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Rough path theory

Classical and new approaches

Contents

Chapter 0

Introduction

0.1 What rough paths are

The notion of rough paths was introduced in a paper by T. Lyons in 1998 [34]. This article lays the foundations of a now well-established theory at the crossroads between analysis, control theory and stochastic calculus, which produces a new kind of 'weak' solutions of differential equations or partial differential equations possessing only a Hölder regularity.

The classical fixed-point theorem due to Cauchy and Lipschitz ensures existence and unicity for solutions $y : \mathbb{R} \to \mathbb{R}^n$ of *ordinary differential equations* of the type

$$dy(t) = V(t, y(t))dt, \quad y(0) = y_0 \in \mathbb{R}^n \tag{1}$$

when $V : \mathbb{R} \to \mathbb{R}^n \to \mathbb{R}^n$ is a function which is Lipschitz continuous in y. The simultaneous development of the study of evolution problems, of stochastic analysis, of nonlinear analysis, of numerical analysis, of control theory, and of applications to the equations of mathematical physics, has led to a considerable extension of the setting of Cauchy and Lipschitz and of the hypotheses under which a solution may be defined.

Staying in the realm of ordinary differential equations (o.d.e.'s), the most famous extension of the Cauchy-Lipschitz theory is due to Di Perna and Lions. The Lipschitz regularity requirement in the variable y is replaced by the weaker requirement that V should be absolutely continuous, together with a control on the L^1_{loc}-norm of the derivative $\partial V/\partial y$ and on the L^∞-norm of the divergence div V. The associated transport equation $\partial_t f(t, y) + V \cdot \nabla_y f(t, y) = 0$ admits then a *weak* solution, making it possible to solve the initial o.d.e. for *almost every initial condition.* This point of view has been used e.g. for the study of hydrodynamic limits of kinetic equations such as the Boltzmann equation. The passage from the initial differential equation to the transport equation is done by assuming a random initial condition, $f(0, t)$, representing the probability density at time 0.

If V is not assumed to be even nearly differentiable in y – typically when V is only *Hölder-continuous* in y – it is well-known that the solution is not unique any more. For instance,

$y' = 2\sqrt{y}$, $y(0) = 0$ has two solutions, 0 and t^2, on $\mathbb{R}_\pm$, and the solutions may be pasted together arbitrarily at 0, yielding four different solutions which are everywhere C^1.

Let us now say a few words on *stochastic differential equations.* Noisy equations are widespread in modelization in physics, chemistry, biology, economics, finance, and so on. Most often people use *white noise* which modelizes a 'blind', uncorrelated force. White noise is also known as the distribution-valued derivative of *Brownian motion.* Noisy extensions of eq. (1) are called *diffusion equations*,

$$dy(t) = V_0(t, y(t))dt + \sum_{i=1}^{d} V_i(t, y(t))dB_i(t), \quad y(0) = y_0 \in \mathbb{R}^n \tag{2}$$

when the driving noise $B(t) = (B_1(t), \ldots, B_d(t))$ is made of d independent Brownian motions. Such equations are much more irregular than deterministic equations since typical trajectories of B are known to be only $(1/2)^-$-Hölder continuous, i.e. α-Hölder for every $\alpha < \frac{1}{2}$. Yet the development of stochastic calculus for Brownian motions, and more generally for semi-martingales, has made it possible to define process-valued solutions of such equations. Such solutions are only almost surely $(1/2)^-$-Hölder continuous, like Brownian motion itself. Furthermore, as well-known, different discretization schemes yield different notions for stochastic calculus and for stochastic differential calculus, including of course the Itô and the Stratonovich calculi.

The point of view of rough path theory draws heavily on the Itô theory, but is written in a deterministic setting; as such, it should be seen as a branch of *control theory.* In the language of control theory, one has a system which is driven by irregular vector fields $\frac{dx_1}{dt}, \ldots, \frac{dx_d}{dt}$, typically modelizing highly oscillatory forces, or some realization of a random force. One wants to give a meaning to equations of the type

$$dy(t) = V_0(t, y(t))dt + \sum_{i=1}^{d} V_i(t, y(t))dx_i(t), \quad y(0) = y_0 \in \mathbb{R}^n \tag{3}$$

The paradigmatic example, though, namely, *fractional Brownian motion* (fBm) , comes from the area of stochastic processes. In order to avoid pathologies, one assumes that the paths $t \mapsto x_i(t)$ are locally α-Hölder for a certain exponent $\alpha \in (0, 1)$, i.e.

$$||x||_\alpha := \sup_{0 \le t \le T} |x(t)| + \sup_{0 \le s,t \le T} \frac{|x_i(t) - x_i(s)|}{|t - s|^\alpha} < \infty. \tag{4}$$

This is the case of typical trajectories of fBm of index α' if $\alpha' > \alpha$. On the other hand, the functions $y \mapsto V_i(t, y(t))$ must be more than Lipschitz continuous (actually, $1/\alpha$-Lipschitz), in order to compensate the lack of regularity of the driving forces with respect to time. Letting $x_0(t) = t$, one sees that dt-term in (3) may be seen simply as a supplementary, regular control. As in the case of ordinary differential equations with non-Lipschitz coefficients – but for totally different reasons – the solution to such an equation is not unique any more (nor does it exist a priori) as soon as $\alpha \le 1/2$. Namely, solving eq. (3) by successive iterations in the way of Picard,

$$y^{(0)}(t) = y_0, \quad y^{(n+1)}(t) = \int_0^t V_0(s, y^{(n)}(s))ds + \sum_{i=1}^{d} V_i(s, y^{(n)}(s))dx_i(s) \tag{5}$$

leads after $n \geq 2$ iterations to *iterated integrals*,

$$x_{(i_1,\ldots,i_n)}(s,t) := \int_s^t dx_{i_1}(t_1) \int_s^{t_1} dx_{i_2}(t_2) \ldots \int_s^{t_{n-1}} dx_{i_n}(t_n) \tag{6}$$

Using Stokes' formula, such quantities may be interpreted as volumes delimitated by the curve $t \mapsto x(t)$; for instance, $x_{(1,2)}(s,t)$ is the *signed area* between the horizontal line through $x(s)$, the vertical line through $x(t)$, and the curve. Now it is well-known that the area bounded by an irregular Jordan curve is an ill-defined notion. Correspondingly, the original point of view of T. Lyons was to *define* the iterated integrals $x_{(i_1,\ldots,i_n)}(s,t)$ as the *limit* in adequate Hölder spaces of the well-defined iterated integrals $x^{\varepsilon}_{(i_1,\ldots,i_n)}$ of some family of approximations x^{ε}, $\varepsilon \to 0$ of x. A fundamental step forward was to prove that knowing *iterated integrals of order* $\leq N = \lfloor 1/\alpha \rfloor$ was sufficient to give an adequate meaning to differential equations such as (3). The next step was to prove that such stacks of iterated integrals were essentially in $1-1$ correspondence with combinatorial objects $\mathbf{x}(s,t) = (\mathbf{x}^1(s,t) = x(t) - x(s), \ldots, \mathbf{x}^N(s,t))$ called *rough path* lifts of x, through the obvious mapping $\mathbf{x}^n_{(i_1,\ldots,i_n)}(s,t) = \lim_{\varepsilon\to 0} x^{\varepsilon}_{(i_1,\ldots,i_n})(s,t)$. Such objects are defined by algebraic axioms called *Chen* (or *multiplicative*) and *shuffle* (or *geometric*) *properties*, which always hold true for the iterated integrals of a smooth path, and supposed to have finite norm in adequate Hölder spaces. Thus, in some sense, these axioms characterize exactly all possible iterated integrals of a given continuous path.

The development of rough path techniques, blended with tools coming from stochastic calculus, has made it possible to apply the original ideas of Lyons to a random setting, where most interesting examples are found for the present time, either to stochastic differential equations driven by fBm, or to stochastic P.D.E.'s driven by space-time white noise [20]. Among the most prominent results, one may cite ergodicity results adapted to this non-Markovian setting []; an extension of Malliavin's theory of hypoelliptic equations satisfying Hörmander's bracket condition []; the use of second-order rough paths to solve some stochastic P.D.E.'s []. These are also one of the ingredients for the construction of solutions to the KPZ equation in [].

The main emphasis in this book is on the bewildering variety of fields which, as it happens, have turned out to be fundamental for a good understanding of the theory. *Sub-Riemannian geometry* is one of these; it has been understood from the beginning that the algebraic axioms in the definition of rough paths were equivalent to giving a lift of the underlying path into a *section of a principal G-bundle*, where G is a Carnot-Carathéodory Lie group. Such Lie groups are archetypal sub-Riemannian manifolds, and the associated metric objects reflect the inhomogeneous distances and norms in use for rough paths. These beautiful aspects have been developed at length in the book by P. Friz and N. Victoir; geometric arguments are very helpful, although not absolutely necessary, in defining the theory of rough paths. Somehow, though, we believe – and argue in Chapter ??? – that further progress will be in the application of rough paths to the study of sub-Riemannian geometry instead of the contrary.

0.2 Fourier normal ordering

We shall mainly focus on a new line of research influenced by *quantum field theory*, *multi-scale analysis* and *algebraic combinatorics*. We started developing these arguments around 2007 when stumbling into the "$\alpha = 1/4$"-barrier problem. In all known examples, in particular for fBm, it seemed impossible to define an explicit rough path over paths of Hölder regularity index $\alpha \leq 1/4$ (though rough path lifts are known to exist by formal arguments); all approximating sequences x^ε gave diverging iterated integrals in the limit $\varepsilon \to 0$. Solutions to this problem came in the first place by realizing that, for such wildly oscillating processes, iterated integrals depend strongly on the highest frequency Fourier components. This is a standard problem in quantum field theory; hence it is no surprise that and the concepts and methods of this field apply to this setting. The fundamental concept here is that of *multi-scale analysis*: decomposing the path x into a sum $\sum_j x^j$ by using a dyadic Fourier partition of unity (with $\log|\mathrm{supp}\mathcal{F}(x^j)| \approx j$), one sees that iterated integrals are *ultra-violet divergent*, and that this divergence comes from well-identified Fourier sectors. The next idea is then to *Fourier normal order* these iterated integrals, i.e. use Fubini's theorem to rewrite iterated integrals as a sum of integrals over some domain of n-forms of the type $dx^{j_1} \ldots dx^{j_n}$, with the *Fourier normal ordering assumption* $j_1 \leq \ldots \leq j_n$. Placing high-frequency components to the right reduces the task of estimating such integrals to previous work on random Fourier series beautifully gathered in a book by J.-P. Kahane [23]. Using the algebraic axioms of rough paths, one understands easily that second-order iterated integrals are in $1-1$-correspondence with quantities depending only on one time-variable (later on to be interpreted as *second-order Fourier normal ordered skeleton integrals*). These quantities are divergent when obtained as the limit of the same quantities for the known approximations x^ε when x is fBm, and are in general divergent for an arbitrary path x and conventional smoothing sequences. But replacing them by any other well-defined quantity (with some minimal Hölder regularity requirements) yields a well-defined rough path over x.

The construction of iterated integrals of arbitrary order requires the introduction of tools coming from algebraic combinatorics. The reason is that iterated integrals reordered by the use of Fubini's theorem turn into integrals over more complicated domains encoded by *trees*. Reinterpreting the algebraic rough path axioms in terms of the *Hopf algebra of decorated trees* (also called *Connes-Kreimer algebra*) and of the *shuffle Hopf algebra* is the key to an *algebraic classification of formal rough paths*, i.e. of quantities satisfying the algebraic axioms but not necessarily the Hölder regularity requirements. One thereby gets a connection to the latest developments in algebraic combinatorics triggered by the seminal paper of Connes and Kreimer in 1999. Originally that paper was on a Hopf algebraic reinterpretation of the Bogolioubov forest formula for renormalization in quantum field theory. Algebraists developed their own concepts, but also interacted from the beginning and to the present day very strongly with people from theoretical physics, numerical analysis and control theory []. This line of research on rough paths is becoming part of this circle of ideas.

Now comes probably the most interesting part for the analyst, namely: select out of this huge class of formal rough paths *actual rough paths* with the required Hölder regularity properties; and see what happens to the results obtained for stochastic differential equations with $\alpha > 1/4$

when one considers such rough paths and some $\alpha \leq 1/4$. The issue of *selecting a rough path* is not totally settled; it may never be for it is mainly a matter of choice. It is actually easy to select a (still huge) class of rough paths, which we called *Fourier-normal ordered rough paths*, with slightly stronger regularity requirements than for conventional rough paths. We prove various fundamental results for this class in Chapter ???, extending those obtained by conventional Malliavin calculus for fBm with $\alpha > 1/4$, and requiring the developments of new tools of stochastic calculus adapted to this setting, again inspired by quantum field theoretic concepts such as *operator product expansions*. Several explicit rough paths have been constructed, using *Fourier domain regularization* [], *renormalization* []. The simplest construction is called *zero tree data* [].

A last class of rough paths, still relying strongly on Fourier normal ordering, has been obtained in [] by constructive (i.e. rigorous) quantum field theoretic arguments. The first construction yields the Lévy area of fBm in terms of a Feynman-Kac type Gibbs measure. The model is just renormalizable, hence requires a very careful construction by *cluster expansions*, relying on combinatorics developed in the 80'es by mathematical physicists []. The action of the renormalization group is very particular, not usual for connoisseurs of the field, since the interaction is 'swallowed up' by the appearance of a infinite mass after a few steps of iteration, which explains why the theory is both free (i.e. it represents a Gaussian field) and not trivial (since the interaction has made it possible to renormalize the Lévy area and make it finite). It may be generalized by combining the Fourier normal ordering algorithm and quantum field theoretic arguments, yielding a rough path over fBm with arbitrary index. The second construction yields a lift of fBm with arbitrary index into a rough path as a solution of a singular differential equation, and relies on a functional integration representation called *response formalism* or *Martin-Siggia-Rose formalism*.

0.3 A guide for the reader

The reader will find some classical aspects and applications of rough path theory rendered in a concise way in Chapters 1 and 2, exemplified by elementary computations. Hopefully this quick introduction into rough path theory will be enough for the non-expert in this theory to find his way through the rest of the book or through the rough path literature, including the books by Lyons and Qian [] and Friz and Victoir [].

In the rest of the book, we tried to introduce the Fourier normal ordering algorithm and our quantum field theoretic constructions by replacing them systematically into a broader context. Chapter 3 is a quick preliminary presentation of the results obtained along these lines and serves as a bridge between the first two chapters and the rest of the book. It turned out to be difficult even to state these results without the appropriate combinatorial and physical language, so the reader will forgive some imprecision at this stage.

Chapter 4 introduces the Hopf algebraic setting by presenting two models closely related to integrals and differential equations, the Hopf algebraic rephrasing of Runge-Kutta numerical schemes of integration by C. Brouder, and D. Kreimer's toy model for renormalization in quantum field theory, which is actually a model of iterated integrals. The *algebraic classification theorem of formal rough paths* is given in the next chapter, together with the first examples of actual rough paths, namely, the *zero tree data* construction and the *Fourier domain regularization*, an ad-hoc regularization procedure, which is superseded by the renormalization procedure of Chapter 7 (indeed, it is very similar to the first endeavours to get finite quantities in quantum field theory by throwing away divergent terms), but may be of some help for the reader as a preparation for the next chapters.

Chapter 6 introduces some ideas and concepts of quantum field theory and renormalization, together with the multi-scale proof of convergence of renormalized Feynman diagrams. It is here for the convenience of the reader who is not a specialist of quantum field theory. Clearly enough, it is impossible to explain away quantum field theory in a few pages, but we hope that our explanations will be enough to follow the construction of renormalized rough paths in Chapter 7.

Chapter 8 is dedicated to the two constructive quantum field theoretic models yielding rough path lifts of fBm. Constructive field theory is a tough subject, requiring at the same time a good understanding of conventional aspects of quantum field theory and the renormalization group, heavy combinatorial tools due to cluster expansions, and sometimes some functional analysis. We shall not be able to reproduce a complete proof of these results, which would have taken up too many pages. We hope at least to have made the main ideas acceptable to a reader who is already acquainted with the philosophy of quantum field theory.

Finally, Chapter 9 is dedicated to the presentation of new tools of stochastic calculus (including *Fourier normal ordered Malliavin calculus*), and to the application of these tools to the study of stochastic differential equations.

Our aim will have been reached if the present book is of some interest and pleasant for readers coming from either of the fields touched upon, probabilists, algebraists, or physicists.

Chapter 1

T. Lyons' theory of rough paths

T. Lyons' theory is in essence a geometric theory. The stack of iterated integrals of a path is beautifully reinterpreted as a section of a principal bundle equipped with a degenerate metric. Rough path theory, in this sense, may be seen as a branch of sub-Riemannian geometry, and borrows some tools from group theory. The deep connection to sub-Riemannian geometry certainly deserves further investigation – rough path theory will, so we think, play a major rôle in the study of sub-Riemannian geometry. However we shall not insist on the geometric and group-theoretic aspects, partly in order to keep the present book reasonably sized, partly because they make the backbone of the excellent, up-to-date book written in 2010 by P. Friz and N. Victoir; partly also because they are not strictly necessary for the definition and the study of rough paths. Rather, we shall mostly review the most down-to-earth computational aspects of the subject in the easiest non-trival case $1/3 < \alpha \leq \frac{1}{2}$. The presentation is borrowed from the work of A. Lejay, who leans himself heavily on previous work by T. Lyons. In the end of the section we introduce rough paths as a stack of quantities loosely related to the underlying path, satisfying algebraic compatibility (Chen and shuffle) and Hölder regularity properties. Finally we comment on the geometric aspects and mention the classical results due to L. Coutin and Z. Qian on fractional Brownian motion.

