Random matrix theory of disordered conductors

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(Joint work with Sven Bachmann and Wojciech De Roeck)

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Phenomenon (observed experimentally and numerically):
Universal conductance fluctuations
For disordered conductors of mesoscopic length ($\sim 1 - 10\mu\mathrm{m}$), with a thin wire geometry (width $\sim 100\mathrm{nm}$)

$$\mathrm{Var}\left(\frac{G}{G_0}\right) = \frac{2}{15\beta}$$

with conductance quantum $G_0 = 2e^2/h$ and symmetry index β

$$\beta = \begin{cases} 1 & \text{time-reversal symmetric} \\ 2 & \text{TR symmetry broken} \\ 4 & \text{TR, but spin-orbit-scattering (not considered here)} \end{cases}$$

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Universality, β-dependence $\Rightarrow$ Random Matrix Theory

Random matrices: First applied to physics by Wigner (1950s) to describe spectra of complex nuclei. Self-adjoint $N \times N$ matrices with random coefficients, $N \to \infty$.

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Prominent example Gaussian orthogonal/unitary ensemble: GOE(N), GUE(N) $\stackrel{\mathrm{d}}{=} X_\beta(1)$ for $\beta = 1, 2$ with

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Eigenvalue process: weak solution of Dyson Brownian Motion

$$\mathrm{d}\lambda_k(t) = \frac{\sqrt{2}}{\sqrt{\beta N}}\mathrm{d}W_k(t) + \frac{1}{N}\sum_{l \neq k} \frac{1}{\lambda_k(t) - \lambda_l(t)}\mathrm{d}t, \qquad k = 1, ..., N.$$

Quasi-one-dimensional wire with N channels ($N = k_F^{d-1} W$):
Scattering matrix $S \in \mathbb{C}^{2N \times 2N}$

$$S \begin{pmatrix} \Phi^+_{\text{left}} \\ \Phi^-_{\text{right}} \end{pmatrix} = \begin{pmatrix} \Phi^-_{\text{left}} \\ \Phi^+_{\text{right}} \end{pmatrix}.$$

Transfer matrix $\mathcal{M} \in \mathbb{C}^{2N \times 2N}$

$$\mathcal{M} \begin{pmatrix} \Phi^+_{\text{left}} \\ \Phi^-_{\text{left}} \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{++} & \mathcal{M}_{+-} \\ \mathcal{M}_{-+} & \mathcal{M}_{--} \end{pmatrix} \begin{pmatrix} \Phi^+_{\text{left}} \\ \Phi^-_{\text{left}} \end{pmatrix} = \begin{pmatrix} \Phi^+_{\text{right}} \\ \Phi^-_{\text{right}} \end{pmatrix}.$$

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Current conservation:

$$\mathcal{M}^* \Sigma_z \mathcal{M} = \Sigma_z, \qquad \Sigma_z = \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}.$$

Time-reversal invariance ($\beta = 1$):

$$\Sigma_x \overline{\mathcal{M}} \Sigma_x = \mathcal{M}, \qquad \Sigma_x = \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix}.$$

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$$\mathcal{M}(s + \mathrm{d}s) = (1 + \mathrm{d}\mathcal{L}(s))\,\mathcal{M}(s)$$
$$\mathrm{d}\mathcal{M}(s) = \mathrm{d}\mathcal{L}(s)\mathcal{M}(s).$$

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Out of all $\mathrm{d}\mathcal{L}$ such that constraints on $\mathcal{M}$ are conserved, DMPK choose $\mathrm{d}\mathcal{L}$ due to maximum entropy assumption, but no microscopic justification.

Maximum entropy assumption $\Rightarrow \mathcal{L}$ Brownian motion with invariant increments

$$\mathcal{U}^* \mathrm{d}\mathcal{L}\mathcal{U} \stackrel{d}{=} \mathrm{d}\mathcal{L},$$

for all unitary matrices $\mathcal{U}$ of the form

$$\mathcal{U} = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}.$$

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$$\Rightarrow \mathcal{L}(s) = \begin{pmatrix} \mathfrak{a}_+(s) & \mathfrak{b}(s) \\ \mathfrak{b}^*(s) & \mathfrak{a}_-(s) \end{pmatrix}$$

