GEOMETRY of GERBES and CONFORMAL FIELD THEORY

Krzysztof Gawędzki, Helsinki, Oct. 2006

- A bit of history
- Why (abelian) gerbes?
- Local data
- Gerbes on Lie groups and WZW models
- Quantization
- Gauge fields twisted by gerbes and branes
- Gerbes and orientifolds
- Conclusions

A bit of history:

non abelian cohomology
and abstract gerbes
(Giraud 1971)

Deligne cohomology
(Deligne, Beilinson, mid 1980's)

Deligne cohomology in topological
field theory and WZW models
(K.G. 1987)

loop spaces, characteristic classes
and geometric quantization
(Brylinski 1991)

(line-)bundle gerbes
(Murray 1994, Murray-Stevenson 1999)

gerbes, anomalies and index
(Carey-Mickelsson-Murray 1995)

gerbes on Lie groups

(Chatterjee-Hitchin 1998)

(Brylinski 2000)

(Meinrenken 2002)

(K.G.-Reis 2003)

gerbe modules, twisted gauge fields and branes

(Kapustin 1999)

gerbes, gerbe modules and twisted K-theory

(Bouwknegt-Carey-Mathai-Murray-Stevenson 2001)

(Mickelsson et al. 2002-)

gerbe modules and WZW branes

(K.G.-Reis 2002)

(K.G. 2004)

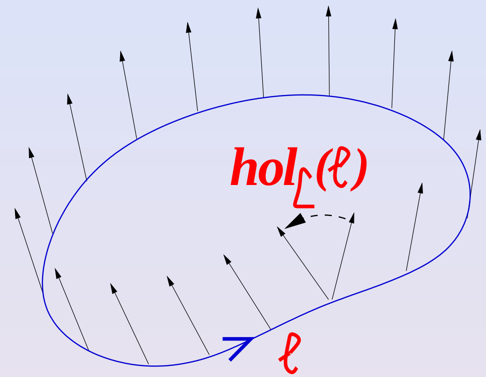
gerbes and orientifolds

(Schreiber-Schweigert-Waldorf 2006)

Line bundles (with connection)

- For an exact 2-form $F = dA$ on M,

$$\int_S F = \int_{\partial S} A$$ **Stokes Theorem**



- If F is closed with periods $\in 2\pi \boldsymbol{Z}$ then

$$\int_S F = \frac{1}{\sqrt{-1}} \log hol_{\mathcal{L}}(\partial S) \mod 2\pi$$

if $\mathcal{L}$ is a line bundle of curvature F with $hol_{\mathcal{L}}(\ell)$ denoting the **holonomy** along the closed curve $\ell : S^1 \longrightarrow M$

- For flat line bundles $\mathcal{L}$ with $F = 0$

$$hol_{\mathcal{L}}(\cdot) \in \pi_1(M)^* = H^1(M, U(1))$$

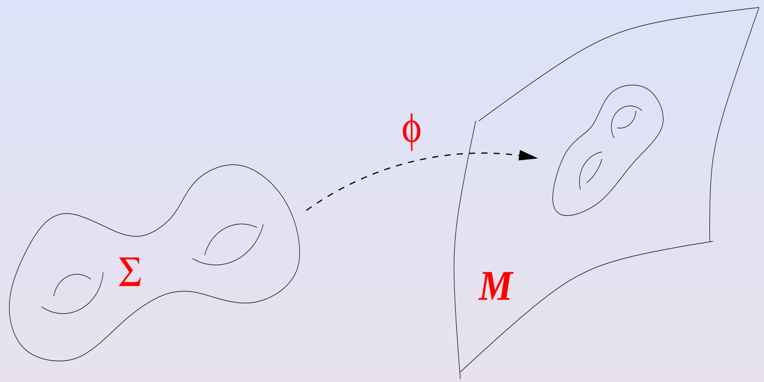
and it determines the isomorphism class $[\mathcal{L}]$ of $\mathcal{L}$

- For fixed F, $[\mathcal{L}' \otimes \mathcal{L}^*] \in H^1(M, U(1))$

(Line-bundle) gerbes (with connection)

- For an exact 3-form $H = dB$ on M,

$$\int_V H = \int_{\partial V} B$$ **Stokes Theorem**



- If H is closed with periods $\in 2\pi \boldsymbol{Z}$ then

$$\int_V H = \frac{1}{\sqrt{-1}} \log hol_{\mathcal{G}}(\partial V) \mod 2\pi$$

if $\mathcal{G}$ is a gerbe of curvature H with $hol_{\mathcal{G}}(\phi)$ denoting the **holonomy** along the closed surface $\phi : \Sigma \to M$