1.1 The Young integral ($\alpha > \frac{1}{2}$)

Consider as in the Introduction a differential equation of the type

$$dy(t) = V_0(y(t))dt + \sum_{i=1}^{d} V_i(y(t))dx_i(t), \tag{1.1}$$

where $x = (x_1, \ldots, x_d)$ is some (deterministic or random) path called the *driving process*, and y, V_0, V_i are in general vector-valued. In this paragraph we are interested in the weakly irregular case – more regular than the usual diffusion equations, in any case – when the Hölder regularity index α of the driving process is $> \frac{1}{2}$. Then, as we shall see, a somewhat clever Riemann-sum approach due to L. C. Young [60] already in the 30'es allows one to give a sense and solve such equations.

We shall first work to give a sense to pathwise integrals such as $\int_s^t \langle f(x(s)), dx(s)\rangle := \sum_{i=1}^d \int_0^t f_i(x(s))dx_i(s)$. Two simple ideas come into play: (i) the integral should satisfy the Chasles relation, namely, $\int_s^t(\cdots) = \int_s^u(\cdots) + \int_u^t(\cdots)$. So one may assume that $|t-s| \ll 1$ and sum over small sub-intervals; (ii) if $|t-s| \ll 1$, then

$$y(s,t) := f(x(s))(x(t)-x(s)) \tag{1.2}$$

is a 'good' approximation of the integral provided $\alpha > \frac{1}{2}$. A trivial but key identity in the next computations is

$$\sum_{i=0}^{n-1} (t_{i+1}^n - t_i^n)^{1+\varepsilon} \to_{n\to\infty} 0 \tag{1.3}$$

for a partition $\Pi_{i=0}^{n-1}[t_i^n, t_{i+1}^n)$ of $[0,1]$ with mesh going to zero. In the following we assume f to be a smooth function, although much weaker regularity assumptions suffice. This will be enough to solve differential equations with smooth coefficients, and contains all significant ideas.

The α-Hölder norms are defined as follows,

$$||x||_{\alpha,[s,t]} := \sup_{s'\in[s,t]} |x(s')| + \sup_{s',t'\in[s,t]} \frac{|x(s')-x(t')|}{|t'-s'|^\alpha}. \tag{1.4}$$

Theorem 1.1 (Young [60, 29]) *Let $x(t)$ be an α-Hölder path with $\alpha > \frac{1}{2}$, and Π^n, $n \ge 1$ be increasing partitions of $[0,1]$, i.e. $\Pi^n = \{0 < t_1^n < \ldots < t_n^n < 1\}$, $\Pi^{n+1} \setminus \Pi^n = \{t_j^{n+1}\}$ for some j, with mesh going to zero. Let*

$$z^{\Pi^n}(t_i^n, t_j^n) := \sum_{i\le k<j} \langle f(x(t_k^n)), x(t_{k+1}^n) - x(t_k^n)\rangle$$

$$z^{\Pi^n}(s,t) = \langle f(s), x(t_i^n) - x(s)\rangle + z^{\Pi^n}(t_i^n, t_j^n) + \langle f(x(t_j^n)), x(t) - x(t_j^n)\rangle$$

if $t_{i-1}^n < s < t_i^n, t_j^n < t < t_{j+1}^n$. Then $z^{\Pi^n}(s,t) \to_{n\to\infty} z(s,t)$, a function independent on the choice of the partition, also denoted by $\int_s^t f(x_u)dx_u$, such that (i) z satisfies the Chasles relation $z(s,t) = z(s,u) + z(u,t)$; (ii) $t \mapsto z(s,t)$ is α-Hölder; (iii) z depends continuously on the path x in the sense of α-Hölder norms, more precisely $|| \int_0^t f(x_u)dx_u - \int_0^t f(\tilde{x}_u)d\tilde{x}_u||_{\alpha,[0,1]} \le C||x-\tilde{x}||^2_{\alpha,[0,1]}$.

Proof.

1. Let us first prove that one may remove one by one the points in Π^n, thus defining a series of subpartitions $\widetilde{\Pi}^{n-1}, \widetilde{\Pi}^{n-2}, \ldots$ so that $z^{\widetilde{\Pi}^m}(s,t) - z^{\widetilde{\Pi}^{m-1}}(s,t) = O((\frac{|t-s|}{m})^{1+\varepsilon})$. Clearly, one may choose some point $t_k^n \in \Pi^n$ such that $t_{k+1}^n - t_{k-1}^n \le \frac{2(t-s)}{\#(\Pi^n\cap[s,t])}$; we let $\widetilde{\Pi}^{n-1} = \Pi^n \setminus \{t_k^n\}$. Then

$$z^{\Pi^n}(s,t) - z^{\widetilde{\Pi}^{n-1}}(s,t) = y(t_{k-1}^n, t_k^n) + y(t_k^n, t_{k+1}^n) - y(t_{k-1}^n, t_{k+1}^n) \tag{1.5}$$

and $y(t^n_{k-1}, t^n_k) + y(t^n_k, t^n_{k+1}) - y(t^n_{k-1}, t^n_{k+1})$ satisfies the *approximate Chasles identity*

$$|y(s,t) - y(s,u) - y(u,t)| = |f(x(s)) - f(x(u))||x(t) - x(u)| \lesssim ||\nabla f||_\infty |t-s|^{1+\varepsilon} \tag{1.6}$$

if $s < u < t$, with $1 + \varepsilon = 2\alpha$.

Since the series $\sum_m m^{-1-\varepsilon}$ is summable, this yields the following approximate identity

$$z^{\Pi^n}(s,t) = \langle f(x(s)), x(t) - x(s)\rangle + O(|t-s|^{1+\varepsilon}) \tag{1.7}$$

2. Let us now prove by compacity that some subsequence of $(z^{\Pi^n}(s,t))_{n\geq 1}$ converges in the sup norm $|| \cdot ||_\infty$. Write $\Pi^n \cap [0,s] = \{t^n_j, \ldots, t^n_k\}$, $\Pi^n \cap [s,t] = \{t^n_{k+1}, \ldots, t^n_l\}$. Then (recall $y(u,v) := f(x(u))(x(v) - x(u))$ is a first order approximation of the integral over a small interval)

$$z^{\Pi^n}(0,t) = z^{\Pi^n}(0,s) + z^{\Pi^n}(s,t) + (y(t^n_k, t^n_{k+1}) - y(t^n_k, s) - y(s, t^n_{k+1}))$$

whence the functions $t \mapsto z^{\Pi^n}(0,t)$ are equicontinuous by using point 1. By Ascoli's theorem, some subsequence $(z^{\Pi^{\phi(n)}}(0,t))_n$ converges. The same equation shows that the limit satisfies Chasles property, i.e. $\lim_n z^{\Pi^{\phi(n)}}(s,t) = Z(0,t) - Z(0,s) =: Z(s,t)$. Equation (1.7) shows that any limit has a finite α-Hölder norm, controled by $||x||_\alpha$.

3. (unicity of the limit) Any two limits of subsequences $\tilde{Z}(s,t)$, $Z(s,t)$ satisfy the approximate identity

$$\tilde{Z}(s,t) - \langle f(x(s)), x(t) - x(s)\rangle = O(|t-s|^{1+\varepsilon}), Z(s,t) - \langle f(x(s)), x(t) - x(s)\rangle = O(|t-s|^{1+\varepsilon}) \tag{1.8}$$

together with the Chasles identity. Hence, by (1.3),

$$|\tilde{Z}(t) - Z(t)| \leq \sum_i |\tilde{Z}(t^n_i, t^n_{i+1}) - Z(t^n_i, t^n_{i+1})| \to_{n\to\infty} 0. \tag{1.9}$$

The proof of the continuity of the integration map follows from the previous identities and is left to the reader. □

In some sense the three identities (1.7,1.8,1.6) are approximate Chasles identities; this is the key concept here.

Let us now consider the case of differential equations.

Theorem 1.2 *Consider the differential equation (1.1), with x an α-Hölder path, $\alpha > \frac{1}{2}$ and initial condition $y(0) = y_0$. Then it may be solved uniquely using Young's integration.*

Proof. The solution will be constructed as the unique fixed point of the Young integration map $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \int \begin{pmatrix} 1 \\ f_i(y) \end{pmatrix} dx + \int \begin{pmatrix} 0 \\ 0 \end{pmatrix} dy = \begin{pmatrix} x(t) \\ \int_0^t \langle f(y(u)), dx(u)\rangle \end{pmatrix}$, where by some translation we have assumed $x(0) = 0, y(0) = 0$.

1. (Existence of fixed point) Consider the sequence (y^n) defined inductively by $y^0(t) = y_0$, $y^{n+1}(t) = \int_0 \langle f(y^n(u)), dx(u)\rangle$ $(n \geq 1)$. By Theorem 1.1 (see eq. (1.7) and (1.6),

$$|y^{n+1}(t) - y^{n+1}(s)| \leq C \left(||x||_{\alpha,[0,1]} |t-s|^\alpha + ||y||_{\alpha,[0,1]} |t-s|^{1+\varepsilon} \right) \tag{1.10}$$

for some constant C depending on f. Thus, for any sufficiently small subinterval $[s,t]$, say of length $\leq \delta$,

$$\left(||y^n||_{\alpha,[s,t]} < 2C||x||_{\alpha,[0,1]} \right) \Longrightarrow \left(||y^{n+1}||_{\alpha,[s,t]} < 2C||x||_{\alpha,[0,1]} \right). \tag{1.11}$$

By summing over subintervals $[t_i, t_{i+1}]$ of length δ spanning the interval $[0,1]$ and using the bound $\sum_{i=j}^k |t_{i+1} - t_i|^\alpha \lesssim \delta^{\alpha-1} |t_k - t_i|^\alpha$, one obtains that the sequence (y^n) is bounded in C^α, hence some subsequence converges. The limit y is a fixed point, hence satisfies the differential equation, as follows from the continuity of the integration map.

2. (Unicity) Let $y, \hat{y}$ be two fixed points, so that $y(t) - \hat{y}(t) = \int_0^t f(y(u))dx(u) - \int_0^t f(\hat{y}(u))dx(u)$. Using once again the approximate Chasles identity (1.6),

$$\begin{aligned} |\int_s^t \langle f(y_u), dx(u)\rangle - \int_s^t \langle f(\hat{y}(u), dx(u)\rangle| &= |\langle f(y(u)) - f(\hat{y}(u)), x(t) - x(s)\rangle| + O(|t-s|^{1+\varepsilon}) \\ &\leq ||\nabla f||_\infty \sup_{u\in[s,t]} |y(u) - \hat{y}(u)| \ \cdot \ ||x||_\alpha |t-s|^\alpha + O(|t-s|^{1+\varepsilon}). \end{aligned} \tag{1.12}$$

Applying this bound on subintervals of a small enough interval $[0,t]$ and iterating n times, one gets $\sup_{u\in[0,t]} |y(u) - \hat{y}(u)|| \lesssim 2^{-n} \to_{n\to\infty} 0$. But then $y(t) = \hat{y}(t)$ and one may apply the same procedure to the next interval $[t, 2t]$ and so on, until one has proved $y = \hat{y}$.

□

1.2 Simplest non-trivial case ($1/3 < \alpha \leq \frac{1}{2}$)

We now assume $x(t) = (x_1(t), x_2(t))$ to be a two-dimensional, α-Hölder path, with $1/3 < \alpha \leq \frac{1}{2}$. To prove unicity in Theorem 1.1, one crucially needed an approximate Chasles identity $|\langle f(x(s)), x(t) - x(s)\rangle - \langle f(x(u)), x(u) - x(s)\rangle - \langle f(x(u)), x(t) - x(u)\rangle| = |\langle f(x(s)) - f(x(u)), x(t) - x(u)\rangle| = O(|t-s|^{1+\varepsilon})$, which is now clearly false in general. This is the sign that the first order approximation of the integral on small intervals is not good enough. Going to order 1 this time, one writes, for $|t-s| \ll 1$,

$$\int_s^t f_{i_1}(x(u)) dx_{i_1}(u) \simeq f_{i_1}(x(s))(x_{i_1}(t) - x_{i_1}(s)) + \sum_{i_2} \nabla_{i_2} f_{i_1}(x(s)) \int_s^t dx_{i_1}(t_1) \int_s^{t_1} dx_{i_2}(t_2). \tag{1.13}$$

In short notation

$$\int_s^t \langle f(x(u)), dx(u)\rangle \simeq \langle f(x(s)), x(t) - x(s)\rangle + \langle \nabla f(x(s)), \int_s^t dx \otimes dx\rangle =: y(s,t). \tag{1.14}$$

In principle, one proves easily with this new definition that $y(s,t)$ satisfies an approximate Chasles relation, provided $\int_s^t dx_{i_1}(t_1) \int_s^{t_1} dx_{i_2}(t_2) = O(|t-s|^{2\alpha})$, a natural bound for an α-Hölder path, apparently.

Unfortunately there is a big gap in the argument – the iterated integral $\int_s^t dx_{i_1}(t_1) \int_s^{t_1} dx_{i_2}(t_2)$ is not defined a priori. Computing the inner integral leaves a term of the form $\int_s^t x_{i_2}(t_1) dx_{i_1}(t_1)$ which is *not* a Young integral if x is not $(\frac{1}{2}+\varepsilon)$-Hölder continuous. Think of the case of Brownian motion; this quantity is closely related to the so-called *Lévy area* (see below), which may be computed by using Itô or Stratonovich stochastic integration. As well-known from arguments due to E. Wong and M. Zakai [59, 24], any sequence of paths of bounded variation converging uniformly to Brownian motion (for instance, piecewise linear approximations) lead in the limit (in an L^2 sense, and even almost surely in the latter case) to the Stratonovich integral $\int_s^t x_{i_2}(t_1) \circ dx_{i_1}(t_1)$.

Since we want a theory as general as possible, valid even in a deterministic setting, we seek a pathwise definition of iterated integrals, hence in some sense generalizations of the Stratonovich integral. The road is long still as we shall see. Contrary to the Brownian case, there is no unicity – no preferred limit, as we shall argue, indeed a huge arbitrariness of choice –, and – in the particular case of fractional Brownian motion with Hurst index $\alpha \leq 1/4$ – families of equally 'natural' approximations leading to different definitions of stochastic calculus. Yet in all known examples of very irregular paths ($\alpha \leq 1/4$), all straightforward constructions (such as piecewise linear approximations) fail [11, 51, **?**].

1.2.1 A. Lejay's area bubbles

The Green-Riemann formula yields a geometric interpretation of iterated integrals:

$$x_{(1,2)}(s,t) := \int_s^t dx_{i_1}(t_1) \int_s^{t_1} dx_{i_2}(t_2)$$

(resp. the antisymmetrized quantity, $x_{(1,2)}(s,t) - x_{(2,1)}(s,t)$, called *Lévy area*) is a measure of the *signed area* between the curve $t \mapsto (x_1(t), x_2(t))$ and the coordinate axes (resp. between the curve and the segment $[x(s), x(t)]$). These two quantities differ simply by the area of a triangle. For the sake of informal discussion, we shall speak in both cases of the *area generated by the path.*

The following construction, due to A. Lejay, shows clearly that the area generated by a path of Hölder regularity $\alpha < \frac{1}{2}$ is extremely sensitive to the chosen 'discretization'. Let us give a very simple illustration. We set $x(t) = (t, 0)$ and see x as an α-Hölder path, with $\alpha < \frac{1}{2}$, although it is smooth of course. The curve $(x_1(t), x_2(t))$ is simply the horizontal axis. Now we color 'even' subintervals of time, $t \in [\frac{2i}{2^n}, \frac{2i+1}{2^n}]$, resp. 'odd' subintervals $t \in [\frac{2i+1}{2^n}, \frac{2i+2}{2^n}]$, in black, resp. in red. We take scissors and cut each red subinterval in three pieces, and substitute the middle subinterval by an Ω-shaped line, i.e. a horizontal segment, followed by a 'bubble', followed again by a horizontal segment. Each bubble is a small clockwise circle located in the upper half-plane and tangent to the horizontal axis, with radius $r_n \approx 2^{-(\alpha+\varepsilon)n}$ for some small ε, $\varepsilon < \alpha$. The paths x^n obtained in this way converges in α-Hölder norm to x when $n \to \infty$ because the variation of x on small red subintervals $[s,t] \subset [\frac{2i+1}{2^n}, \frac{2i+2}{2^n}]$ is of order $|t-s|^{\alpha+\varepsilon}$. On the other hand, the area generated by the bubbles is of order $2^n \times (2^{-(\alpha+\varepsilon)n})^2 \to_{n\to\infty} \infty$. By a small variation of this argument, alternating clockwise (with negative area) and anti-clockwise (with positive area) circles leads to a totally arbitrary area in the limit $n \to \infty$. The same argument applies to any path x, see [29] for details.

There are two lessons to be learnt out of this construction: (i) two different approximation schemes may lead to different areas; (ii) unclever schemes with two many accumulating clockwise or anti-clockwise 'bubbles' do not allow to define an area in the limit. Apparently the piecewise linear approximation scheme of Coutin and Qian for fBm [11] has this drawback for $\alpha \le 1/4$ (see below).

We took the liberty of calling this phenomenon 'area bubbles', *bulles d'aire* in French (a pun actually, impossible to translate as all puns).

1.2.2 Axiomatization of rough paths of second order

Drawing on the lessons of the preceding paragraph, we shall make the following crucial *assumption.* To standardize notations we write $x_i(s,t) := x_i(t) - x_i(s)$ for the increments of the path.