distributed like

$$\begin{aligned} \mathfrak{a}_+ &\stackrel{\mathrm{d}}{=} \mathfrak{a}_- \stackrel{\mathrm{d}}{=} iX_2/\sqrt{2N} \\ \mathrm{Re}\,\mathfrak{b} &\stackrel{\mathrm{d}}{=} \mathrm{Im}\,\mathfrak{b} \stackrel{\mathrm{d}}{=} X_1/(2\sqrt{N+1}) \quad (\beta = 1) \\ \mathfrak{b} &\stackrel{\mathrm{d}}{=} Y_2/\sqrt{N} \quad (\beta = 2). \end{aligned}$$

all blocks independent except $\mathfrak{a}_- = \overline{\mathfrak{a}_+}$ for $\beta = 1$.

This invariance allows for an autonomous evolution of the eigenvalues of T_k of t^*t, t the transmission matrix

$$t = \mathcal{M}_{++} - \mathcal{M}_{+-}\mathcal{M}_{--}^{-1}\mathcal{M}_{-+}.$$

The DMPK equation for the transmission eigenvalues T_k

$$\mathrm{d}T_k(s) = v_k(T(s))\mathrm{d}s + D_k(T(s))\mathrm{d}B_k(s),$$

B_k, $k = 1, ..., N$ independent Brownian motions,

$$v_k = -T_k + \frac{2T_k}{\beta N + 2 - \beta}\left(1 - T_k + \frac{\beta}{2}\sum_{j \neq k}\frac{T_k + T_j - 2T_kT_j}{T_k - T_j}\right)$$

$$D_k = \sqrt{4\frac{T_k^2(1 - T_k)}{\beta N + 2 - \beta}}.$$

$\beta \geq 1$, $\beta = 1$ time-reversal invariant, $\beta = 2$ not time-reversal invariant.

Despite singular repulsion terms:

Theorem

Let $\beta \geq 1$, $N \in \mathbb{N}$, and $T_k(0) = 1$, $k = 1, ..., N$. Then there is a unique continuous process $(T(s))_{s\geq 0}$ such that $0 < T_1(s) < ... < T_N(s) < 1$ for all $s > 0$ and $(T(s))_{s>0}$ is a strong solution to the DMPK equation.

Sketch of proof

Nondegenerate $T(0) \in \Delta_N :\Leftrightarrow 0 < T_1(0) < ... < T_N(0) < 1$:
Use smooth cut-off χ_R

$$\chi_R(T) = 1 \text{ if } \mathrm{dist}(T, \partial\Delta_N) \geq 1/R$$

$\Rightarrow$ SDE with Lipschitz coefficients (existence and uniqueness!)

$$v_k^{(R)}(T) = \chi_R(T) v_k(T)$$
$$D_k^{(R)}(T) = \chi_R(T) D_k(T).$$

Stopping time S_R to control how long solutions to cut-off and full SDE coincide.

Lyapunov function

$$f(T) = \sum_{k=1}^{N} \left(-2\log(|T_k|) - 2\log(|1 - T_k|) - \sum_{l=1, l\neq k}^{N} \log(|T_k - T_l|) \right)$$

$s \mapsto f(T(s))$ is “almost” a supermartingale $\Rightarrow$ Markov estimate

$$\mathbb{P}(S_R \leq s)\log R \leq f(T(s \wedge S_R)) \leq f(T(0)) + Cs$$
$$\Rightarrow S_R \to \infty \quad \text{a.s.} \quad (R \to \infty).$$

Degenerate initial conditions:

$\tilde{T}$, $\hat{T}$ two solutions of DMPK equation with

$$\tilde{T}_k(0) < \hat{T}_k(0), \qquad k = 1, ..., N$$

then

$$\tilde{T}_k(s) < \hat{T}_k(s), \qquad k = 1, ..., N$$

for all $s > 0$.

$\Rightarrow$ Approximate $T_k(0) \equiv 1$ from below. Limit $(T(s))_{s\geq 0}$ can be proven to be continuous, $T(s) \in \Delta_N$ for $s > 0$ and is solution of the DMPK equation for $s > 0$.

Uniqueness by ordering property and Gronwall estimate.

With the T_k process, the statistics of the conductance g can be studied via the Landauer-Büttiker formula

$$g_N = \sum_{k=1}^{N} T_k.$$

Theorem

For all $p \geq 1$

$$\lim_{N\to\infty} \frac{\mathbb{E}(g_N^p(s))}{N^p} = \frac{1}{(1+s)^p}$$

uniformly for s from compact intervals.