- For flat gerbes $\mathcal{G}$ with $H = 0$

$$hol_{\mathcal{G}}(\cdot) \in H^2(M, U(1))$$

and it determines the isomorphism class $[\mathcal{G}]$ of $\mathcal{G}$

- For fixed H, $[\mathcal{G}' \otimes \mathcal{G}^*] \in H^2(M, U(1))$

Local data

Let $(\mathcal{O}_i)$ be a (good) open covering of M

- $\mathcal{L}$ is represented by (A_i, g_{ij}) s.t.

$$
\begin{aligned}
&F = dA_i && \text{on} \quad \mathcal{O}_i\,, \\
&A_j - A_i = \sqrt{-1}\ d\log g_{ij} && \text{on} \quad \mathcal{O}_{ij} \equiv \mathcal{O}_i \cap \mathcal{O}_j\,, \\
&g_{ij}\, g_{ik}^{-1}\, g_{jk} = 1 && \text{on} \quad \mathcal{O}_{ijk}
\end{aligned}
$$

- $\mathcal{G}$ is represented by (B_i, A_{ij}, g_{ijk}) s.t.

$$
\begin{aligned}
&H = dB_i && \text{on} \quad \mathcal{O}_i\,, \\
&B_j - B_i = dA_{ij} && \text{on} \quad \mathcal{O}_{ij}\,, \\
&A_{ij} - A_{ik} + A_{jk} = \sqrt{-1}\ d\log g_{ijk} && \text{on} \quad \mathcal{O}_{ijk}\,, \\
&g_{ijk}\, g_{ijl}^{-1}\, g_{ikl}\, g_{jkl}^{-1} = 1 && \text{on} \quad \mathcal{O}_{ijkl}
\end{aligned}
$$

Physical interpretation:

- $\mathcal{A}(\ell) = \mathrm{e}^{-\frac{1}{2}\|d\ell\|^2}\ hol_{\mathcal{L}}(\ell)$ is the **Feynman amplitude** of a **particle** in the electromagnetic field F

- $\mathcal{A}(\phi) = \mathrm{e}^{-\frac{1}{2}\|d\phi\|^2}\ hol_{\mathcal{G}}(\phi)$ is the **Feynman amplitude** of a **closed string** in the Kalb-Ramond field H

Example:

$M = G$ is a simple compact **Lie group**

$$H_k = \frac{1}{12}\left\langle g^{-1}dg,\, [g^{-1}dg, g^{-1}dg]\right\rangle_k$$ is a bi-invariant 3-form

where $g^{-1}dg$ is the Maurer-Cartan form and $\langle\cdot\,,\,\cdot\rangle_k$ is the Killing form on $\boldsymbol{g} = Lie(G)$ normalized by $\langle\alpha^\vee, \alpha^\vee\rangle_k = \frac{k}{\pi}$ for short coroots

The number k is called the **level**

For G **simply connected**:

The periods of H_k are in $2\pi\boldsymbol{Z}$ iff $k \in \boldsymbol{Z}$. The corresponding gerbe $\mathcal{G}^k$ is unique (modulo isom.). It was **constructed**:

- for $SU(2)$ by *K.G. (1986)*
- for $SU(N)$ by *Chatterjee-Hitchin (1998)*
- for G compact simply connected by *Meinrenken (2002)*

For G **non simply connected** *(K.G.-Reis 2003)*:

Let $G = \tilde{G}/Z$ for $Z \subset center(\tilde{G})$ and $\tilde{G}$ simply connected

- There is an **obstruction** $[u] \in H^3(Z, U(1))$ to the existence of gerbe $\mathcal{G}^k$ on G with curvature H_k
- Triviality of $[u]$ selects the levels k s.t. $\mathcal{G}^k$ exists
- $\mathcal{G}^k$ is then unique except for $G = SO(4n)/\boldsymbol{Z}_2$ with $\mathcal{G}^k_\pm$ (**discrete torsion**)

Wess-Zumino-Witten (WZW) model

$\equiv$ string in a group manifold G with the Feynman amplitude involving $hol_{\mathcal{G}^k}$

- A prototype example of the **Conformal Field Theory**

Geometric quantization of the bulk (≡ closed string) WZW theory

- By **transgression**

gerbe $\mathcal{G}$ on M $\quad --\,--\!> \quad$ line bundle $\mathcal{L}_{\mathcal{G}}$ on the loop space LM