Assumption. Assume there *exist* quantities $x_{(i_1,i_2)}(s,t)$, $i_1, i_2 = 1,2$, such that the stack $\boldsymbol{x} := ((x_{i_1}(s,t))_{i_1=1,2}, (x_{(i_1,i_2)}(s,t))_{i_1,i_2=1,2})$ is an *α-Hölder rough path over* x, namely:

(i) (Hölder regularity properties) x is α-Hölder and $|x_{(i_1,i_2)}(s,t)| \le C|t-s|^{2\alpha}$;

(ii) (algebraic compatibility properties) $\boldsymbol{x}$ satisfies the Chen property,

$$x_{(i_1,i_2)}(s,t) - x_{(i_1,i_2)}(s,u) - x_{(i_1,i_2)}(u,t) - x_{i_1}(u,t) \cdot x_{i_2}(s,u) = 0 \tag{1.15}$$

and the shuffle property

$$x_{(i_1,i_2)}(s,t) + x_{(i_2,i_1)}(s,t) = x_{i_1}(s,t) \cdot x_{i_2}(s,t) \tag{1.16}$$

Several comments are in order.

(i) $x_{(i_1,i_2)}$ may only improperly be said to be 2α-Hölder since it is *not* the increment of a function of one variable. Namely, were it the increment $f(t) - f(s)$ of a function of one variable, it would obviously satisfy the *Chasles property* $x_{(i_1,i_2)}(s,t) - x_{(i_1,i_2)}(s,u) - x_{(i_1,i_2)}(u,t) = 0$ instead of the Chen property (1.15).

(ii) The correcting product term $x_{i_1}(u,t) \cdot x_{i_2}(s,u)$ in the Chen property is of geometric origin, see Fig. 1.1, although the Chen property may be checked by a straightforward computation left to the reader.

The Chen property states that the (non-antisymmetrized) area generated by the path between s and t, minus the same quantity between s and u, minus the same quantity

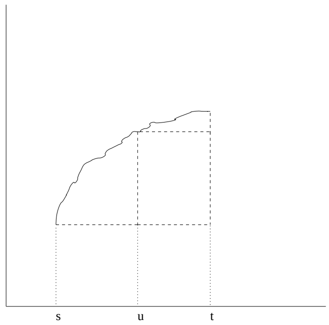


Figure 1.1: Chen property.

between u and t, is equal to the area of the rectangle on the figure. Shifting from non-antisymmetrized area to the antisymmetrized area, or to any linear combination of the two, always leaves some correcting term interpreted as a sum or difference of areas of one or two rectangles. So this *defect of additivity of the area* is an intrinsic defect of geometric origin, having many different interpretations scattered over the text (see subsections 1.3 and 5.1 in particular). It must be taken into account in any definition of a rough path.

(ii) The *shuffle property* follows from the Fubini theorem: from the identity $\int_s^t dx_{i_2}(t_2) \int_s^{t_2} dx_{i_1}(t_1) = \int_s^t dx_{i_1}(t_1) \int_{t_1}^t dx_{i_2}(t_2)$ follows

$$\int_s^t dx_{i_1}(t_1) \int_s^{t_1} dx_{i_2}(t_2) + \int_s^t dx_{i_2}(t_2) \int_s^{t_2} dx_{i_1}(t_1) = \int_s^t \int_s^t dx_{i_1}(t_1) dx_{i_2}(t_2) \qquad (1.17)$$

provided the sum of the integrals over the two complementary simplices $t \ge t_1 \ge t_2 \ge s$ and $t \ge t_2 \ge t_1 \ge s$ is equal to the integral over the rectangle $[s,t]^2$. The reader may have recognized the integration by parts formula from which one deduces the Itô formula [47]. The Itô term is absent for the Stratonovich integral, in which case the usual rules of infinitesimal calculus – in particular, the fundamental formula $F(X(t)) - F(X(0)) = \int F(X(s))dX(s)$ – apply. Since the emphasis here is on Stratonovich-like integrals, we choose to include this property. On the other hand, many constructions (and in particular the results of this subsection) still hold true with appropriate modifications for rough paths which *do not* satisfy the shuffle property (see in particular [29] where this property is not assumed, and the usual Itô correction term is discussed in connection with rough path theory).

For a path $x = (x_1, x_2)$ with two components, the shuffle property automatically reduces the number of 'independent' second-order components to one, e.g. $x_{(1,2)}$, since $x_{(1,2)}(s,t) + x_{(2,1)}(s,t)$ sum up to $x_1(s,t) \cdot x_2(s,t)$, and $x_{(1,1)}(s,t) = \frac{1}{2}(x_1(s,t))^2$.

(iii) as follows e.g. from A. Lejay's area bubble argument, rough paths are not unique. Hence we shall also speak of *a lift* of a path. To a path seen as an α-Hölder path with $1/3 < \alpha \le 1/2$, one associates lifts of order 2. To more irregular paths one shall need to associate higher-order lifts (see below).

In the rough path literature one often speaks of *multiplicative* property instead of *Chen* property, resp. *geometric property* instead of *shuffle property*. The terminology here is motivated by the connection to the literature on Hopf algebras, see below. The name 'multiplicative' is certainly appropriate, see below, but 'geometric', on the other hand, is rather confusing.

With these hypotheses the following lemma is easy to prove (use Taylor's formula), for f sufficiently regular (say, C^2):

Lemma 1.1 *(see [29], lemma 3.1). Let $\boldsymbol{x}$ be a rough path over x, with x α-Hölder, $\alpha > 1/3$. Approximate $\int_s^t \langle f(x(u)), dx(u)\rangle$ by*

$$\begin{aligned} y(s,t) &:= \langle f(x(s)), x(t)-x(s)\rangle + \sum_{i_1,i_2} \nabla_{i_2} f_{i_1}(x(s)) \int_s^t dx_{i_1}(t_1) \int_s^{t_1} dx_{i_2}(t_2) \\ &= \langle f(x(s)), x(t)-x(s)\rangle + \langle \nabla f(x(s)), \mathbf{x}^2(s,t)\rangle \end{aligned} \tag{1.18}$$

Then $y(s,t)$ satisfies the approximate Chasles identity,

$$|y(s,t) - y(s,u) - y(u,t)| = O(|t-s|^{1+\varepsilon}). \tag{1.19}$$

'Proof'. Forgetting about terms of order 2, $y(s,t)-y(s,u)-y(u,t) = (f(x(u))-f(x(s)))(x(t)-x(u))$ by the trivial 'Chen property of order 1', i.e. the Chasles property $x(t)-x(s) = (x(t)-x(u))-(x(u)-x(s))$. The Chen property, involving $x_{(i_1,i_2)}(u,t)$ instead of the increment $x(t)-x(u)$, is required to bound the terms of order 2; the correcting, product term cancels with the term of order 1 contributing an unwanted $O(|t-s|^{2\alpha})$ (see introduction to subsection 1.2). Then the Hölder property (i) for the area $x_{(i_1,i_2)}(u,t)$ shows that the contribution of the area term to $y(s,t)-y(s,u)-y(u,t)$ is of order $O(|t-s|^{3\alpha}) = O(|t-s|^{1+\varepsilon})$. □

Using this bound, one easily extends Theorem 1.1 (with this new definition of the approximate integrals $y(s,t)$) to the rough path setting with $\alpha > 1/3$. The proof of Theorem 1.2 may also be adapted to this setting provided the successive approximations y^n (see proof of the theorem) are lifted, so that the fixed-point theorem may be applied to pairs $(\boldsymbol{x}, \boldsymbol{y})$. The lift of y is here canonical, once a lift of x has been chosen. Namely, for $|t-s| \ll 1$, a natural choice of approximate lift of the approximate integral $y(s,t)$ is (compare with eq. (1.18)) $y_{(i_1,i_2)}(s,t) := f_{i_1}(x(s)) f_{i_2}(x(s)) x_{(i_1,i_2)}(s,t)$. The equivalent of the approximate Chasles relation for this quantity of order 2 is an *approximate Chen relation*,

$$|y_{(i_1,i_2)}(s,t) - y_{(i_1,i_2)}(s,u) - y_{(i_1,i_2)}(u,t) - y_{i_1}(s,u)\cdot y_{i_2}(u,t)| = O(|t-s|^{1+\varepsilon}). \tag{1.20}$$

The true Chen relation for the bundle $((y_i^n), (y^n_{(i_1,i_2)}))$ follows when the mesh goes to zero as in the proof of Theorem 1.1. For the proofs the reader is deferred to T. Lyons' original papers [34, 35].

1.3 An introduction to the general case

From the preceding considerations, the reader may easily guess the main lines for the extension to lowest Hölder regularity indices α. The approximate integrals $y(s,t)$ require the use of Taylor's formula to order $N-1$, where $N+1$ is the smallest integer strictly larger than $1/\alpha$, so that error terms such as $y(s,t)-y(s,u)-y(u,t)$ are of order $O(|t-s|^{(N+1)\alpha}) = O(|t-s|^{1+\varepsilon})$. In other terms, $N = \lfloor 1/\alpha \rfloor$, $\lfloor \cdot \rfloor$ = integer part, except if $1/\alpha$ is an integer (then $N = 1/\alpha + 1$); $N = 2$ in the preceding subsection. What is less obvious is the combinatorial aspect. What are the generalized Chen and shuffle properties ? These are clearly defined and lie at the basis of all rough path constructions (at least when $\alpha > 1/4$, see below). But there are many ways to state them. We shall mainly see in this paragraph how they arise from down-to-earth computations. One of the main original contributions of T. Lyons, much elaborated over later on by P. Friz and N. Victoir (see the book [16]), was to understand a bundle of quantities satisfying the Chen and shuffle properties (we intentionally leave aside the Hölder regularity properties, see §1.2.2 above, because they are not required at this formal level) as a section of a G-valued principal bundle for some nilpotent group G. As already mentioned, we shall be content with some brief comments on this important aspect. Finally, the Chen and shuffle properties refer to the Chen and shuffle Hopf algebras, which are of interest to people from algebraic combinatorics, sometimes with a specific interest in renormalization theory (or conversely), see e.g. [12, 14, 18, 39, 40, 13], and from numerical analysis, see e.g. [33] and references inside. They are also hidden in Ecalle's work on resurgence, see e.g. [42].

Let us first introduce the notion of signature of a path, due to Chen [7] (see the book by F. Baudoin [2] for beautiful developments).

Definition 1.2 (signature of a path) *Let $x : \mathbb{R} \to \mathbb{R}^d$ be a smooth path. Then the signature of x is the bundle of its iterated integrals*

$$X(s,t) := (1, (x_{i_1}(s,t))_{i_1=1,\ldots,d}, (x_{(i_1,i_2)}(s,t))_{i_1,i_2=1,\ldots,d}, \ldots), \tag{1.21}$$

where

$$x_{(i_1,\ldots,i_n)}(s,t) := \int_s^t dx_{i_1}(t_1) \int_s^{t_1} dx_{i_2}(t_2) \ldots \int_s^{t_{n-1}} dx_{i_n}(t_n). \tag{1.22}$$

Restricting to iterated integrals of order $\le N$, one obtains a finite bundle called truncated signature of order N.

Now we start with some formal but elementary arguments. Let $e_1, \ldots, e_d$ be a basis of $\mathbb{R}^d$.

Definition 1.3 (tensor algebra) *The tensor algebra of $\mathbb{R}^d$ is the infinite-dimensional non-commutative algebra $T\mathbb{R}^d := \mathbb{R} \oplus \left(\oplus_{n=1}^{\infty} T^n\mathbb{R}^d\right)$, $T^n\mathbb{R}^d := (\mathbb{R}^d)^{\otimes n}$, generated by the identity 1 and by the tensor products $e_{i_1} \otimes \ldots \otimes e_{i_n}$ of arbitrary order. The product is simply the concatenation product, $(e_{i_1} \otimes \ldots e_{i_n}) \otimes (e_{i_{n+1}} \otimes \ldots e_{i_m}) := e_{i_1} \otimes \ldots \otimes e_{i_m}$.*

Then $X(s,t)$ may be embedded into TV as the element

$$X(s,t) = 1 + \sum_{i_1=1}^{d} x_{i_1}(s,t) e_{i_1} + \sum_{i_1,i_2=1}^{d} x_{(i_1,i_2)}(s,t) e_{i_1} \otimes e_{i_2} + \dots \tag{1.23}$$

For any smooth path, the following fundamental Chen property holds:

$$X(0,t) = X(u,t) \otimes X(0,u). \tag{1.24}$$

Apart from the tensor product $\otimes$, [1] this looks very much like the concatenation formula for paths, $x([0,t]) = x([0,u]) \cdot x([u,t])$. The infinitesimal version of this formula is $dX(0,t) = dx_t \otimes X(0,t)$. Projecting to the component along $T^n\mathbb{R}^d$, the Chen property reads in components

$$\begin{aligned} x_{(i_1,\dots,i_n)}(0,t) &= x_{(i_1,\dots,i_n)}(0,u) + x_{(i_1,\dots,i_n)}(u,t) \\ &+ \sum_{j=1}^{n-1} x_{(i_1,\dots,i_j)}(u,t) x_{(i_{j+1},\dots,i_n)}(0,u). \qquad \text{(Chen property)} \end{aligned} \tag{1.25}$$

An informal proof goes as follows,

$$\begin{aligned} &\left(\int_0^t dx_1(t_1) \int_0^{t_1} dx_2(t_2) \dots \int_0^{t_{n-1}} dx_n(t_n)\right) - \left(\int_0^u dx_1(t_1) \int_0^{t_1} dx_2(t_2) \dots \int_0^{t_{n-1}} dx_n(t_n)\right) \\ &\qquad - \left(\int_u^t dx_1(t_1) \int_0^{t_1} dx_2(t_2) \dots \int_0^{t_{n-1}} dx_n(t_n)\right) \\ &= \int_u^t dx_1(t_1) \left(\int_0^{t_1} dx_2(t_2) \dots \int_0^{t_{n-1}} dx_n(t_n) - \int_u^{t_1} dx_2(t_2) \dots \int_u^{t_{n-1}} dx_n(t_n)\right) \\ = \ & \left(\int_u^t dx_1(t_1)\right) \left(\int_0^u dx_2(t_2) \dots \int_0^{t_{n-1}} dx_n(t_n)\right) \\ &\quad + \int_u^t dx_1(t_1) \left[\left(\int_0^t dx_2(t_2) \dots \int_0^{t_{n-1}} dx_n(t_n)\right) - \left(\int_0^u dx_2(t_2) \dots \int_0^{t_{n-1}} dx_n(t_n)\right) \right. \\ &\qquad \left. - \left(\int_u^t dx_2(t_2) \dots \int_0^{t_{n-1}} dx_n(t_n)\right)\right] \\ &= \dots = \sum_{j=1}^{n-1} \left(\int_u^t dx_1(t_1) \dots \int_u^{t_{j-1}} dx_j(t_j)\right) \left(\int_0^u dx_{j+1}(t_{j+1}) \dots \int_0^{t_{n-1}} dx_n(t_n)\right) \end{aligned} \tag{1.26}$$

by an easy induction. The infinitesimal version,

$$d\left(\int_0^t dx_{i_1}(t_1) \int_0^{t_1} dx_{i_2}(t_2) \dots \int_0^{t_{n-1}} dx_{i_n}(t_n)\right) = dx_{i_1}(t) \cdot \int_0^t dx_{i_2}(t_2) \dots \int_0^{t_{n-1}} dx_{i_n}(t_n)$$

is elementary.

[1] and an apparently unfortunate ordering (which may be remedied, either by reversing the order of integration in the iterated integrals or simply by changing the order of the variables, $X(t,0)$ instead of $X(0,t)$!)

The shuffle property (as in the second order case) is a generalized Chasles identity stating that the integral over a hyper-rectangle is equal to the sum of the integrals over any partition into simplices $\{t > t_1 > \ldots > t_n > s\}$, and follows formally by using Fubini's theorem. Explicitly,

$$x_{(i_1,\ldots,i_{n_1})}(s,t)x_{(j_1,\ldots,j_{n_2})}(s,t) = \sum_{\boldsymbol{k}\in \mathrm{Sh}(\boldsymbol{i},\boldsymbol{j})} x_{(k_1,\ldots,k_{n_1+n_2})}(s,t), \tag{1.27}$$

where $\mathrm{Sh}(\boldsymbol{i},\boldsymbol{j})$ ranges over the 'shuffles' [2] of the ordered lists $\boldsymbol{i} = (i_1,\ldots,i_{n_1})$ and $\boldsymbol{j} = (j_1,\ldots,j_{n_2})$, i.e. of permutations of the concatenated list $(\boldsymbol{i},\boldsymbol{j})$ preserving the order of the sublists $\boldsymbol{i}$ and $\boldsymbol{j}$.

