$$\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} = \frac{\sigma}{\tau}\vec{R}$$

Now dimensional analysis is possible,
the only paramter is $D^{1/2}/\tau$.

$$[D^{1/2}/\tau] = T^{-3/2} \qquad\qquad [D] = T^{-1}$$

whence

$$\lambda \propto (D^{1/2}/\tau)^{2/3} = D^{1/3}\tau^{-2/3}$$

Self consistent when $\lambda\tau \propto (D\tau)^{1/3} = St^{1/3} \gg 1$.

Positive even order moments of pair separation

Start by recalling

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}+\mathcal{U}\psi = E\psi$$

Introduce

$$c^n_{k,l} = \langle \psi^k \dot{\psi}^{n-k} \bar{\psi}^l \dot{\bar{\psi}}^{n-l} \rangle \qquad \langle |\psi|^{2n} \rangle = c^n_{n,n}$$

Then for any n the $c^n_{k,l}$ verify the closed system

$$\begin{aligned}
\frac{\mathrm{d}c^n_{k,l}}{\mathrm{d}t} = \quad & k c^n_{k-1,l} - E(n-k) c^n_{k+1,l} \\
& + l c^n_{k,l-1} - E(n-l) c^n_{k,l+1} \\
& + \tfrac{(n-k)(n-k-1)}{2} (\beta_L - \beta_N) c^n_{k+2,l} \\
& + \tfrac{(n-l)(n-l-1)}{2} (\beta_L - \beta_N) c^n_{k,l+2} \\
& + (n-k)(n-l)(\beta_L + \beta_N) c^n_{k+1,l+1}
\end{aligned}$$

Application to real turbulence within Kolmogorov phenomenology

We consider heavy particles, ie. $\tau/t_\eta = St \gg 1$.
Inertial drift velocity $\vec{w}(t) = \vec{v}(t) - \vec{u}(\vec{r}(t), t)$.
By dimensional analysis $w \propto \sqrt{\epsilon\tau}$. more explicit arguments are also possible Note also that w changes on the timescale τ.
Important timescale: η/w, traversal time of Kolmogorov scale by inertial particle

$$\frac{t_\eta}{\eta/w} \sim \frac{t_\eta \epsilon^{1/2}}{\eta}\tau^{1/2} = (\tau/t_\eta)^{1/2} = St^{1/2} > 1$$

Start from

$$\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} + \frac{1}{\tau}\frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = \frac{\sigma}{\tau}\vec{R}$$

This equation can be averaged over time interval Δt.
If $\Delta t \ll \lambda^{-1}$ then only σ is averaged. Note that is a self-consistency condition that ultimately needs to be checked.
Interesting case is when we average a large number of independent σ.

Simplest case $St^{1/2} \ggg 1$.

Take $\eta/w \ll \Delta t \ll t_\eta \ll \tau$. Velocity field is effectively frozen during Δt and w is constant.

$$\bar\sigma_{ij} = \int_t^{t+\Delta t} \frac{\mathrm{d}t'}{\Delta t}\,\sigma_{ij}(\vec r(t'),t') = \int_t^{t+\Delta t} \frac{\mathrm{d}t'}{\Delta t}\,\sigma_{ij}(\underbrace{\vec r(t)+(t'-t)\vec w}_{=\vec r(t')}, \underset{\substack{\uparrow \\ \sigma(\cdot,t')=\sigma(\cdot,t)}}{t})$$

$\bar\sigma_{ij}$ will be Gaussian as average of a large number of independent elements. $\longrightarrow$ As usual we deduce the diffusion coefficient of the equivalent white noise as

$$C_{ijmn} = \int_{-\infty}^{\infty} \mathrm{d}t' \,\langle \sigma_{ij}(\vec 0,0)\sigma_{mn}(t'\vec w,0)\rangle = 2\tilde D w^{-1}\tilde C_{ijmn}(\hat w)$$

One can derive $\tilde D$ and $\tilde C$ from the 2-pt structure function of the velocity field under the hypotheses of incompressibility and isotropy:

$$\tilde D = \frac{1}{2(d-1)} \int_0^\infty \frac{S_2(r)}{r^2}\,\mathrm{d}r$$

$$\begin{aligned}\tilde C_{ijmn}(\hat w) = {} & d\delta_{im}(\delta_{jn} - \hat w_j \hat w_n) - \delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj} \\ & - 3\hat w_i\hat w_j\hat w_m\hat w_n + \hat w_i\hat w_m\delta_{jn} + \hat w_i\hat w_j\delta_{mn} + \hat w_m\hat w_j\delta_{in} \\ & + \hat w_i\hat w_n\delta_{mj} + \hat w_m\hat w_n\delta_{ij}\end{aligned}$$

Special degeneracy

C_{ijmn} has a special degeneracy:

$$\hat{z}_n C_{ijmn}(\hat{z}) = 0$$

consequently also $\hat{z}_j C_{ijmn}(\hat{z}) = 0$ because $C_{ijmn} = C_{mnij}$

Indeed

$$\hat{z}_n C_{ijmn} = \int_{-\infty}^{\infty} dt \frac{\partial}{\partial t} \langle \nabla_j u_i(\vec{0}) u_m(t\hat{z}) \rangle = 0$$

$\rightarrow$ $w_j \langle \sigma_{ij}(t_1) \sigma_{mn}(t_2) \rangle = 0$ for any i, m and n

$\rightarrow$ one can set $\sigma_{ij} w_j = 0$

$\rightarrow$ no local stretching in the direction of $\vec{w}$

$\rightarrow$ evolution of components of $\vec{R}, \vec{V}$ transverse to $\vec{w}$ is independent of longitudinal ones, described by isotropic $d-1$ dimensional Kraichnan model! (with $D = \tilde{D} w^{-1}$ and $\wp = 0$)

$\rightarrow$ if $\lambda_{\mathrm{Kr}} \gg \tau^{-1}$ then λ_{Kr} should be the "real" Lyapunov exponent

In general at large t

$$\int_0^t \nabla_j u_i(t' \hat{w}) dt'/t \sim t^{-1/2}$$

due to short-correlated increments. However

$$\int_0^t \hat{w} \cdot \nabla u_i(t' \hat{w}) dt'/t = [u_i(t\hat{w}) - u_i(\vec{0})]/t \sim t^{-2/3} \ll t^{-1/2}$$

Subdiffusive growth can be explained by the anti-correlation of the relevant increments (similar to the situation holding for fractional Brownian motion with Hurst exponent less than $1/2$). This special smallness then produces zero in the considered order of approximation that corresponds to white-noise description.

Asymptotic analysis

Consider again regime of $\lambda\tau \gg 1$

As above, we have $\lambda \propto D^{1/3}\tau^{-2/3}$

However now $D = \tilde{D}w^{-1} \propto \tau^{-1/2}$ ($\tilde{D}$ indep. of τ)

$\rightarrow$
$$\lambda \propto \tilde{D}^{1/3}\epsilon^{-1/6}\tau^{-5/6}$$

Self consistent when $\lambda\tau \propto \tilde{D}^{1/3}\epsilon^{-1/6}\tau^{1/6} \sim St^{1/6} \gg 1$.