- The space of **quantum states** of the bulk WZW theory:

$$\mathcal{H} \;=\; \Gamma(\mathcal{L}_{\mathcal{G}^k}) \qquad \leftarrow \quad \text{space of sections}$$

with a geometric action of the double affine algebra $\hat{\boldsymbol{g}} \oplus \hat{\boldsymbol{g}}$ of level k

- The **spectrum** of irreducible highest weight reps. $\hat{V}_\lambda$ of $\hat{\boldsymbol{g}}$

$$\mathcal{H} \;=\; \bigoplus_{\lambda,\lambda'} M_{\lambda\lambda'} \otimes \hat{V}_\lambda \otimes \hat{V}_{\lambda'}$$

found by identifying the highest weight subspaces $M_{\lambda\lambda'} \subset \Gamma(\mathcal{L}_{\mathcal{G}^k})$

$-\!->$ **partition functions** *(Felder-K.G.-Kupiainen 1987)*

Gauge fields twisted by gerbes *(Kapustin 1999)*

Let $\mathcal{G}$ be a gerbe on M with local data (B_i, A_{ij}, g_{ijk})

- A vector bundle $\mathcal{E}$ twisted by $\mathcal{G}$ with a gauge field is given by $n\times n$-matrix valued local data $(\boldsymbol{A}_i, \boldsymbol{G}_{ij})$ s.t.

$$\boldsymbol{A}_j = \boldsymbol{G}_{ij}^{-1}\boldsymbol{A}_i\boldsymbol{G}_{ij} - \sqrt{-1}\,\boldsymbol{G}_{ij}^{-1}d\boldsymbol{G}_{ij} + A_{ij}\mathbf{1} = 0 \quad \text{on} \quad \mathcal{O}_{ij}$$

$$\boldsymbol{G}_{ij}\,\boldsymbol{G}_{jk} = g_{ijk}\,\boldsymbol{G}_{ik} \quad \text{on} \quad \mathcal{O}_{ijk}$$

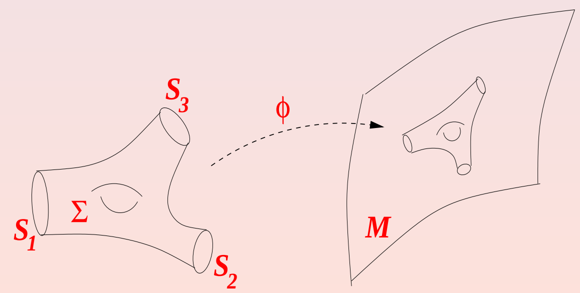
- $\mathcal{E}$ is also called a $\mathcal{G}$**-module** (of rank n)

- Let $W_{\mathcal{E}}(\ell) = \mathrm{tr}\,(hol_{\mathcal{E}}(\ell))$ be the **Wilson loop** "observable"
  Due to the twist by $\mathcal{G}$ the phase of $W_{\mathcal{E}}(\ell)$ is ambiguous

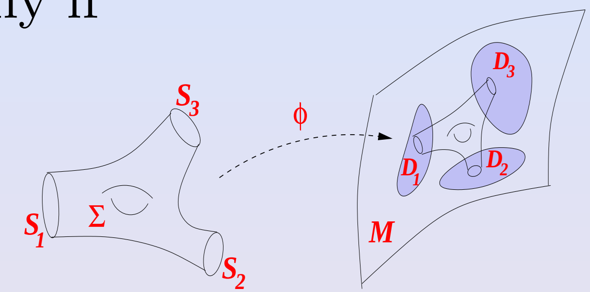
- **But:** if $\phi : \Sigma \to M$ with $\partial\Sigma = \bigsqcup_\alpha S_\alpha$ and $\mathcal{E}_\alpha$ are $\mathcal{G}$-modules then for $\ell_\alpha = \phi|_{S_\alpha}$

$$hol_{\mathcal{G}}(\phi)\ \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha)\,,$$

is unambiguous !!!



- Problem with the $\mathcal{G}$-modules: they exist only if the curvature H of $\mathcal{G}$ is exact !!!