Let us give some hints of the underlying group-theoretic interpretation. The afore-mentioned defect of additivity of the antisymmetrized (so-called Lévy) area $\mathcal{A}(s,t) := \frac{1}{2}(x_{(i_1,i_2)}(s,t) - x_{(i_2,i_1)}(s,t))$,

$$\mathcal{A}(s,t) - \mathcal{A}(s,u) - \mathcal{A}(u,t) = \frac{1}{2}\left(x_{i_1}(u,t)x_{i_2}(s,u) - x_{i_2}(u,t)x_{i_1}(s,u)\right),$$

disappears if one considers $\boldsymbol{x}(s,t) := (x_1(s,t), x_2(s,t), \mathcal{A}(s,t))$ as the ordinary Lie algebraic coordinates in the Heisenberg group $G^{(2)}$ with product $(x_1,y_1,z_1)\cdot(x_2,y_2,z_2) = (x_1+y_1, x_2+y_2, z_1+z_2+\frac{1}{2}(x_1y_2 - x_2y_1)$. The readers who are not familiar with this group (which appears recurrently in symplectic geometry and quantum mechanics, and is also well-known to Lie group specialists as the simplest non-trivial nilpotent Lie group) may simply consider it as the group of upper-triangular matrices of the form $\exp\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$ and consider the group product $\cdot$ as the product of matrices. The product formula is then a consequence of the famous Baker-Campbell-Hausdorff formula, $\exp(A+B) = \exp(A+B+\frac{1}{2}[A,B]+\ldots)$. Then $\boldsymbol{x}(s,t) = \boldsymbol{x}(u,t)\cdot\boldsymbol{x}(s,u)$, and in particular (using the matrix inverse), $\boldsymbol{x}(s,t) = \boldsymbol{x}(0,t)\cdot\boldsymbol{x}(0,s)^{-1}$, so that $\boldsymbol{x}(s,t)$ appears as a (multiplicative) increment of the $G^{(2)}$-valued path $t \mapsto \boldsymbol{x}(0,t)$. In geometric terms, $t\mapsto \boldsymbol{x}(0,t)$ is a section of the trivial principal bundle over $\mathbb{R}$ with fiber $G^{(2)}$. In the general case, the truncated signature may be interpreted similarly as a section of the trivial $G^{(N)}$-principal bundle over $\mathbb{R}$, where $G^{(N)}$ is now the *free nilpotent Lie group of order* N with Lie algebra generated by $e_1,\ldots,e_d$, i.e. $G^{(N)} = \exp\mathfrak{g}^{(N)}$, where:

(i) exp is the formal exponential in the tensor algebra $T\mathbb{R}^d$, $\exp(X) = 1 + X + \frac{1}{2!}X\otimes X + \frac{1}{3!}X\otimes X\otimes X + \ldots$;

(ii) $\mathfrak{g}^{(N)}$ (the Lie algebra of the group $G^{(N)}$) is the *free nilpotent Lie algebra of order* N. Informally, it is the Lie algebra generated by successive brackets $e_{i_1}, [e_{i_1}, e_{i_2}] := e_{i_1}\otimes e_{i_2} - e_{i_2}\otimes e_{i_1}, [e_{i_1},[e_{i_2},e_{i_3}]],\ldots$ up to brackets of order N, setting to zero all higher-order Lie brackets.

The group $G^{(N)}$ is naturally equipped with a sub-Riemannian (i.e. degenerate) metric obtained by setting $d\boldsymbol{x}^2 = dx^2$, i.e. the length of a path $\boldsymbol{x} = (1, x(t), (x_{(i_1,i_2)})(t))_{i_1,i_2},\ldots) \in G^{(N)}$

[2] in French, 'shuffle' is battement (de cartes).

is by definition $\int |dx(t)|$, the length of the underlying $\mathbb{R}^d$-valued path x. One easily shows (by formal compacity arguments) that $G^{(N)}$ is a geodesic space. The associated geodesic distance is called the *Carnot-Carathéodory* distance. If x is a smooth path, its bundle of usual iterated integrals (up to some arbitrary order n) is called the *canonical lift* of x to $G^{(n)}$.

Let us now turn to irregular paths; so x is only assumed to be α-Hölder for some $\alpha \in (0,1)$ (in practice $\alpha \leq 1/2$, otherwise the theory is void). A *formal rough path of order* N over x is a data bundle of order N, $\boldsymbol{x} = ((x_{i_1}), \ldots, (x_{(i_1,\ldots,i_N)}))$ satisfying the Chen and shuffle property. This formal notion will turn out to be useful later on, when we give a general construction of formal rough paths. Finally one give the general definition of a rough path:

Definition 1.4 *An α-Hölder rough path is a data bundle $((x_{i_1}), \ldots, (x_{(i_1,\ldots,i_N)}))$ satisfying the Chen and shuffle properties for $n = 1, \ldots, N$, together with the Hölder regularity properties,*

$$|x_{(i_1,\ldots,i_n)}(s,t)| = O(|t-s|^{n\alpha}), \quad n = 1, \ldots, N. \tag{1.28}$$

Turning once again to the group-theoretic interpretation (see [16] for details), the following fact is crucial, since it shows that the componentwise regularity properties of a rough path combine into a single regularity property for the associated group-valued path: a path $\boldsymbol{x}$ is α-Hölder for the geodesic distance on $G^{(N)}$ if and only if $x_{(i_1,\ldots,i_n)}(s,t)$, $1 \leq i_1, \ldots, i_n \leq d$, $1 \leq n \leq N$, satisfy the Hölder property (1.28). From general arguments on metric spaces, one then proves that any $G^{(N)}$-valued α-Hölder path may be approximated in the sense of the supremum norm $|| \cdot ||_\infty$ by bounded variation paths $(\boldsymbol{x}_j)_{j\in\mathbb{N}}$ (themselves canonical lifts, see above, of bounded variation paths $x_j : \mathbb{R} \to \mathbb{R}^d$), forming a bounded sequence for the α-Hölder norm. Let $\alpha' < \alpha$. Then, by Ascoli's theorem, one shows the existence of some subsequence which converges for the α'-Hölder norm. In other words, letting $\alpha' < \alpha$, *every α-Hölder rough path $\boldsymbol{x}$ above a path x is a limit in the α'-Hölder norm of the bundle of iterated integrals of a sequence of regular approximations of x.* This important general result which we call the *approximation theorem* shows the validity of the above axiomatization of rough paths, see Def. 1.4. The regularity loss in the replacement of α by $\alpha' < \alpha$ is unavoidable in the general case, and leads to the definition of the so-called *space of geometric rough paths* as the completion – in the sense of α-Hölder norms – of the space of regular paths.

A general theorem due to T. Lyons and N. Victoir [36] shows that any α-Hölder path x may be lifted into an α-Hölder rough path (actually only α'-Hölder path, $\alpha' < \alpha$, if $1/\alpha \in \mathbb{N}$). To prove this statement, the authors see x as a section of the quotient bundle with fiber $\exp\left(\oplus_{n\geq 1} T^n\mathbb{R}^d\right) / \exp\left(\oplus_{n\geq 2} T^n\mathbb{R}^d\right) \simeq \mathbb{R}^d$, and find a lift of x into a section $\boldsymbol{x}$ of the principal bundle with fiber $G^{(N)}$ with the correct Hölder regularity. Such sections exist (they are actually extremely arbitrary, as any lift of sections in geometry), but the construction, far from being canonical, uses the axiom of choice. This second important result we call the *existence theorem.* Both the approximation theorem and the existence theorem suffer in general from being extremely non-explicit, relying on the far from well-understood, difficult metric properties of the Carnot-Carathéodory spaces $G^{(N)}$ – for an overview of the issues of sub-Riemannian geometry the reader may refer to the book by Montgomery [43] – and on arbitrary lifts.

If one now assumes to have exhibited a lift $\boldsymbol{x}$ of some α-Hölder path x (say, by the existence theorem or any more explicit construction), together with a sequence of approximations (by the approximation theorem), one may solve and study differential equations driven by $\boldsymbol{x}$. The solutions do not depend on the choice of approximations, only on the choice of lift $\boldsymbol{x}$, but the essence of the results, say in the book by P. Friz and N. Victoir, is to resort to some sequence of regular approximations x^ε and show that the solutions of the associated regular differential equations converge nicely when $\varepsilon \to 0$, yielding at the same time explicit bounds for the errors of the numerical schemes.

1.4 Example of fractional Brownian motion

The only examples to which rough path theory has been applied so far are those for which explicit approximation sequences are known. Brownian motion and Stratonovich stochastic differential equations may be considered as the fundamental example. As previously stated, Itô integration enters the larger framework of rough paths that do not possess the shuffle property. The other 'classical' examples are reversible Markov processes [], diffusions on fractals [] and fractional Brownian motion (fBm). In all these cases of random driving processes, one restricts to $\alpha > 1/4$; the barrier at $\alpha = 1/4$ is a real one; going beyond it requires totally different methods as we shall see. Let us develop to some length the case of fBm which is our main example here.

Definition 1.5 (fractional Brownian motion) *Fractional Brownian motion with Hurst (or Hölder regularity) index $\alpha \in (0,1)$ is the centered, Gaussian process $B_t, t \geq 0$ with covariance kernel $\mathbb{E}[B_s B_t] = \frac{1}{2}(s^{2\alpha} + t^{2\alpha} - |t-s|^{2\alpha})$.*

Without any further consequences, one may extend $B_t, t \geq 0$ to negative times by setting instead $\mathbb{E}[B_s B_t] = \frac{1}{2}(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha})$, which we shall assume in the sequel. Also, we shall need to consider a vector-valued process, denoted by $B(t)$, $B(t) := (B_1(t), \ldots, B_d(t))$, with independent, identically distributed fBm components $B_1, \ldots, B_d$. Brownian motion is obtained as the particular case $\alpha = \frac{1}{2}$. The inequality $\mathbb{E}[|B_t - B_s|^2] \lesssim |t-s|^{2\alpha}$ (actually an equality), together with the standard Kolmogorov-Centsov lemma and classical estimates on Gaussian processes, show that, for every $\varepsilon > 0$, $t \mapsto B(t)$ has a.s. $(\alpha - \varepsilon)$-Hölder trajectories. By a suitable modification of the process over a subset of zero measure, one may assume that all trajectories are α'-Hölder for some fixed $\alpha' < \alpha$. The interest for our purposes lies then in the case $\alpha < \frac{1}{2}$ (which we shall generally assume in the sequel). A by now standard result by L. Coutin and Z. Qian states the following.

Theorem 1.3 *(see[11])*

Let $B^n(t)$ be the piecewise linear approximation of $B(t)$ coinciding with $B(t)$ at dyadic points $k \cdot 2^{-n}, n \in \mathbb{Z}$. Let $\alpha' < \alpha$ and $T > 0$. Then:

(i) *if* $1/3 < \alpha \le 1/2$, *the associated rough paths* $(B^n, (B^n_{(i_1,i_2)}))$ *converge in* L^2 *for the so-called* α'-Hölder rough path norm, $||\boldsymbol{x}||_{\alpha'} := ||\boldsymbol{x}||_\infty + \sup_{s,t\in[-T,T]}(\frac{|x(t)-x(s)|}{|t-s|^{\alpha'}} + \sum_{i_1,i_2\le d} \frac{|x_{(i_1,i_2)}(s,t)|}{|t-s|^{2\alpha'}})$.

(ii) *a similar results holds if* $1/4 \le \alpha < 1/2$, *if one includes* $\sum_{i_1,i_2,i_3\le d} \frac{|x_{(i_1,i_2,i_3)}(s,t)|}{|t-s|^{3\alpha'}}$ *in the definition of the rough path norm;*

(iii) *if* $\alpha \le 1/4$, *then* $\mathbb{E}|B^n_{(1,2)}(s,t)|^2$ *diverges when* $n \to \infty$.

Points (i), (ii) are positive results, point (iii) is a negative one. The case of Brownian motion ($\alpha = \frac{1}{2}$) is already known from Wong-Zakai-type theorems. *Positive results* have been the starting point for a number of papers on stochastic differential equations driven by fBm with $\alpha \in (1/4, 1/2)$ (sometimes restricting to $\alpha \in (1/3, 1/2)$ for simplicity and trusting that they also hold for $\alpha \in (1/4, 1/3]$). Further investigations have shown that the *negative result* for $\alpha \le 1/4$ is indeed a no-go theorem [51, 52]; the Lévy areas $B^n_{(i_1,i_2)}$, $i_1 \ne i_2$ fluctuate wildly in the limit $n \to \infty$; renormalizing it by the vanishingly small scale factor $2^{n/2(4\alpha-1)}$ (actually, a logarithmic factor when $\alpha = 1/4$) leads to a Brownian motion when $n \to \infty$, by using central limit theorem arguments.

Chapter 2

Gubinelli's algebraic rough path theory

This different presentation of rough path theory appeared originally in 2004 in a paper by M. Gubinelli [19], two years after the book by T. Lyons. Although formally equivalent to the theory by T. Lyons, P. Friz and N. Victoir, algebraic rough path theory rests entirely on the axiomatic definition of rough paths, see Definition 1.4 and does not require the use of any approximation scheme, nor of any kind of Lie group-theoretic or geometric arguments; it is more algebraic in spirit, hence its name. Also (at least for $\alpha > 1/3$), the proof of Cauchy-Lipschitz type theorems for differential equations is a little shorter than the proof sketched in the previous section. We shall be able to give somewhat sketchy but complete arguments in this case.

2.1 Sewing map and Gubinelli's integral

The starting point is a natural but elegant algebraic elaboration on the defect of additivity of the Lévy area, probably inspired by the Cech cohomology. Let $C_j(\mathbb{R},\mathbb{R}^d)$ be the vector space of continuous functions $f:\mathbb{R}^j\to\mathbb{R}^d$, and $\delta_j : C_j(\mathbb{R},\mathbb{R}^d)\to C_{j+1}(\mathbb{R},\mathbb{R}^d)$ the linear map defined by $(\delta_j f)(t_1,\ldots,t_{j+1}) := \sum_{i=1}^{j+1}(-1)^{i+1} f(t_1,\ldots,\check{t}_i,\ldots,t_{j+1})$. The sequence of maps

$$0\to C_1(\mathbb{R},\mathbb{R}^d)\stackrel{\delta_1}{\to} C_2(\mathbb{R},\mathbb{R}^d)\stackrel{\delta_2}{\to}\cdots\stackrel{\delta_{j-1}}{\to} C_j(\mathbb{R},\mathbb{R}^d)\stackrel{\delta_j}{\to}\cdots$$

is an *exact differential complex*. The notion is standard in algebraic topology and may be found in any textbook, but the knowledge of algebraic topology is useless in this context. Let us just explain the words. A *differential complex* means that $\delta_{j+1}\circ\delta_j\equiv 0$, as the reader may check by a straightforward computation. In particular, $\delta_2\circ\delta_1\equiv 0$ is equivalent to the Chasles property, $x(s,t)-x(s,u)-x(u,t)=0$ for increments, i.e. for functions x of two arguments such that $x(t_1,t_2)=x(t_2)-x(t_1)$. As we have seen in the previous section, the Chasles property is typical for integrals or for paths, or in other words for the first-order components of rough paths. On the other hand, $(s,t)\mapsto x_{(i_1,i_2)}(s,t)$ does not satisfy the Chasles property, but the

Chen property. Still, $\delta_3\circ\delta_2\equiv 0$ implies a property for the defect of additivity $(\delta_2 x)(t,u,s):=x(s,t)-x(s,u)-x(u,t)$, namely, $\sum_{i=1}^4(-1)^{i+1}(\delta_2 x)(t_1,\ldots,\check{t}_i,\ldots,t_4)=0$. An element h in Z_j :=Ker δ_j is a called a *cocycle* in cohomology theory, whereas a *coboundary* is an element of $B_j:=\mathrm{Im}\ \delta_{j-1}$. Because $\delta_j\circ\delta_{j-1}\equiv 0$, one has the obvious inclusion $B_j\subset Z_j$. This complex is *exact*, which means that $B_j=Z_j$ (simply set $\bar{h}(t_1,\ldots,t_{j-1}):=h(t_0,t_1,\ldots,t_{j-1})$ for any fixed reference point $t_0\in\mathbb{R}$: an elementary computation yields $\delta_{j-1}\bar{h}=h$). Note however that one may replace $\tilde{h}$ with $\tilde{h}+\delta_{j-2}f$ for any $f\in C^{j-2}(\mathbb{R},\mathbb{R}^d)$, hence $\tilde{h}$ is not unique. In the sequel we shall only be concerned with the case $j=3$, and look more specifically for pre-images g of h under δ_2 which belong to C_2^{1+}, where C_2^{1+} is the space of functions $g=g(t_1,t_2)$ such that $\sup_{t_1,t_2\in[-T,T]}\frac{|g(t_1,t_2)|}{|t_1-t_2|^{1+\varepsilon}}<\infty$ for some $\varepsilon>0$. Such pre-images do not necessarily exist, but in any case they are unique: namely, $\delta g_1=\delta g_2$, $g_1,g_2\in Z_2^{1+}$ implies that $(g_1-g_2)(t_1,t_2)=f(t_1)-f(t_2)$ for some $(1+\varepsilon)$-Hölder function; but then $f'\equiv 0$, hence f is a constant and $g_1-g_2=0$. The first non-trivial result is the following *lemme de la couturière*, where (in analogy with the case of functions of two variables) we define the space C_3^{1+} to be the subspace of functions $h\in C_3$ such that, for some $\varepsilon>0$, $||h||_{C_3^{1+\varepsilon}}<\infty$ [1].