Idea of proof: Itô's formula for g_N, compactness argument for the functions

$$s \mapsto \frac{\mathbb{E}(g_N^p(s))}{N^p},$$

limit is unique solution of a "hierarchy of ODEs".

Hilbert space for Anderson model on a tube with circumference N

$$\mathcal{H} = l^2(\mathbb{Z} \times \mathbb{Z}_N) = l^2(\mathbb{Z}) \otimes \mathbb{C}^N\,.$$

Hamiltonian

$$H = H_{\mathrm{kin}} + \lambda V$$

Random potential: $V(x, z)$ i.i.d. for $z = 1, ..., N$, $x = 1, ..., L$.

$$\mathbb{E}(V(x,z)) = 0, \;\; \mathbb{E}(V(x,z)^2) = 1$$

Kinetic Hamiltonian:

$$\begin{aligned}(H_{\mathrm{kin}}\Psi)(x,z) =& \Psi(x+1,z) + \Psi(x-1,z) \\ &+ h_1\left[\mathrm{e}^{\mathrm{i}\gamma}\Psi(x,z+1) + \mathrm{e}^{-\mathrm{i}\gamma}\Psi(x,z-1)\right] \\ &+ h_2\left[\mathrm{e}^{\mathrm{i}\gamma}\Psi(x-1,z+1) + \mathrm{e}^{-\mathrm{i}\gamma}\Psi(x+1,z-1))\right] \\ =:& \left(P^*\Psi_{x+1} + H_\perp\Psi_x + P\Psi_{x-1}\right)(z)\end{aligned}$$

Theorem

For $E \neq 0$, $|E| < 2$, $0 \neq \gamma < \pi/N$ and h_1, h_2 sufficiently small

$$H_{\mathrm{kin}} \Psi = E \Psi$$

has $2N$ plane wave solutions

$$\Psi_{k,\nu}(x,z) = \frac{1}{\sqrt{N}} e^{ik_\nu^\sigma x} e^{\frac{2\pi i}{N}\nu z}$$

with $\nu \in \mathbb{Z}_N$, $\sigma \in \{+,-\}$. For $\gamma \neq 0$, the longitudinal wave numbers k_ν^σ are non degenerate in the sense that

$$\varsigma_1 k_{\nu(1)}^{\sigma_1} + \varsigma_2 k_{\nu(2)}^{\sigma_2} + \varsigma_3 k_{\nu(3)}^{\sigma_3} + \varsigma_4 k_{\nu(4)}^{\sigma_4} = 0 \pmod{2\pi}$$

for signs $\varsigma_1, \ldots, \varsigma_4$ only in the trivial case, and if $\gamma = 0$ also for

$$(\varsigma_1, \sigma_1, \nu(1)) = (\varsigma_2, -\sigma_2, -\nu(2)) \quad \textit{and} \quad (1,2) \leftrightarrow (3,4)$$

One-layer transfer matrix:

$$\begin{pmatrix}\Psi_{x+1}\\ \Psi_x\end{pmatrix} = \begin{pmatrix}(P^*)^{-1}(E - H_\perp - \lambda V_x) & -(P^*)^{-1}P\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_x\\ \Psi_{x-1}\end{pmatrix}$$

Multi-layer transfer matrix:

$$\begin{pmatrix}\Psi_{L+1}\\ \Psi_L\end{pmatrix} = T_L^\lambda \cdots T_1^\lambda \begin{pmatrix}\Psi_1\\ \Psi_0\end{pmatrix}$$

In kinetic eigenbasis:

$$M^\lambda(L) := M_L^\lambda \cdots M_1^\lambda = K^{-1}\, T_L^\lambda \cdots T_1^\lambda\, K$$

Cancel oscillatory terms:

$$A^\lambda(\lfloor \lambda^{-2}s \rfloor) := (M^0(\lfloor \lambda^{-2}s \rfloor))^{-1} M^\lambda(\lfloor \lambda^{-2}s \rfloor).$$