Solution:

- A pair $\mathcal{D} = (D, \mathcal{E})$ is called a **$\mathcal{G}$-brane** if $D \subset M$ and $\mathcal{E}$ is a $\mathcal{G}|_D$-module

- If $\phi : \Sigma \to M$ and $\phi(S_\alpha) \subset D_\alpha$ for $S_\alpha \subset \partial\Sigma$ then the product $hol_{\mathcal{G}}(\phi) \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha)$ is well defined and

$$\mathcal{A}(\phi) = \mathrm{e}^{-\frac{1}{2}\|d\phi\|^2} hol_{\mathcal{G}}(\phi) \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha),$$

gives the Feynman amplitude of an **open string** with ends moving in the branes D_α and coupled to the (twisted) "**Chan-Paton**" gauge fields

Example of the WZW model

- The $\mathcal{G}^k$-branes $\mathcal{D} = (D, \mathcal{E})$ are called **symmetric** if they conserve the diagonal symmetry $\hat{\boldsymbol{g}} \subset \hat{\boldsymbol{g}} \oplus \hat{\boldsymbol{g}}$. They have the conjugacy classes

$$D \;=\; \mathcal{C}_\lambda \equiv \{\, h\, \mathrm{e}^{\lambda} h^{-1} \mid h \in G \,\}$$

as supports and the curvature of $\mathcal{E}$ given by the 2-form

$$F \;=\; 2 \left\langle h^{-1} dh,\, \mathrm{Ad}_{\mathrm{e}^{\lambda}} (h^{-1} dh) \right\rangle_k$$

(λ is a weight viewed as an element of $\boldsymbol{g}$)

Classification of sym. $\mathcal{G}^k$-branes *(K.G. 2004)*:

- For **simply connected** G:

$$\mathcal{D} \;=\; (\mathcal{C}_\lambda, n\,\mathcal{E}^1)$$

with $(\mathcal{C}_\lambda, \mathcal{E}^1)$ a unique symmetric brane of rank 1 supported by $\mathcal{C}_\lambda$ and $n\,\mathcal{E}^1 \equiv \underbrace{\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}^1}_{n \text{ times}}$

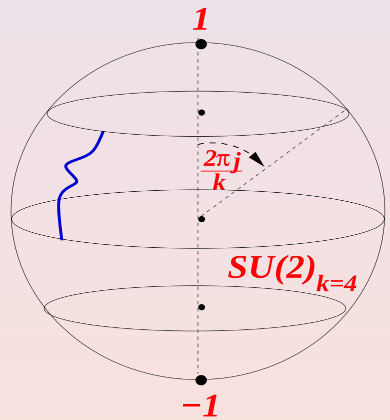
- For **non simply connected** $G = \tilde{G}/Z$:

$$\mathcal{D} = (\mathcal{C}_\lambda, n_1\mathcal{E}_1^1 \oplus \cdots \oplus n_\ell \mathcal{E}_{n_\ell}^1)$$

with $|Z_\lambda|$ different symmetric branes $(\mathcal{C}_\lambda, \mathcal{E}_i^1)$ of rank 1 supported by $\mathcal{C}_\lambda \cong \tilde{\mathcal{C}}_\lambda / Z_\lambda$ for $Z_\lambda \subset Z$ with the holonomy differing by the characters of $Z_\lambda \cong \pi_1(\mathcal{C}_\lambda)$

Example:

- For $G = SU(2) \cong S^3$ the admitted conjugacy classes $\mathcal{C}_j$ correspond to spins (weights) $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$
They are the spheres S^2 under the angles $\frac{2\pi j}{k}$
Each carries one symmetric $\mathcal{G}^k$-brane of rank 1



- For $G = SO(3) \cong \boldsymbol{R}P^3$ level k has to be even
Are admitted the conjugacy classes $\mathcal{C}_j \cong \tilde{\mathcal{C}}_j$ for $j < \frac{k}{4}$ carrying one rank 1 symmetric $\mathcal{G}^k$-brane
and $C_{\frac{k}{4}} \cong \tilde{\mathcal{C}}_{\frac{k}{4}} / \boldsymbol{Z}_2$ carrying two symmetric rank 1 $\mathcal{G}^k$-branes

- In principle there is an **obstruction** $[v] \in H^2(Z_\lambda, U(1))$ to the existence of rank 1 $\mathcal{G}^k|_{C_\lambda}$-module $\mathcal{E}^1$ on $\mathcal{C}_\lambda \cong \tilde{\mathcal{C}}_\lambda / Z_\lambda$ but $H^2(Z_\lambda, U(1)) = 0$ for cyclic groups $Z_\lambda = \boldsymbol{Z}_m$

Exceptional case:

- For $G = SO(4n)/\boldsymbol{Z}_2 = Spin(4n)/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$ and $\mathcal{C}_\lambda \cong \tilde{\mathcal{C}}_\lambda / Z_\lambda$ with $Z_\lambda = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ the **obstruction** $[v] \in H^2(Z_\lambda, U(1)) = \boldsymbol{Z}_2$ forbids the existence of a rank 1 $\mathcal{G}^k_-|_{\mathcal{C}_\lambda}$-module and

$$\mathcal{D} = (\mathcal{C}_\lambda, n\,\mathcal{E}^2)$$

where $(\mathcal{C}_\lambda, \mathcal{E}^2)$ is a unique symmetric rank 2 $\mathcal{G}^k_-$-brane
$-->$ generation of a **non-abelian gauge symmetry**

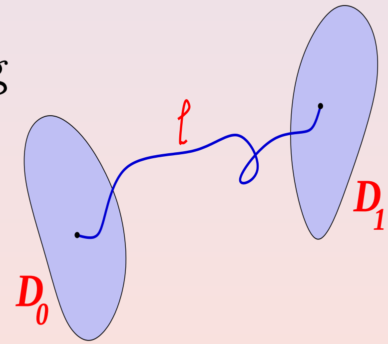
Geometric Quantization of boundary ($\equiv$ open string) WZW theorie:

- By **transgression**

gerbe $\mathcal{G}$ on M and a pair $(\mathcal{D}_0, \mathcal{D}_1)$ of $\mathcal{G}$-branes	$---\!>$	vector bundle $\mathcal{E}_{\mathcal{D}_0}^{\mathcal{D}_1}$ on the space of curves IM

where $\quad IM = \{\, \ell : [0,1] \to M \mid \ell(0) \in D_0,\ \ell(1) \in D_1 \,\}$

- The space of **quantum states** of the string spanning the branes D_0 and D_1 is



$$\mathcal{H}_{\mathcal{D}_0}^{\mathcal{D}_1} = \Gamma(\mathcal{E}_{\mathcal{D}_0}^{\mathcal{D}_1}) \qquad \leftarrow \quad \begin{matrix}\text{space}\\ \text{of sections}\end{matrix}$$

with a geom. action of the affine algebra $\hat{\boldsymbol{g}}$ of level k

- The **spectrum** of irreps. $\hat{V}_\lambda$ of $\hat{\boldsymbol{g}}$

$$\mathcal{H}_{\mathcal{D}_0}^{\mathcal{D}_1} \;=\; \bigoplus_{\lambda} M_{\mathcal{D}_1}^{\mathcal{D}_1}{}_{\lambda} \otimes \hat{V}_\lambda$$

found from the highest weight subspaces $M_{\mathcal{D}_1}^{\mathcal{D}_1}{}_{\lambda} \subset \Gamma(\mathcal{E}_{\mathcal{D}_0}^{\mathcal{D}_1})$

$---$> **partition functions** $\mathcal{Z}_{\mathcal{D}_1}^{\mathcal{D}_2}(\tau)$
boundary operator product
(K.G.-Reis 2002, K.G. 2004)

Orientifolds

- Let $\kappa : M \to M$ be an involution s.t. $\kappa^* H = -H$
  Let $\mathcal{G}$ be a gerbe an M of curvature H and $\mathcal{G}^*$ its dual (with curvature $-H$). A **Jandle structure** is a triple (κ, ι, η) where ι and η are isomorphisms

$$\kappa^* \mathcal{G} \overset{\iota}{\cong} \mathcal{G}^*, \qquad \iota^2 \overset{\eta}{\cong} Id$$

  (Schreiber-Schweigert-Waldorf 2005). It permits to define the holonomy of $\phi : \Sigma \to M$ for **non-orientable** Σ and the Feynman amplitudes of **non-orientable strings**

- Jandle structures on gerbes $\mathcal{G}^k$ over $G = \tilde{G}/Z$ are obstructed by $H^3(\Gamma, U(1))$ and classified by $H^2(\Gamma, U(1))$ with $\Gamma = \boldsymbol{Z}_2 \ltimes Z$ where the generator of $\boldsymbol{Z}_2$ acts by inversion on Z and $U(1)$

- $H^p(\Gamma, U(1))$ may be calculated from the Lyndon-Hochschild-Serre spectral sequence (*K.G.-Schweigert-Suszek-Waldorf*, in writing)

Conclusions

- (Line-bundle) gerbes represent the background Kalb-Ramond fields in a non-trivial topology
- Gerbe modules represent twisted Chan-Paton gauge fields on branes to which couple the ends of open strings
- The classification of such structures on Lie groups leads to the classification of the WZW models and facilitates their solution on the classical and the quantum level
- It explains the origin and permits an explicit calculation of the finite group cohomology entering the solution
- Open problem: the dynamics of gerbes and their modules and its relations to the renormalization group flows