Theorem 2.1 (lemme de la couturière or sewing lemma [19]) *Let $h\in Z_3\cap C_3^{1+}$. Then there exists exactly one function $g\in C_2^{1+}$ such that $\delta_2 g=h$. We write: $g:=\Lambda h$ and call Λ the* sewing map*.*

Proof. Unicity is automatic by the previous argument. We must still construct such a function g. Introduce the dyadic partition $[s,t[=\amalg_{i=0}^{2^n-1}[t_i^n,t_{i+1}^n)$, where $t_i^n:=s+\frac{i}{2^n}(t-s)$, and set $g^n(s,t):=\tilde{g}(s,t)-\sum_{i=0}^{2^n-1}\tilde{g}(t_i^n,t_{i+1}^n)$, where $\tilde{g}$ is any function in C^2 such that $\delta\tilde{g}=h$ (for instance $\bar{h}$, see above, will do). Then $g^{n+1}(s,t)-g^n(s,t)=\sum_{i=0}^{2^n-1}h(t_{2i}^{n+1},t_{2i+1}^{n+1},t_{2i+2}^{n+1})$ (computation) is $O(2^{-\varepsilon n})$ for some $\varepsilon>0$ by the hypothesis on h, so the series $\sum_n(g^{n+1}(s,t)-g^n(s,t))$ converges and there exists a limit $g(s,t):=\lim_{n\to\infty}g^n(s,t)$; by the same argument, $g\in C_2^{1+}$ and $\delta_2 g=h$. $\square$

The Young integral may easily be written in terms of the sewing operator Λ. Let x,y be α-Hölder paths with $\alpha>\frac{1}{2}$. Then the approximate integral (see previous section) $z(s,t):=y(s)(x(t)-x(s))$ satisfies the approximate Chasles property $(\delta_2 z)(t,u,s)=(z(t)-z(u))(x(u)-x(s))=O(|t-s|^{1+\varepsilon})$; in other words, $\delta_2 z\in C_3^{1+}$. Let $g:=(\Lambda\circ\delta_2)(z)$ as in the lemme de la couturière. Then $\delta_2 g=\delta_2 z$, but g (contrary to z which satisfies only an α-Hölder-type property) belongs to C_2^{1+}. We claim that the Young integral $\int_s^t y(u)dx(u)$ may be identified with $z(s,t)-g(s,t)=(\mathrm{Id}-\Lambda\circ\delta_2)(z)(s,t)$. Note to begin with the nice properties of the operator $\mathrm{Id}-\Lambda\circ\delta_2$: on the one hand, $\delta_2\circ(\mathrm{Id}-\Lambda\circ\delta_2)\equiv 0$ by definition, so $z-g$ satisfies the Chasles relation; on the other hand, $(\Lambda\circ\delta_2)(z)=g\in C_2^{1+}$, so that $|\sum_{i=0}^{n-1}g(s+\frac{i}{n}(t-s),s+\frac{i+1}{n}(t-s))|\lesssim n\cdot n^{-1-\varepsilon}\to_{n\to\infty}0$. Hence one gets $(z-g)(s,t)=\lim_{n\to\infty}y(s+\frac{i}{n}(t-s))(x(s+\frac{i+1}{n}(t-s))-x(s+\frac{i}{n}(t-s)))$, a Riemann sum known from previous arguments to converge to the Young integral when $n\to\infty$.

[1] The norm $||\cdot||_{C_3^\gamma}$ is defined in general as some homogeneous norm of degree γ such that $||f(s,u)g(u,t)||_{C_3^\gamma}<\infty$ if f, resp. g is α, resp. β-Hölder with $\alpha+\beta=\gamma$. For instance $||h||_{C_3^\gamma}=\inf\{\sum_i||h_i||_{\rho_i,\gamma-\rho_i};\ h=\sum_i h_i, 0<\rho_i<\gamma\}$, with $||h||_{\alpha,\beta}:=\sup_{s,u,t}\frac{|h(s,u,t)|}{|u-s|^\alpha|t-u|^\beta}$.

The previous argument extends with little changes to the case $\alpha \in (1/3, 1/2]$ as soon as one has lifted x to a rough path $\boldsymbol{x} := ((x_{i_1}(s,t)), (x_{(i_1,i_2)}(s,t))$. We must assume the function y to be a path *controled by* x: this means by definition that the following (Φ, R)-*decomposition* holds, namely,

$$y(t) - y(s) = \langle \Phi(s), (x(t) - x(s)) \rangle + R(s,t), \tag{2.1}$$

where Φ is some $d \times d$-matrix of α-Hölder functions and R – a kind of remainder term – satisfies a 2α-Hölder bound, $||R||_{C_2^{2\alpha}([0,T])} := \sup_{s,t\in[0,T]} \frac{|R(s,t)|}{|t-s|^{2\alpha}} < \infty$. In some sense $\Phi(s)$ is a discrete gradient of y, so the equivalent of the improved approximate integral of order 2 introduced in the previous section is $z(s,t) := \langle y(s), x(t) - x(s) \rangle + \sum_{i_1,i_2} \Phi_{i_1,i_2}(s) x_{(i_1,i_2)}(s,t) = \langle y(s), x(t) - x(s) \rangle + \langle \Phi(s), \boldsymbol{x}^2(s,t) \rangle$, where $\boldsymbol{x}^2(s,t) = (x_{(i_1,i_2)}(s,t))_{i_1,i_2=1,\ldots,d}$. One computes

$$\begin{aligned} \delta(\langle y(s), x(t) - x(s) \rangle (s,u,t) &= \langle x(u) - x(t), y(u) - y(s) \rangle \\ &= \langle \Phi(s), (x(t) - x(u)) \otimes (x(u) - x(s)) \rangle + \langle R(s,u), x(t) - x(u) \rangle, \end{aligned} \tag{2.2}$$

$$\delta(\langle \Phi(s), \boldsymbol{x}^2(s,t) \rangle)(s,u,t) = \langle \Phi(t) - \Phi(u), \boldsymbol{x}^2(s,u) \rangle + \langle \Phi(s), (x(t) - x(u)) \otimes (x(u) - x(s)) \rangle, \tag{2.3}$$

so that $\delta(\langle y(s), x(t) - x(s) \rangle + \langle \Phi(s), \boldsymbol{x}^2(s,t) \rangle) = \langle \Phi(u) - \Phi(t), \boldsymbol{x}^2(s,u) \rangle + \langle R(s,u), x(t) - x(u) \rangle$. This expression is by construction in C_3^{1+}, so that we may set

$$\begin{aligned} \int_s^t \langle y(u), dx(u) \rangle &:= \langle y(s), x(t) - x(s) \rangle + \langle \Phi(s), \boldsymbol{x}^2(s,t) \rangle \\ &\quad - \Lambda \left(\langle \Phi(t) - \Phi(u), \boldsymbol{x}^2(s,u) \rangle + \langle R(s,u), x(t) - x(u) \rangle \right), \end{aligned} \tag{2.4}$$

a formula equivalent to the guessed Riemann-sum type expression

$$\begin{aligned} \int_s^t \langle y(u), dx(u) \rangle &:= \lim_{n\to\infty} \sum_{i=0}^{n-1} \langle y(s + \frac{i}{n}), x(s + \frac{i+1}{n}) - x(s + \frac{i}{n}) \rangle \\ &\quad + \langle \Phi(s + \frac{i}{n}), \boldsymbol{x}^2(s + \frac{i}{n}, s + \frac{i+1}{n}) \rangle \end{aligned} \tag{2.5}$$

in which the remainder term R does not appear.

2.2 Controled paths and differential equations

If one wants to solve differential equations driven by the rough path $\boldsymbol{x} = (x, \boldsymbol{x}^2)$ by some fixed point theorem, one needs to check that the successive approximations y^n of the solution (see proof of Theorem 1.2) remain in the space $\mathcal{Q}(x)$ of paths controled by x (say, with some fixed initial condition at $t = 0$), and to introduce a norm $|| \cdot ||_{\mathcal{Q}(x)}$ on $\mathcal{Q}(x)$ such that the integration map on small enough time intervals $[0, T]$ is a contraction. A natural choice is $||y||_{\mathcal{Q}(x,[0,T])} := ||y||_{\alpha,[0,T]} + ||\Phi||_{\alpha,[0,T]} + ||R||_{C_2^{2\alpha}([0,T])}$. One must prove that $\mathcal{Q}(x)$ is stable under integration, $y \mapsto \int \langle y, dx \rangle$, and under functional transformation $y \mapsto V(y)$, where V is one of the (smooth by assumption) coefficients of the differential equation. Let us give two separate arguments. The exposition is borrowed from [49].

(i) (stability of the space of controled processes under integration)

The preceding arguments readily imply that the (Φ, R)-decomposition of $\int\langle y, dx\rangle$ is $\big(y; \langle\Phi(s), \boldsymbol{x}^2(s,t)\rangle - \Lambda\left(\langle\Phi(t)-\Phi(u), \boldsymbol{x}^2(s,u)\rangle + \langle R(s,u), x(t)-x(u)\rangle\right)\big)$; with some extra efforts one proves that the norm $||\int\langle y, dx\rangle||_{\mathcal{Q}(x))}$ is essentially 'controled' by $||y||_{\mathcal{Q}(x)}$.

(ii) (stability of the space of controled processes under functional transformations)

Let $\tilde{y} := V(y)$ with V smooth. Then

$$\tilde{y}(t) - \tilde{y}(s) = \langle\nabla V(y(s)), (y(t)-y(s)\rangle + [(V(y(t) - V(y(s)) - \langle\nabla V(y(s)), y(t)-y(s)\rangle]$$

decomposes as the couple

$$(\langle\nabla V(y(s)), \Phi(s)\rangle; \langle\nabla V(y(s)), R(s,t)\rangle + [(V(y(t)) - V(y(s)) - \langle\nabla V(y(s)), y(t)-y(s)\rangle]\,).$$

All terms in the decomposition, and also $\tilde{y}$, have Hölder norms 'controled' by $||y||_{\mathcal{Q}(x)}$, *except* the product term $\langle\nabla V(y(s)), y(t)-y(s)\rangle$ which is controled by the *square* of the norm $||y||_{\mathcal{Q}(x)}$.

All together, these bounds imply the following type of inequality,

$$||\int_0^t V(y(u))dx(u) - \int_0^t V(\hat{y}(u))dx(u)||_{\mathcal{Q}(x,[0,T])} \leq \\ c_1(T)||y-\hat{y}||_{\mathcal{Q}(x,[0,T])} + c_2(T)||y-\hat{y}||_{\mathcal{Q}(x,[0,T])}\ \max\left(||y||_{\mathcal{Q}(x,[0,T])}, ||\tilde{y}||_{\mathcal{Q}(x,[0,T])}\right) \tag{2.6}$$

for some coefficients $c_1(T), c_2(T) \to_{T\to 0} 0$. This makes it possible to apply a fixed point theorem on a small enough interval.

The case $\alpha \leq 1/3$ is sketched in [49]. The very definition of the space of controled processes becomes dull, and no new ideas come up, so we felt confident that we could skip this generalization.

The appearance of this squared norm (N-th norm in general), due to the use of Taylor expansions in functional transformations, makes it impossible to prove a *global existence theorem* for solutions of rough differential equations, say when the coefficients of the equation have bounded derivatives up to a sufficient order without being bounded, e.g. when $V(x)$ has linear growth at infinity. On the other hand, such diffusion equations – when the driving process is Brownian motion – are known to be globally well-defined; in some sense this is the first theorem in the study of stochastic differential equations, see e.g. [8]. In that case, the space of adapted processes, equipped with the norm $||y||_{[0,T]} := \left(\mathbb{E}\int_0^T |y|^2(s)ds\right)^{\frac{1}{2}}$, plays the rôle of the space of controled processes, and $\left(\mathbb{E}\left(\int_0^T V(y(s))dB_s\right)^2\right)^{\frac{1}{2}} = \left(\int_0^T V^2(y(s))ds\right)^{\frac{1}{2}}$ is controled by $||y||$ itself, which is crucial for global estimates of the solutions. As we shall see, similar linear controls are possible for a restricted class of rough paths called *Fourier normal ordered paths*, introduced by the author, which are the subject of the next sections.

Chapter 3

Fourier normal ordered rough paths: a preliminary overview

Despite the general, somewhat abstract existence and approximation theorems due to T. Lyons and N. Victoir, several important questions were pending starting from 2004, in particular after the 'no-go' theorems proved by Coutin-Qian and Unterberger concerning the 'natural' Lévy area of fBm with $\alpha \le 1/4$: (1) is there an explicit, constructive way of defining a rough path over fBm with $\alpha \le 1/4$? More generally, is it possible to produce explicit rough paths over any arbitrary Hölder path ? (2) If so, may these rough paths be obtained by an explicit approximation theorem ? (3) Can one produce convincing arguments that some rough path is "natural", i.e. obtained by a geometrically and probabilistically meaningful procedure ? To these three questions, a pure mathematician would quite naturally add a fourth one: (4) Can one give a general, explicit procedure yielding all rough paths over a given path ? In other words, can one classify *all* possible lifts of a given path ?

The point of view developed by the author in a series of papers [55, 54, 56, 53, 57], together with L. Foissy [15] and J. Magnen [37, 38], is that (i) the underlying sub-Riemannian geometry used to approach rough paths is too degenerate and too singular to give satisfactory answers to these questions. Somehow additional, non-geometric data should be provided in order to 'lift' this degeneracy and select some particular rough path; (ii) without the help of geometry, the algebraic axioms of rough paths (in particular, the shuffle property) are difficult to verify, thus calling for a new algebraic reformulation which would make it easier to construct explicit rough paths. The first problem is of analytic nature and amounts to the following: how to avoid the singularities that seem to appear naturally when one tries to define the area of an irregular path ? The second one is a priori of purely algebraic nature; it concerns the larger class of *formal rough paths* (see above), obtained by disregarding the Hölder regularity requirements.

Using a new combinatorial method that we called *Fourier normal ordering*, and leaning heavily on the philosophy and tools of quantum field theory, together with combinatorial Hopf-algebraic arguments, we solved questions (1) and (4), and contributed a first answer to (3). The 'definitive' answer (if this ever makes sense) to (3) is not yet settled though, although in progress. We have no answer to (2), at least for the moment. Let us summarize the results obtained so far and present some perspectives. The following points follow approximately the chronological development of the subject and serve also as a basis for the next sections.

3.1 The case of the Lévy area

People who had thought about the problem of definition of the Lévy area (for fBm with $\alpha \le 1/4$) knew it was easy to define a *formal Lévy area*. The argument is very elementary and totally general, and may be found in Gubinelli's work [19]. If $x(t) = (x_1(t), x_2(t))$ is some path, and $x_{(i_1,i_2)}(s,t)$ some formal lift of x satisfying the Chen and shuffle property, then $\tilde{x}_{(i_1,i_2)}(s,t) := x_{(i_1,i_2)}(s,t) + \varepsilon_{i_1,i_2}(f(t) - f(s))$, with ε being the antisymmetric tensor $\varepsilon_{1,2} = -\varepsilon_{2,1} = 1, \varepsilon_{i,i} = 0$, also satisfies these two properties for any function f. The Chen property remains true because (in Gubinelli's terminology) $\delta_2 \tilde{x}_{(i_1,i_2)} = \delta_2 x_{(i_1,i_2)}$ thanks to the fact that $f(t) - f(s) = \delta_1 f(s,t)$ is itself an increment; the shuffle property also remains true because of the alternating signs of the antisymmetric tensor. Now assume x is α-Hölder for some $\alpha \le 1/2$, and replace the a priori ill-defined second-order integral

$$x_{(1,2)}(s,t) = \int_s^t dx_1(t_1) \int_s^{t_1} dx_2(t_2) = -(x_1(t) - x_1(s))x_2(s) + \int_s^t dx_1(t_1)x_2(t_1) \qquad (3.1)$$

– sum of a boundary term and of an increment satisfying the Chasles relation – by $\tilde{x}_{(1,2)}(s,t) := x_{(1,2)}(s,t) - (f(t) - f(s))$, where $f(t) - f(s) = \int_s^t dx_1(t_1)x_2(t_1)$. Then $\tilde{x}_{(1,2)} = -(x_1(t) - x_1(s))x_2(s)$, a boundary term which does not contain iterated integrals any more, is well-defined and candidate for a Lévy area. Unfortunately, $\tilde{x}_{(1,2)}$ does not possess the required Hölder regularity (obviously, it is only α-Hölder).

The new thing is that this too naive idea can be made to work by introducing *Fourier normal ordering*. In a nutshell (compare with (3.1)), the substitution $x_{(1,2)} \to \tilde{x}_{(1,2)}$ is meaningful only when applied to the iterated integral of a Fourier normal ordered quantity of the type $\sum_{|j_1| \le |j_2|} a_{1,j_1} a_{2,j_2} e^{\mathrm{i} j_1 t_1} e^{\mathrm{i} j_2 t_2}$ (for a 2π-periodic path with components $x_i(t) = \sum_j a_{i,j} e^{\mathrm{i} j t}$), or more generally $\int_{|\xi_1| < |\xi_2|} d\xi_1 d\xi_2 \, a_1(\xi_1) a_2(\xi_2) e^{\mathrm{i}(\xi_1 t_1 + \xi_2 t_2)}$. Such quantities are called *Fourier normal ordered*. Applying to such expressions the decomposition (3.1) and discarding the increment term leaves out a boundary term which is 2α-Hölder (at least in the case of fBm), as required. Originally the proof of this rather elementary estimate (see () in Appendix) was borrowed from the book by J.-P. Kahane on random series of functions [23]. In general, Hölder estimates are easiest to obtain (using equivalent Besov norms) in terms of a smooth dyadic Fourier partition of the functions, $x_i(t) := \sum_{j_i \in \mathbb{Z}} x_i^{j_i}(t)$, where the support of the Fourier transform of x^j is contained, say, in $[-2^{j+2}, -2^{j-1}] \cup [2^{j-1}, 2^{j+2}]$; thus one must check the convergence of a certain double series ranging in $-\infty < j_1 \le j_2 < +\infty$. Now the other series with complementary index set $-\infty < j_2 \le j_1 < +\infty$ is again Fourier normal ordered if one applies first Fubini's formula; the second boundary term is thus very similar to the first one and bounded in the same way. Summing the two boundary terms yields quantities satisfying the Chen and shuffle properties by the above general argument, but this time the Hölder regularity properties are satisfied.