Theorem

If h_1 and h_2 depend on λ so that

$$h_1(\lambda) \to 0\,, \qquad h_2(\lambda) \to 0,$$

and

$$\lambda^{-2}\mathrm{cha}(\gamma, h_1(\lambda), h_2(\lambda)) \to \infty\,,$$

as $\lambda \to 0$, then the process $\left(A^{\lambda}\left(\lfloor \lambda^{-2} s \rfloor\right)\right)_{s\geq 0}$ converges in distribution to the process $(\mathcal{A}(s))_{s\geq 0}$ on the path space of $\mathbb{C}^{2N\times 2N}$-valued processes endowed with Skorhod topology. For $\gamma \geq 0$, $(\mathcal{A}(s))_{s\geq 0}$ is given as the unique solution for $s \geq 0$ to

$$\mathrm{d}\mathcal{A}(s) = \mathrm{d}\mathcal{Z}_{\gamma}(s)\mathcal{A}(s)$$
$$\mathcal{A}(0) = 1.$$

Sketch of proof

$$A^\lambda(x) - A^\lambda(x-1) = \lambda Z_x A^\lambda(x-1),$$
$$Z_x = (M^0(x))^{-1} R_x M^0(x)$$

with

$$R_x = \mathrm{i} \begin{pmatrix} \frac{1}{\sqrt{|v^+|}} Q^* V_x Q \frac{1}{\sqrt{|v^+|}} & \frac{1}{\sqrt{|v^+|}} Q^* V_x Q \frac{1}{\sqrt{|v^+|}} \Pi \\ -\Pi \frac{1}{\sqrt{|v^+|}} Q^* V_x Q \frac{1}{\sqrt{|v^+|}} & -\Pi \frac{1}{\sqrt{|v^+|}} Q^* V_x Q \frac{1}{\sqrt{|v^+|}} \Pi \end{pmatrix},$$

and

$$M^0(x) = \begin{pmatrix} \exp\left(\mathrm{i} x k^+\right) & 0 \\ 0 & \Pi \exp\left(\mathrm{i} x k^-\right) \Pi \end{pmatrix}.$$

Define the process

$$Z_\gamma^\lambda(s) = Z^\lambda(s) = \lambda \sum_{x=0}^{\lfloor \lambda^{-2} s \rfloor} Z_x \,.$$

Square variation process is given by

$$\mathbb{E}\left[\left(Z^\lambda(s)^\sharp\right)_{mn}\left(Z^\lambda(s)\right)_{pr}\right]$$

$$= \lambda^2 \sum_{x=1}^{\lfloor \lambda^{-2} s \rfloor} \exp\left(\mathrm{i}x\left(-k^{\sigma_m}_{\sigma_m\nu_m} + k^{\sigma_n}_{\sigma_n\nu_n} - k^{\sigma_p}_{\sigma_p\nu_p} + k^{\sigma_r}_{\sigma_r\nu_r}\right)\right) \mathbb{E}\left[(R_x^\sharp)_{mn}(R_x)_{pr}\right].$$

Donsker's invariance principle and rapidly oscillating phases imply

$$\left(Z_\gamma^\lambda(s)\right)_{s\geq 0} \xrightarrow{\mathrm{d}} (\mathcal{Z}_\gamma(s))_{s\geq 0} \qquad (\lambda \to 0)$$

on path space. Estimates for jumps of A^λ and convergence theory for martingales $\Rightarrow$ Theorem.

$$\mathcal{Z}_\gamma(s) = \begin{pmatrix} \tilde{\mathfrak{a}}_+(s) & \tilde{\mathfrak{b}}(s) \\ \tilde{\mathfrak{b}}^*(s) & \tilde{\mathfrak{a}}_-(s) \end{pmatrix}$$

Does $\mathcal{Z}_\gamma$ look like $\mathcal{L}$?

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- for $\gamma > 0$ ($\beta = 2$), only $\tilde{\mathfrak{a}}$ and $\mathfrak{a}$ differ, which does not change dynamics of T_k.

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Does $\mathcal{Z}_\gamma$ look like $\mathcal{L}$?

- for $\gamma > 0$ ($\beta = 2$), only $\tilde{\mathfrak{a}}$ and $\mathfrak{a}$ differ, which does not change dynamics of T_k.
- for $\gamma = 0$ ($\beta = 1$), diagonal of $\tilde{\mathfrak{b}}$ scales wrong, no independent T_k dynamics, influence on conductance statistics remains to be understood, probably small.

Thank you for your attention!