3.2 The general algebraic construction for formal rough paths

Shifting from the Lévy area to the general case requires non-trivial combinatorial arguments which are explained in section 5 below (see section 4 for a presentation based on more elementary examples). The complications come from the fact that expressions of order n of the type $\int_{t>t_1>\ldots>t_n>s} dx_{i_1}^{j_1}(t_1)\ldots dx_{i_n}^{j_n}(t_n)$, except in the already Fourier normal-ordered case $j_1<\ldots<j_n$, must be rewritten by means of Fubini's theorem, permuting the order of integration *so that the Fourier frequencies should appear in increasing order, starting from the leftmost (outer) integral, till the rightmost (inner) integral.* But then the integration domain is not a simplex any more, but some a priori complicated union of simplices. Fortunately the resulting integrals may be encoded by trees, as already appears in a previous work by D. Kreimer [26] inspired by expansion formulas in quantum field theory. A. Connes and D. Kreimer issued a series of papers in the years 1999-2000 [9, 10] about the Hopf algebraic structure of trees, motivated in particular by a deeper mathematical understanding of the famous Bogolioubov forest formula allowing a coherent subtraction of singularities in the Feynman diagrams obtained by formally expanding the interaction. This Hopf algebra, soon after called Connes-Kreimer Hopf algebra, is by now pervasive in algebraic combinatorics. The Chen and shuffle properties appear naturally in this context, and the Hopf algebraic framework provides both synthetic formulas for our general construction, and the necessary tools for the proofs. Let us simply describe informally the tree encoding for iterated integrals for the moment. For a trunk tree with only one branch, the definition coincides with the usual one. Trees are conventionally represented and thought with the root *downwards.*

Definition 3.1 *Let $\mathbb{T}$ be a tree. Index its vertices as $1,\ldots,n$, so that the indices increase when one follows up a path from the root, 1, to a leaf. Attach a component index $\ell(i)=1,\ldots,d$ to each vertex, and choose some reference time t_0. Denoting by i^- the ancestor of the vertex i in $\mathbb{T}$, the corresponding tree iterated integral of $x=(x_1,\ldots,x_d)$ is $\int_{t_0}^{t} dx_{t_1}(\ell(1)) \int_{t_0}^{t_{2^-}} dx_{t_2}(\ell(2))\ldots\int_{t_0}^{t_{n^-}} dx_{t_n}(\ell(n))$.*

Later on, in section 5, this iterated integral will be denoted by $I_x^{t,t_0}(\mathbb{T})$.

The main result is roughly as follows (precise statements will be given later on, see ???, since they require a little bit of tree combinatorics even to be stated).

Classification theorem. *Associate to every tree $\mathbb{T}$ with $1\le n\le N$ vertices and time t an* integration procedure *$\phi^t_{\cdot}(\mathbb{T})$, i.e. a 'natural' linear rule $dx_{i_1}^{j_1}\otimes\ldots\otimes dx_{i_n}^{j_n}\mapsto\phi^t_{dx_{i_1}^{j_1}\otimes\ldots\otimes dx_{i_n}^{j_n}}(\mathbb{T})$ associating to a Fourier normal ordered (signed) measure a* tree data*, and such that $\phi^t_{dx_i}(\bullet)=x_i(t)$ for the trivial tree $\bullet$ with one vertex. Then a canonical and explicit reconstruction algorithm builds a formal rough path over a path x out of the tree data associated to the path.*

All terms are defined in section 5. In a nutshell, an integration procedure should simply satisfy the obvious product rule which states that the integral of a tensor measure over a product domain should be equal to the product of the integrals over each projection. The reconstruction algorithm holds in particular when x is smooth and $\phi^t_{dx_{i_1}^{j_1}\otimes\ldots\otimes dx_{i_n}^{j_n}}(\mathbb{T})$ are obtained by canonical integration over the domain encoded by $\mathbb{T}$. Actually, the requirement that one should reconstruct thus the canonical lift of x for any smooth path x fixes the reconstruction algorithm uniquely and yields straightforward formulas for it. The difficulty is to prove that the same algorithm yields a formal rough path for other, arbitrary integration procedures. The original proof in [55] is half-algebraic, half-analytic (it uses an argument of 'density' of the space of smooth paths in the space of all continuous paths). A second proof given in [15] relies entirely on Hopf-algebraic structures; the classification theorem comes out as a consequence of an explicit isomorphism between two combinatorial Hopf algebras.

Note that the condition $\phi^t_{dx_i}(\bullet) = x_i(t)$ ensures that the rough path thus obtained is a lift of the original $\mathbb{R}^d$-valued path x.

3.3 Elementary integration procedures

The previous construction is purely algebraic: it does not guarantee that the obtained formal rough paths satisfy the Hölder regularity requirements. The following (apparently somewhat ad-hoc) integration procedure does build real rough paths over any α-Hölder path. In the next theorem, one writes $j \twoheadrightarrow i$ if i, j are vertices of a tree $\mathbb{T}$ and one encounters i on the way down from j to the root.

Theorem 3.1 (integration procedure by domain regularization) *(see [55] for the general case, and [54] for the specific case of fBm)*

Let $x = (x_1, \ldots, x_d)$ be an α-Hölder path. Choose some constant $C_{reg} > 0$. Define $\phi^t_{dx^{j_1}_{i_1}\otimes\ldots\otimes dx^{j_n}_{i_n}}(\mathbb{T})$, $n \geq 2$ to be the canonical tree iterated integral as in Definition 3.1 if, for every vertex i, $|\xi_i + \sum_{j\twoheadrightarrow i} \xi_j| > C_{reg} \sup_{j\twoheadrightarrow i} |\xi_j|$ whenever $\xi_k \in [2^{j_k-1}, 2^{j_k+2}]$, $k = 1, 2, \ldots, n$.. Otherwise let $\phi^t_{dx^{j_1}_{i_1}\otimes\ldots\otimes dx^{j_n}_{i_n}}(\mathbb{T}) = 0$.

Then the integration procedure, followed by the reconstruction algorithm, produces a rough path over x.

In the case of fBm the dyadic Fourier partition is not necessary, and the sum over $j_1 \leq \ldots \leq j_n$ is replaced advantageously by a multiple integral over $|\xi_1| < \ldots < |\xi_n|$. Then the natural condition is simply: $|\xi_i + \sum_{j\twoheadrightarrow i} \xi_j| > C_{reg} \sup_{j\twoheadrightarrow i} |\xi_j|$.

By the triangular inequality, the above condition is never realized if $C_{reg} > N$. This case is perfectly allowable; it amounts to setting to zero *all non-trivial tree data*, i.e. with $n \neq 1$. Then huge simplifications occur, all branching tree integrations cancelling out of the sum. One gets the following formula:

Theorem 3.2 (zero tree data) *(see [?] and [15]) Let x be an α-Hölder with regularly varying*

Fourier transform. Then the rough path above x constructed from zero tree data writes

$$x_{(\ell(1),\ldots,\ell(n))}(s,t)=\sum_{k=0}^{n}(-1)^{n-k}\int\cdots\int_{|\xi_1|>\ldots>|\xi_k|,|\xi_k|<\ldots|\xi_n|} \left[\prod_{j=1}^{k-1} e^{\mathrm{i}t\xi_j}\mathcal{F}x_{\ell(j)}(\xi_j)d\xi_j\right] (e^{\mathrm{i}t\xi_k}-e^{\mathrm{i}s\xi_k})\mathcal{F}x_{\ell(k)}(\xi_k)d\xi_k\left[\prod_{j=k+1}^{n} e^{\mathrm{i}s\xi_j}\mathcal{F}x_{\ell(j)}(\xi_j)d\xi_j\right]. \tag{3.2}$$

If ones removes the assumption on the Fourier transform of x, then a similar formula is valid, up to the necessity of introducing a smooth dyadic Fourier partitioning $x=\sum_{j\in\mathbb{Z}}x^j$. The exact formula now involves infinite series over scale indices j.

Formula (3.2) is in particular valid for fractional Brownian motion, by replacing $\mathcal{F}\Gamma_{\ell(j)}(\xi_j)d\xi_j$ with $c_\alpha\frac{|\xi_j|^{1/2-\alpha}}{i\xi_j}dW_{\xi_j}(\ell(j))$ for some constant c_α (see [?] for the Fourier transform of fBm and [?] for this formula). A very similar formula (relying the Volterra kernel representation of fBm instead of its Fourier transform) has been obtained by D. Nualart and S. Tindel in [45] and proved without appealing to the combinatorial apparatus of Fourier normal ordering:

Theorem 3.3 (zero tree data – Volterra kernel setting) *(see Nualart and Tindel [45])*

The technical condition $|\xi_i+\sum_{j\twoheadrightarrow i}\xi_j|>C_{reg}\sup_{j\twoheadrightarrow i}|\xi_j|$ (which prevents small denominators in the Fourier formula for tree iterated integrals) avoids taking care of the singularities that cause e.g. the Lévy area for fBm with $\alpha\leq 1/4$ to diverge. The obvious similarity with the much more comprehensive treatment of singularities of integrals arising in quantum field theory prompted the following construction.

3.4 Quantum-field theoretic constructions: renormalized rough paths

A fundamental idea in theoretical physics is that the description of phenomena should change with the scale at which they are observed. This is well-known since the early days of the kinetic theory of gases. Looking at a fluid (say, at rest) at a macroscopic level, one distinguishes essentially its pressure, its volume and its temperature; these are related by equations of state such as the famous perfect gas or Van der Waals equation. At a microscopic level on the other hand, one sees a virtually infinite number of constantly colliding particles. Proving the macroscopic equations from the microscropic ones is a very difficult task. Usually one starts from a simplified intermediate, continuous, random level of description (sometimes called mesocopic level), either kinetic equations for the probability density such as the Boltzmann equation, or interacting particle systems. In both cases translation-invariant stationary states usually include Boltzmann-Gibbs type equilibrium measures. If one assumes that the systems first thermalizes locally before it becomes homogeneous, then it may be correctly approximated by Ginzburg-Landau equations of the type $d\phi(x,t) = -\frac{\partial \mathcal{H}}{\partial \phi}(x,t)dt + \sqrt{T}dW(x,t)$, where $H(\phi) := \int \left\{ |\nabla\phi|^2(x) + \lambda V(\phi)(x) \right\} dx$ is a Ginzburg-Landau functional, λ is a small parameter in front of the interaction term $\lambda V(\phi)$ perturbing the free (quadratic) Hamiltonian $H_0(\phi) := \int |\nabla\phi|^2(x)dx$, and $dW(x,t)$ is white-noise. Then the system is allowed to relax to non-homogeneous equilibrium thermal states of the form $d\mu(\phi) = \frac{1}{Z}e^{-\mathcal{H}(\phi)}$, where Z is some constant. In a mean-field limit the space-dependence disappears and one obtains a macroscopic equation of state. Such mean-field limit (also called Lebowitz-Penrose limit after their work [28]) may be rendered rigorous by a block-spinning procedure. Assuming the range of the interaction is rescaled by a factor $1/\gamma \to \infty$, they rewrite an effective Hamiltonian on a lattice with next-neighbour distances rescaled by a large factor $\gamma^{-\alpha}, 0 < \alpha < 1$ allowing the use of a local central limit theorem.

A similar philosophy is at work in high-energy phyics too. There the continuous description by a Ginzburg-Landau type functional (un to a Wick rotation, trading real exponentials for imaginary ones) is believed to be exact, and the one-step, discrete block-spinning procedure just described is conveniently replaced by a continuous, inductive procedure involving a dyadic Fourier partition of the fields. The transformation of the parameters of the effective Hamiltonian under the successive changes of scales is known since the major contributions of K. Wilson in the 70es as *renormalization group.* The amount by which the parameters are renormalized may be computed from a controlled expansion of the exponential $e^{-\lambda\int V(\phi)(x)dx} = 1 - \lambda \int V(\phi)(x)dx + \frac{\lambda^2}{2!}\left(\int V(\phi)(x)dx\right)^2 + \ldots$ The fields are usually singular in the high-frequency limit (i.e. they are distribution-valued), which implies that the computations involving the powers of $V(\phi)$ yield a profusion of singular integrals. However, in good cases, the singularities compensate, and the net compound effect of the flow of the renormalization group from highest scale (i.e. highest Fourier frequency) to lowest scale (i.e. lowest Fourier frequency) is to *regularize the integrals.*

While all this may seem all too obscure to non-specialists, we felt it necessary to explain that the regularization of Feynman integrals (which is the main object of this paragraph) is not originally just about finding some appropriate algorithm to cancel out divergences: the consistency of the subtractions makes the theory at lowest scale just a partial resummation of the original theory written at highest scale.

Let us now turn back to the discussion on rough paths where we left it at the end of the previous subsection. The above introduction to the philosophy of renormalization has hopefully convinced the reader of the interest not to select any arbitrary integration procedure, but to *renormalize* the initially diverging tree iterated integrals. It turns out that these may be rewritten as Feynman diagrams of a special type, to which the ordinary renormalization procedure called BPHZ (Bogolioubov-Parasiuk-Hepp-Zimmermann) prescription [22] applies. This is a recursive method to discard nested divergences, depending on the choice of a regularization scheme for diagrams without sub-divergences, which we choose here to be evaluation at zero momentum. Once again, without saying more, it is difficult to write down a theorem. Let us simply state:

Renormalization theorem. *Consider* $x \overset{d}{=} B$ *(fBm) with* $\alpha \le 1/4$*, and* $\alpha' < \alpha$*. Replace the canonical, diverging integration procedure* $\phi^t_{\cdot}(\mathbb{T})$ *by the renormalized integration procedure* $\mathcal{R}\phi^t_{\cdot}(\mathbb{T})$*. Then the Fourier normal ordered integrals* $\sum_{k_1 \le \ldots \le k_n} \mathcal{R}\phi^t_{dx_1^{k_1} \otimes \ldots \otimes dx_n^{k_n}}(\mathbb{T})$ *are* $n\alpha'$*-Hölder, and the reconstruction algorithm yields a rough path over* x*.*

`general construction of (non-formal) rough paths`

There is, however, a major drawback in the Renormalization theorem. Renormalization is applied in our case to iterated integrals of a Gaussian process, not to deterministic integrals arising from the perturbative expansion of a Hamiltonian. The subtraction of divergences is not generated by a partial resummation of the series of perturbations since the initial theory is free ($V = 0$), i.e. Gaussian, and there is nothing to expand. Hence the procedure (although it captures extremely well the precise nature of the singularities) is again somewhat arbitrary. To respect the philosophy of the renormalization group, one should instead start from an interacting theory ($V \neq 0$). The idea seems puzzling at first: the theory (fBm) is a priori fixed; by changing the theory, the best one can do, so it seems, is to define a rough path over a different process. But the argument is wrong: thinking of A. Lejay's area bubbles, one understand perfectly that some limit of interacting processes – where small-scale details of the process are appropriately tuned – may be free and possess a Lévy area which is different from what one would have expected. This is exactly the spirit of the following ultra-violet (i.e. small-scale) limits. We shall give two rough statements. The law of fBm is denoted by $d\mu(\phi)$, where $\phi(t) - \phi(s) \overset{d}{=} B(t) - B(s)$ (ϕ coincides in law with fBm apart from its initial value $\phi(0)$). Cutting dyadic Fourier components of scale $j > \rho$ for some integer ρ yields a smooth approximation $\phi^{\to\rho}$ of ϕ, and an ultra-violet cut-off measure $d\mu^{\to\rho}(\phi)$.

First field-theoretic construction. *Assume* $\alpha \in (\frac{1}{6}, \frac{1}{4})$*. Consider for* $\lambda > 0$ *small enough the family of probability measures (also called:* $(\phi, \partial\phi, \sigma)$-model*)*

$$\mathbb{P}_{\lambda,V,\rho}(\phi_1, \phi_2) = e^{-\frac{1}{2}c'_\alpha \int\int_{V\times V} dt_1 dt_2 |t_1 - t_2|^{-4\alpha} \mathcal{L}_{int}(\phi_1^{\to\rho}, \phi_2^{\to\rho})(t_1,t_2) - \int \mathcal{L}^{\to\rho}_{bdry}} d\mu^{\to\rho}(\phi_1) d\mu^{\to\rho}(\phi_2), \quad (3.3)$$

where $\mathcal{L}_{int}(\phi_1, \phi_2)$ *is an interaction term which is quadratic in the area* $\mathcal{A}$ *generating by* ϕ*,*

$$\mathcal{L}_{int}(\phi_1, \phi_2)(t_1, t_2) = \lambda^2 (\partial\mathcal{A})(t_1)(\partial\mathcal{A})(t_2), \quad (3.4)$$

and $\mathcal{L}^{\to\rho}_{bdry}$ *is some singular 'Fourier boundary term' multiplied by an exponentially evanescent factor in the limit* $\rho \to \infty$*, which cures unwanted difficulties due to the ultra-violet cut-off. Then* $(\mathbb{P}_{\lambda,V,\rho})_{V,\rho}$ *converges in law when* $|V|, \rho \to \infty$ *to some measure* $\mathbb{P}_\lambda$*, and the associated iterated integrals*

$$\int_s^t d\phi_{i_1}^{\to\rho}(t_1) \int_s^{t_1} d\phi_{i_2}^{\to\rho}(t_2), \ldots, \int_s^t d\phi_{i_1}^{\to\rho}(t_1) \int_s^{t_1} d\phi_{i_2}^{\to\rho}(t_2) \ldots \int_s^{t_{n-1}} d\phi_{i_n}^{\to\rho}(t_n), \ldots$$

converge in law to a rough path over B*.*

Second field-theoretic construction. ???

Chapter 4

The Hopf algebra setting

This section is meant as a motivating introduction to the structure of tree algebras and to renormalization, with two elementary examples: the first one is about Runge-Kutta methods for solving differential equations; the second one is about D. Kreimer's toy model [25] for renormalization. We shall not work out in details these two models: we shall only try to convey the main ideas, and take the opportunity to introduce some prerequisites on the Connes-Kreimer Hopf algebra that will be required in the next sections. The discussion on Runge-Kutta methods is based on an article by C. Brouder [4]; it may be that the general combinatorial numerical methods developed in this article, combined with Fourier normal ordering and suitable renormalization methods, will lead eventually to new developments on rough stochastic *partial* differential equations. As for D. Kreimer's toy model, it has the advantage for our purposes to discuss iterated integrals (of a different type than ours) and to be more directly accessible than the Feynman diagram renormalization.

4.1 Runge-Kutta methods and renormalization (after C. Brouder)

Assume one wants to solve an *ordinary* differential equation $dy = V(y(t))dt$, where $y : \mathbb{R} \to \mathbb{R}^e$ and $V = (V^1, \ldots, V^e) : \mathbb{R}^e \to \mathbb{R}^e$ is a vector field on $\mathbb{R}^e$. The difference with eq. (1.1) is that the driving process x is now trivial. The emphasis here is not on the existence of the solution (ensured by the Cauchy-Lipschitz theorem), but on numerical schemes. Very general numerical methods, known as *Runge-Kutta numerical methods*, are defined as follows. Fix the step value of the scheme to be h, $h \to 0$, and assume by induction that the $(n-1)$-th step value of the solution y_{n-1} has been computed. Then, for a scheme with m sub-steps, y_n is defined as the solution of the following system of nonlinear equations,

$$Y_i = y_{n-1} + h \sum_{j=1}^{m} a_{i,j} V(Y_j) \quad (i = 1, \ldots, m), \qquad y_n = y_{n-1} + h \sum_{j=1}^{m} b_j V(Y_j). \tag{4.1}$$

The matrix $(a_{i,j})_{i,j=1,\ldots,m}$ and the vector $(b_j)_{j=1,\ldots,m}$ are called the Runge-Kutta data associated to this scheme. For instance the Euler scheme $Y_1 \leftarrow y_{n-1}, y_n \leftarrow y_{n-1} + V(Y_1)$ corresponds to

$a_{1,1} = 0, b_1 = 1$. If the matrix $a = (a_{i,j})_{i,j=1,\ldots,m}$ is lower triangular, then the above equations can be solved explicitly and the method is explicit (otherwise it is implicit).

In a famous paper appeared in the early 60es, J. C. Butcher showed how to express the Y_i and y_n in terms of the Taylor coefficients of the vector field V at x_{n-1}. Of course the solution is not linear in V, so there appear differential polynomials of the form $\delta(V) := P(V^i, \partial_{j_1} V^i, \partial^2_{j_1,j_2} V^i, \ldots)$. Without a graphical representation the result is an untractable infinite series. Butcher indexed these polynomials by rooted trees [1]. As in the previous section, we choose to draw the trees with their root downward. If $\mathbb{T}_1, \ldots, \mathbb{T}_i$ are trees, $\mathbb{T} := [\mathbb{T}_1, \ldots, \mathbb{T}_k]$ (the merge of $\mathbb{T}_1, \ldots, \mathbb{T}_k$) is by definition the tree obtained by joining the roots of $\mathbb{T}_1, \ldots, \mathbb{T}_k$ to a supplementary vertex which becomes the root of $\mathbb{T}$. The set of rooted trees is generated from the trivial tree $\bullet$ with one vertex by repeated mergings. Then the vector of differential polynomials $\delta_{\mathbb{T}}(V) = (\delta^i_{\mathbb{T}}(V))_{i=1,\ldots,e}$ is defined inductively by

$$\delta^i_{\bullet}(V) = V^i; \qquad \delta^i_{[\mathbb{T}_1,\ldots,\mathbb{T}_k]}(V) := \partial^k_{j_1,\ldots,j_k} V^i \ \cdot \ \delta^{j_1}_{\mathbb{T}_1} \cdots \delta^{j_k}_{\mathbb{T}_k}. \tag{4.2}$$

For instance the differential polynomial encoded by , resp. is $\sum_{j,k} \partial^2_{j,k} V^i \ \cdot \ V^j V^k$, resp. $\sum_{j,k} \partial_j V^i \ \cdot \ \partial_k V^j \ \cdot \ V^k$.

We still need to define some combinatorial coefficients before we give formulas. Define $\phi(\mathbb{T}) \in \mathbb{R}^e$ inductively as

$$\phi^i(\bullet) = 1; \qquad \phi^i([\mathbb{T}_1, \ldots, \mathbb{T}_k]) = \sum_{j_1,\ldots,j_k} a_{i,j_1} \cdots a_{i,j_k} \phi^{j_1}(\mathbb{T}_1) \cdots \phi^{j_k}(\mathbb{T}_k); \qquad \phi(\mathbb{T}) = \sum_{i=1}^m b_i \phi^i(\mathbb{T}). \tag{4.3}$$

Finally $\sigma(\mathbb{T})$ is the symmetry factor of $\mathbb{T}$, namely the number of automorphisms of the tree. One can prove that $\alpha(\mathbb{T}) := \frac{|V(\mathbb{T})|!}{\mathbb{T}!\sigma(\mathbb{T})}$, where: $V(\mathbb{T})$ (resp. $|V(\mathbb{T})|$) is the set (resp. the number) of vertices of $\mathbb{T}$, $\mathbb{T}!$ is defined recursively by

$$\bullet ! = 1; \qquad [\mathbb{T}_1, \ldots, \mathbb{T}_i] = |V([\mathbb{T}_1, \ldots, \mathbb{T}_k])| \mathbb{T}_1! \cdots \mathbb{T}_k! \tag{4.4}$$

and $\alpha(\mathbb{T})$ is the number of inequivalent *heap orderings* of $\mathbb{T}$, i.e. the number of ways of defining a one-to-one labeling of the n vertices of $\mathbb{T}$ by $1, 2, \ldots, n$ such that labels decrease along the path going from any vertex to the root (we shall meet this notion once again in Chapter ???.

We are now ready for Butcher's formula:

Proposition 4.1 (J. C. Butcher [5]) *The solution to the Runge-Kutta equations (4.1) for $Y_i, i = 1, \ldots, m$ and y_n is*

$$Y_i = y_{n-1} + \sum_{\mathbb{T}} \frac{h^{|V(\mathbb{T})|}}{\sigma(\mathbb{T})} \sum_{j=1}^m a_{i,j} \phi^j(t) \delta_{\mathbb{T}}(V)(y_{n-1}); \qquad y_n = y_{n-1} + \sum_{\mathbb{T}} \frac{h^{|V(\mathbb{T})|}}{\sigma(\mathbb{T})} \phi(\mathbb{T}) \delta_{\mathbb{T}}(V)(y_{n-1}). \tag{4.5}$$

[1] A tree is a connected non-embedded graph without cycles. Choosing one vertex yields a rooted tree. As opposed to e.g. planar trees, the descendents of a vertex are not ordered; hence a rooted tree is exactly specified by the number of descendents of each vertex.

The last sum $y_{n-1} + \sum_{\mathbb{T}} \frac{h^{|V(\mathbb{T})|}}{\sigma(\mathbb{T})}\phi(\mathbb{T})\delta_{\mathbb{T}}(V)(y_{n-1}) =: B(\phi, V, h)(y_{n-1})$ *is called a* B-series *in honour of Butcher.*

For the solution of the differential equation, one proves easily that $\phi(\mathbb{T}) = \frac{1}{\mathbb{T}!}$. More precisely, the Runge-Kutta method is of order n, i.e. gives an nth order approximation of the actual flow if and only if $\phi(\mathbb{T}) = \frac{1}{\mathbb{T}!}$ for every tree $\mathbb{T}$ with $|V(\mathbb{T})| \leq n$.

The tree encoding is natural in the sense that natural operations on flows of vector fields and on Runge-Kutta schemes translate into natural operations on trees. Let us give two examples:

Example 1 (composition of Runge-Kutta schemes). Consider the following natural composition of numerical schemes,

$$\tilde{Y}_i = y_{n-1} + h\sum_{j=1}^{m} a_{i,j}V(Y_j) \quad (i = 1, \dots, m), \tilde{y}_n = y_{n-1} + h\sum_{j=1}^{m} b_j V(\tilde{Y}_j); \tag{4.6}$$

$$Y_i = \tilde{y}_n + h\sum_{j=1}^{m'} a'_{i,j}V(Y_j) \quad (i = 1, \dots, m'), y_n = \tilde{y}_n + h\sum_{j=1}^{m'} b'_j V(Y_j). \tag{4.7}$$

The Runge-Kutta data are the matrix $\begin{pmatrix} (a_{ij})_{i,j} & 0 \\ 0 & (a'_{ij})_{i,j} \end{pmatrix}$ and the vector (b, b'). In terms of B-series, the solutions of the Runge-Kutta equations is the composition $B(\phi, h, V) \circ B(\phi', h, V)$.

Now the interesting point is that $B(\phi, h, V) \circ B(\phi', h, V) = B(\phi'', h, V)$, where $\phi'' := \phi * \phi'$ is the *convolution* of ϕ, ϕ' for the convolution product coming from the underlying Connes-Kreimer Hopf algebra structure of trees. For this we need extra definitions.

Definition 4.2 (Hopf algebra of rooted trees) *(i) A forest is a formal commutative product of trees (including the empty forest). Graphically it is represented as a disjoint union of trees. The vector space of all formal linear combinations of forests is denoted by* $\mathbf{H}$.

(ii) If w *is a descendant of* v *(i.e.* w *is above* v*) then one writes* $w \twoheadrightarrow v$. *One says that* v *is connected to* w *(a symmetric relation) if either* $w = v$, $w \twoheadrightarrow v$ *or* $v \twoheadrightarrow w$. *A subset of vertices* $\boldsymbol{v} \subset V(\mathbb{T})$ *is an* admissible cut *if* $(v, w \in \boldsymbol{v}, v \neq w) \Rightarrow (v$ *is not connected to* $w)$. *If* $\boldsymbol{v}$ *is admissible, which we write* $\boldsymbol{v} \models V(\mathbb{T})$, *then* $Roo_{\boldsymbol{v}}\mathbb{T}$ *is the subforest with vertices* $\{w \in V(\mathbb{T}); \exists v \in \boldsymbol{v}, v \twoheadrightarrow w\}$, *while* $Lea_{\boldsymbol{v}}\mathbb{T}$ *is the subforest with the complementary set of vertices. Note that* $Roo_{\boldsymbol{v}}\mathbb{T}$ *is a tree if* $\mathbb{T}$ *is a tree.*

(iii) The Hopf algebra of rooted trees is the vector space $\mathbf{H}$ *equipped with the following product,*

$$(\mathbb{T}_1\mathbb{T}_2 \cdots \mathbb{T}_k)\left(\mathbb{T}'_1\mathbb{T}'_2 \cdots \mathbb{T}'_{k'}\right) = \mathbb{T}_1\mathbb{T}_2 \cdots \mathbb{T}_k\mathbb{T}'_1\mathbb{T}'_2 \cdots \mathbb{T}'_{k'} \tag{4.8}$$

and the coproduct Δ,

$$\Delta(\mathbb{T}) = \sum_{v \models V(\mathbb{T})} Roo_{\boldsymbol{v}}\mathbb{T} \otimes Lea_{\boldsymbol{v}}\mathbb{T}. \tag{4.9}$$

For instance,

$$\Delta(\vee) = \vee \otimes 1 + 1 \otimes \vee + \mathbf{:} \otimes \bullet + \mathbf{:} \otimes \bullet + \bullet \otimes \bullet\bullet \tag{4.10}$$

(iv) **H** *has an antipode* $\bar{S}$*, defined inductively by*

$$\bar{S}(1) = 1, \quad \bar{S}(\mathbb{T}) = -\mathbb{T} - \sum_{\boldsymbol{v} \models V(\mathbb{T}), \boldsymbol{v} \neq \emptyset} Roo_{\boldsymbol{v}}\mathbb{T} \cdot \bar{S}(Lea_{\boldsymbol{v}}\mathbb{T}). \tag{4.11}$$

The reader who has never met Hopf algebras may read the first chapters of the book by Kassel ??? or? for a systematic presentation and for fundamental examples including tensor algebras, enveloping algebras, the Hopf algebra of a finite group, quantum groups ... In any case, a coproduct structure on a vector space $\mathbf{H}$, $\Delta : \mathbf{H} \to \mathbf{H} \otimes \mathbf{H}$ is defined by dualizing the axioms for a product; using the so-called Sweedler's notation, $\Delta(\mathbb{T}) = \sum \mathbb{T}' \otimes \mathbb{T}''$, *co*-associativity states that $\sum \mathbb{T}' \otimes \Delta(\mathbb{T}'') = \sum \Delta(\mathbb{T}') \otimes \mathbb{T}''$ in $\mathbf{H}^{\otimes 3}$. For a Hopf algebra, Δ should be an algebra morphism. Finally, there should exist a *counit* $\varepsilon : \mathbf{H} \to \mathbb{R}$ and an *antipode* $S : \mathbf{H} \to \mathbf{H}$, both algebra morphisms, such that, if $\Delta(\mathbb{T}) = \sum \mathbb{T}' \otimes \mathbb{T}''$, then $\sum S(\mathbb{T}') \otimes \mathbb{T}'' = \sum \mathbb{T}' \otimes S(\mathbb{T}'') = \varepsilon(\mathbb{T})$. In the case of $\mathbf{H}$, $\varepsilon : \mathbf{H} \to \mathbb{R}$ gives the coefficient of the empty forest.

One trivially makes a set of coefficients $(\phi(\mathbb{T}))_{\mathbb{T}}$ into an algebra morphism $\phi : \mathbf{H} \to \mathbb{R}$ by letting ϕ be linear and satisfy $\phi(\mathbb{T}_1 \cdot \mathbb{T}_2) = \phi(\mathbb{T}_1)\phi(\mathbb{T}_2)$. $\mathbb{R}$-valued algebra morphisms are usually called *characters*. Then, for two characters ϕ_1, ϕ_2, the convolution product $\phi_1 * \phi_2$ is the character defined as

$$\phi_1 * \phi_2(\mathbb{T}) := \sum \phi_1(\mathbb{T}')\phi_2(\mathbb{T}''). \tag{4.12}$$

Somehow the antipode plays the rôle of an inverse. This is apparent when one looks at the convolution group generated by non-zero characters since ϕ^{-1} (the inverse of ϕ for the convolution product) writes $\phi^{-1}(\mathbb{T}) = \phi(S(\mathbb{T}))$.

Example 2 (time-reversal).

Let time flow backwards and inverse the Runge-Kutta formulas. Then one obtains the new scheme $\phi^{-1}(\mathbb{T}) = \phi(S(\mathbb{T}))$.

4.2 D. Kreimer 's toy model

In this subsection one is interested in a special type of (canonical) tree iterated integrals $\tilde{I}_{\mathbb{T}}(x)$ of the smooth path $x = (x^k)_{k=1,\dots}$ defined by its Fourier transform, $\mathcal{F}\left(\frac{dx^k}{dt}\right)(\xi) = \mathbf{1}_{\xi>0}\xi^{-j_k\varepsilon}$ for some real parameters $j_1, j_2, \dots$, in the singular limit $\varepsilon \to 0^+$. Anticipating on section 5, these are actually – up to a time reversal – *tree skeleton integrals*, $\tilde{I}_{\mathbb{T}}(x) = \overline{\mathrm{Sk\,I}}_x^0(\mathbb{T})$. Loosely speaking, conjugating by Fourier transform takes integration into multiplication by $\frac{1}{\mathrm{i}\xi}$. For iterated integrals, the variable ξ_i should be equal to the sum of all Fourier variables associated to vertices lying on the downward path from i to the root. Exact formulas are to be read in subsection 5.1 but do not matter too much at this point.

We let the integrals depend on an extra parameter c by translating the Fourier transform of the path $\hat{x}(\cdot) \rightsquigarrow \hat{x}(\cdot - c)$. In D. Kreimer's work, these integrals are meant to be simplified Feynman integrals, and the tree structure encodes the nesting of the subdivergences, while c

plays the rôle of an external momentum for diagrams with only one external leg. Again this is not essential for our purposes.

For trees with two and three vertices, one obtains for instance:

$$\tilde{I}_{\substack{2\\1}}(x)(c) = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \frac{\xi_1^{-j_1\varepsilon}}{\xi_1+c}\,\frac{\xi_2^{-j_2\varepsilon}}{(\xi_1+\xi_2)+c}$$

$$\tilde{I}_{{}_{2}\vee^{3}_{1}}(x)(c) = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \int_0^\infty d\xi_3 \frac{\xi_1^{-j_1\varepsilon}}{t_1+c}\,\frac{\xi_2^{-j_2\varepsilon}}{(\xi_1+\xi_2)+c}\,\frac{\xi_3^{-j_3\varepsilon}}{(\xi_1+\xi_3)+c};$$

$$\tilde{I}_{\substack{3\\2\\1}}(x)(c) = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \int_0^\infty d\xi_3 \frac{\xi_1^{-j_1\varepsilon}}{\xi_1+c}\,\frac{\xi_2^{-j_2\varepsilon}}{(\xi_1+\xi_2)+c}\,\frac{\xi_3^{-j_3\varepsilon}}{(\xi_1+\xi_2+\xi_3)+c}$$

These expressions are ultra-violet divergent when $\varepsilon \to 0$. To get a finite result, the first idea that comes into one's mind is to extract finite parts :

$$\tilde{I}_{\bullet_1}(x)(c) = \int_0^\infty \frac{\xi^{-j_1\varepsilon}}{\xi+c}dx = B(j_1\varepsilon, 1-j_1\varepsilon)c^{-j\varepsilon} \sim \frac{1}{j_1\varepsilon} + O(1)\log c.$$

One may for instance subtract the singular part in $\mathbb{C}[\varepsilon^{-1}]$ (in a way similar to dimensional regularization for Feynman graphs); or subtract a counterterm, e.g. $\tilde{I}_{\bullet_1}(x)(c=1)$.

Consider now second-order integrals, $\tilde{I}_{\substack{2\\1}}(x)(c)$. The difference

$$\int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \frac{\xi_1^{-j_1\varepsilon}}{\xi_1+c}\left(\frac{\xi_2^{-j_2\varepsilon}}{(\xi_1+\xi_2)+c} - \frac{\xi_2^{-j_2\varepsilon}}{\xi_1+\xi_2}\right) = B(j_2\varepsilon, 1-j_2\varepsilon)\int_0^\infty d\xi_1 \frac{\xi_1^{-j_1\varepsilon}}{\xi_1+c}\left((\xi_1+c)^{-j_2\varepsilon} - \xi_1^{-j_2\varepsilon}\right)$$

is ultra-violet convergent when $\varepsilon \to 0$. Hence (up to regular terms), $\tilde{I}_{\substack{2\\1}}(x)(c)$ factorizes as

$$\begin{aligned} B(j_2\varepsilon, 1-j_2\varepsilon)\int_0^\infty \frac{\xi^{-(j_1+j_2)\varepsilon}}{\xi+c}d\xi &= \left(\int_0^\infty \frac{\xi^{-(j_1+j_2)\varepsilon}}{\xi+c}d\xi\right)\tilde{I}_{\bullet_2}(x)(c=1) \\ &\sim \left(\frac{1}{(j_1+j_2)\varepsilon} + O(1)\log c\right)\left(\frac{1}{j_2\varepsilon} + O(1)\right) \end{aligned} \tag{4.13}$$

Unfortunately, it is apparently not sufficient to subtract the value of this expression at $c = 1$, since the result still has a simple pole. Hence subtracting the counterterm $\tilde{I}_{\substack{2\\1}}(x)(c=1)$ does not make $\tilde{I}_{\substack{2\\1}}(x)(c)$ convergent. One must perform successively the following two operations (for the sake of notations we abbreviate $\tilde{I}_{\mathbb{T}}(x)$ to $\tilde{I}_{\mathbb{T}}$):

(i) prepare $\tilde{I}_{\substack{2\\1}}$ by extracting first the subdivergences of nested trees, in this simple case

$$\bar{\mathcal{R}}(\tilde{I}_{\substack{2\\1}})(c) := \tilde{I}_{\substack{2\\1}}(c) - \tilde{I}_{\bullet_1}(c)\tilde{I}_{\bullet_2}(x)(c=1).$$

(ii) regularize the result of the preceding operation by subtracting the global divergence, also called overall counterterm, e.g.

$$\mathcal{R}(\tilde{I}_{\bullet\!\vert\,{}^{2}_{1}})(c) := \bar{\mathcal{R}}(\tilde{I}_{\bullet\!\vert\,{}^{2}_{1}})(c) - \bar{\mathcal{R}}(\tilde{I}_{\bullet\!\vert\,{}^{2}_{1}})(c=1).$$

Theorem 4.1 *The following renormalized iterated integrals, defined recursively by*

(i) (preparation)

$$\bar{\mathcal{R}}(\tilde{I}_{\mathbb{T}})(c) := \tilde{I}_{\mathbb{T}}(c) + \sum_{\boldsymbol{v} \models \mathbb{T}} \tilde{I}_{\mathrm{Roo}_{\boldsymbol{v}}(\mathbb{T})}(c) \quad \cdot \prod_{\mathbb{T}' \in Lea_{\boldsymbol{v}}(\mathbb{T})} \left[-\bar{\mathcal{R}}(\tilde{I}_{\mathbb{T}'})(c=1) \right], \tag{4.14}$$

where $\mathbb{T}'$ *ranges among the different tree components of the forest* $\mathrm{Roo}_{\boldsymbol{v}}(\mathbb{T})$*, and*

(ii) (regularization)

$$\mathcal{R}(\tilde{I}_{\mathbb{T}})(c) := \bar{\mathcal{R}}(\tilde{I}_{\mathbb{T}})(c) - \bar{\mathcal{R}}(\tilde{I}_{\mathbb{T}})(c=1) \tag{4.15}$$

are finite.

This algorithm is a simplified version of the famous BPHZ (Bogolioubov-Parasiuk-Hepp-Zimmermann) algorithm in quantum field theory which makes Feynman diagrams convergent (see section 5 below).

Chapter 5

The algebraic classification theorem

Two Hopf algebras come into play when one wants to encode the Chen and shuffle properties: the *Hopf algebra of decorated rooted trees* $\mathbf{H}^d$ or *Connes-Kreimer Hopf algebra* (a simple variant of the Hopf algebra of rooted trees defined above) on the one hand, the *shuffle algebra* on the other.

Definition 5.1 (Connes-Kreimer Hopf algebra of decorated rooted trees) *A decorated rooted tree is a rooted tree whose vertices are decorated by labels. Taking formal linear combinations and disjoint unions of decorated rooted trees (called decorated rooted forest or simply forests in our context), and extending trivially the above definition of product, coproduct and antipode, yields the Hopf algebra* $\mathbf{H}^d$ *of decorated rooted trees.*

The upper index d in $\mathbf{H}^d$ may refer to 'decorated'. As a matter of fact, the decorations will be component indices of the path $x = (x_1, \ldots, x_d)$, so they always range in the set $\{1, \ldots, d\}$.

Tree iterated integrals are precisely indexed by decorated trees. As a rule, we shall denote by $\ell : V(\mathbb{T}) \to \{1, \ldots, d\}$ the decorations of the vertices of a tree $\mathbb{T}$. We recall from Definition 3.1

$$I_x^{ts}(\mathbb{T}) = \int_{t_0}^{t} dx_{t_1}(\ell(1)) \int_{t_0}^{t_2-} dx_{t_2}(\ell(2)) \ldots \int_{t_0}^{t_n-} dx_{t_n}(\ell(n)). \tag{5.1}$$

In this section, the notation $I_x^{ts}(\mathbb{T})$ stands for the canonical iterated integrals of the path x between s and t.

Definition 5.2 (shuffle algebra) *(i) Let* $\mathbf{Sh}^d$ *be the shuffle algebra with decorations in* $\{1, \ldots, d\}$, *i.e. the set of words* $(i_1 \ldots i_n)$, $i_1, \ldots, i_n \in \{1, \ldots, d\}$, *with product*

$$(i_1 \ldots i_{n_1}) ⧢ (j_1 \ldots j_{n_2}) = \sum_{\boldsymbol{k} \in Sh(\boldsymbol{i}, \boldsymbol{j})} (k_1 \ldots k_{n_1+n_2}). \tag{5.2}$$

An element of $\mathbf{Sh}$ *is naturally represented as a trunk tree decorated by* $\ell = (\ell(1), \ldots, \ell(n))$ *from the root to the top. For instance* $(i_1 i_2 i_3) = \begin{smallmatrix} i_3 \\ i_2 \\ i_1 \end{smallmatrix}$ *is decorated by* $\ell(j) = i_j$, $j = 1, 2, 3$.

(ii) **Sh** *equipped with the restriction of the coproduct* Δ *of* **H** *to trunk trees, and with the antipode* $S((i_1 \dots i_n)) = -(i_n \dots i_1)$*, is a Hopf algebra. It holds:* $\Delta((i_1 \dots i_n)) = \sum_{k=0}^{n} (i_1 \dots i_k) \otimes (i_{k+1} \dots i_n)$.

One of the links between these two algebras is given by the following Proposition.

Proposition 5.3 (projection morphism) *Let* $\theta : \mathbf{H} \to \mathbf{Sh}$ *be the projection Hopf morphism given by associating to a tree* $\mathbb{T}$ *the sum of the trunk trees* $\mathfrak{t}$ *with same decorations such that*

$$(v \twoheadrightarrow w \text{ in } \mathbb{T}) \Rightarrow (v \twoheadrightarrow w \text{ in } \mathfrak{t}). \tag{5.3}$$

For instance $\theta({}^{b}\mathbb{V}_{a}{}^{c}) = \begin{smallmatrix} c \\ b \\ a \end{smallmatrix} + \begin{smallmatrix} b \\ c \\ a \end{smallmatrix}$.

We consider from now on some rough path $x_{(i_1,\dots,i_n)}$ over x and denote it by $J_x^{ts}(i_1, \dots, i_n)$ (recall $I_x^{ts}(\cdot)$ are the *canonical* iterated integrals, whenever x is smooth). Indexing the $J_x^{ts}(i_1, \dots, i_n)$ by trunk trees $\mathbb{T} \in \mathbf{Sh}$ with decoration $\ell(j) = i_j$, $j = 1, \dots, n$, properties (ii) and (iii) in Definition 1.4 are equivalent to

(ii)bis

$$J_x^{ts}(\mathbb{T}) = \sum_{\boldsymbol{v} \models V(\mathbb{T})} J_x^{tu}(\mathrm{Roo}_{\boldsymbol{v}}(\mathbb{T})) J_x^{us}(\mathrm{Lea}_{\boldsymbol{v}}(\mathbb{T})), \quad \mathbb{T} \in \mathbf{Sh}; \tag{5.4}$$

in other words, $J_x^{ts} = J_x^{tu} * J_x^{us}$ for the shuffle convolution defined in subsection 5.2;

(iii)bis

$$J_x^{ts}(\mathbb{T}) J_x^{ts}(\mathbb{T}') = J_x^{ts}(\mathbb{T} \pitchfork \mathbb{T}'), \quad \mathbb{T}, \mathbb{T}' \in \mathbf{Sh}. \tag{5.5}$$

In other words, J_x^{ts} is a character of **Sh**.

Such a functional indexed by *trunk trees* extends easily to a general *tree-indexed functional* or *tree-indexed rough path* by setting $\bar{J}_x^{ts}(\mathbb{T}) := J_x^{ts} \circ \theta(\mathbb{T})$. Since θ is a Hopf algebra morphism, one gets immediately the generalized properties

(ii)ter $\bar{J}_x^{ts} = \bar{J}_x^{tu} * \bar{J}_x^{us}$ for the convolution of **H**, i.e.

$$\bar{J}_x^{ts}(\mathbb{T}) = \sum_{\boldsymbol{v} \models V(\mathbb{T})} \bar{J}_x^{tu}(\mathrm{Roo}_{\boldsymbol{v}} \mathbb{T}) \bar{J}_x^{us}(\mathrm{Lea}_{\boldsymbol{v}} \mathbb{T}); \tag{5.6}$$

(iii)ter $\bar{J}_x^{ts}(\mathbb{T}) \bar{J}_x^{ts}(\mathbb{T}') = \bar{J}_x^{ts}(\mathbb{T}.\mathbb{T}')$, in other words, $\bar{J}_x^{ts}$ is a character of **H**.

Properties (ii), (iii) and their generalizations are satisfied for the usual integration operators I_x^{ts} and their tree extension $\bar{I}_x^{ts}$, provided x is a smooth path so that iterated integrals make sense [19].

Let us give an explicit formula for tree iterated integrals. Let $\mathbb{T}$ be e.g. a tree, and index its vertices as $1,\ldots,n$, so that $(i \twoheadrightarrow j) \Rightarrow (i > j)$. Denoting by i^- the ancestor of the vertex i in $\mathbb{T}$, one has

$$\bar{I}_x^{ts}(\mathbb{T}) = \int_s^t dx_{t_1}(\ell(1)) \int_s^{t_{2^-}} dx_{t_2}(\ell(2)) \ldots \int_s^{t_{n^-}} dx_{t_n}(\ell(n)). \tag{5.7}$$

Remark 5.4 *Note that (5.7) obviously does not depend on the choice of the vertex indexation. We call this* invariance under indexation of the vertices*, or* naturality property*. We may rephrase it saying that $\bar{I}_x^{ts}(\mathbb{T})$ depends only on the* topology *of $\mathbb{T}$. The same property applies to every natural construction and is required in Definition 5.9.*

Suppose now one wishes to construct a rough path over x, and concentrate on the algebraic properties (ii), (iii) of Definition 1.4. Assume one has constructed characters of $\mathbf{Sh}$, $J_x^{ts_0}$, $t \in [0,T]$ with s_0 fixed – in other words, a one-time functional satisfying the usual shuffle property (iii) –, such that $J_x^{ts_0}(i) = x_t(i) - x_{s_0}(i)$, then one immediately checks that $J_x^{ts} := J_x^{ts_0} * (J_x^{ss_0} \circ \bar{S})$ satisfies properties (ii)bis and (iii)bis. Namely (by the definition of the antipode) $J_x^{ts} := J_x^{ts_0} * (J_x^{ss_0} \circ \bar{S})$ is equivalent to the Chen property $J_x^{ts} * J_x^{ss_0} = J_x^{ts_0}$. So the only difficult part consists in defining some regularized character of $\mathbf{Sh}$ satisfying the regularity properties (i).

5.1 Fourier transform and skeleton integrals

Instead of regularizing iterated integrals, $I_x^{ts_0} \rightsquigarrow J_x^{ts_0}$ with s_0 fixed, we choose to regularize *skeleton integrals*, $\mathrm{Sk}\,\mathrm{I}_x^t$, which are analogues of iterated integrals but depending naturally on a single argument, defined by using Fourier transform.

Definition 5.5 (skeleton integral) *Let*

$$\mathrm{Sk}\,\mathrm{I}_x^t(a_1 \ldots a_n) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \prod_{j=1}^n \mathcal{F}x'_{\xi_j}(a_j) d\xi_j \;\cdot\; \int^t dt_1 \int^{t_1} dt_2 \ldots \int^{t_{n-1}} dt_n e^{\mathrm{i}(t_1\xi_1 + \ldots + t_n\xi_n)}, \tag{5.8}$$

where, by definition, $\int^x e^{\mathrm{i}y\xi} dy = \frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi}$. It may be checked that $\mathrm{Sk}\,\mathrm{I}_x^t$ is a character of $\mathbf{Sh}$ – or, in other words, satisfies the shuffle property –, just as for usual iterated integrals.

The projection θ yields immediately a generalization of this notion to tree skeleton integrals, compare with eq. (5.7),

$$\overline{\mathrm{Sk}\,\mathrm{I}}_x^t(\mathbb{T}) = \mathrm{Sk}\,\mathrm{I}_x^t \circ \theta(\mathbb{T}) = \int^t dx_{t_1}(\ell(1)) \int_s^{t_{2^-}} dx_{t_2}(\ell(2)) \ldots \int^{t_{n^-}} dx_{t_n}(\ell(n)). \tag{5.9}$$

An explicit computation yields ([54], Lemma 4.5):

$$\overline{\mathrm{Sk\,I}}^t_x(\mathbb{T}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \prod_{j=1}^n \mathcal{F}x'_{\xi_j}(\ell(j))d\xi_j \ \cdot \ \frac{e^{\mathrm{i}t(\xi_1+\ldots+\xi_n)}}{\prod_{i=1}^n [\xi_i + \sum_{j \twoheadrightarrow i} \xi_j]}. \tag{5.10}$$

5.2 Fourier normal ordering for smooth paths

We begin by the following

Definition 5.6 (Fourier projections and measure-splitting) *(i) Let μ be some signed measure with compact support, typically, $\mu = \mu_{(x,\ell)}(dt_1, \ldots, dt_n) = \otimes_{j=1}^n dx_{t_j}(\ell(j))$. Then*

$$\mu = \sum_{\sigma \in \Sigma_n} \mu^\sigma \circ \sigma^{-1}, \tag{5.11}$$

where

$$\mathcal{P}^\sigma : \mu \mapsto \mathcal{F}^{-1}\left(\mathbf{1}_{|\xi_{\sigma(1)}| \le \ldots \le |\xi_{\sigma(n)}|} \mathcal{F}\mu(\xi_1, \ldots, \xi_n)\right) \tag{5.12}$$

is a Fourier projection, and μ^σ is defined by

$$\mu^\sigma := \mathcal{P}^{\mathrm{Id}}(\mu \circ \sigma) = (\mathcal{P}^\sigma \mu) \circ \sigma. \tag{5.13}$$

The set of all measures whose Fourier transform is supported in $\{(\xi_1, \ldots, \xi_n); |\xi_1| \le \ldots \le |\xi_n|\}$ will be denoted by $\mathcal{P}^+ Meas(\mathbb{R}^n)$. Thus $\mu^\sigma \in \mathcal{P}^+ Meas(\mathbb{R}^n)$.

(ii) More generally, if T is a tree,

$$\mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^n) = \{\mu; \boldsymbol{\xi} \in \mathrm{supp}(\mathcal{F}\mu) \Rightarrow ((i \twoheadrightarrow j) \Rightarrow (|\xi_i| > |\xi_j|))\}. \tag{5.14}$$

This definition applies in particular to the tensor measures $\mu = \mu_{(x,\ell)} = \otimes_{i=1,\ldots,n} dx_{t_i}(\ell(i)) = \otimes_{i=1,\ldots,n} x'_{t_i}(\ell(i))dt_i$ if $\ell = (\ell(1), \ldots, \ell(n))$ is the decoration of a trunk tree. Note that even though μ is a *tensor measure* in this case, the projected measures μ^σ are *not*. This forces us to extend the previous definitions of I^{ts}_x, J^{ts}_x, $\bar{I}^{ts}_x$, $\bar{J}^{ts}_x$, $\mathrm{Sk\,I}^{ts}_x$, $\overline{\mathrm{Sk\,I}}^{ts}_x$ to measure-indexed characters. This is straightforward. However, one must then trade decorated trees (or forests) for so-called *heap-ordered trees* (or *forests*), i.e. trees without decoration but with indexed vertices $1, \ldots, n$ such that

$$(i \twoheadrightarrow j) \Rightarrow (i > j). \tag{5.15}$$

For instance,

$$\bar{I}^{ts}_\mu(\mathbb{T}) = \int_s^t \int_s^{t_2-} \ldots \int_s^{t_n-} d\mu(t_1, \ldots, t_n). \tag{5.16}$$

Remark. Recall from Remark 5.4 that iterated integrals depend only on the topology of the tree, which means that

$$\bar{I}^{ts}_\mu(\mathbb{T}) = \bar{I}^{ts}_{\mu\circ\sigma}(\sigma^{-1}.\mathbb{T}) \tag{5.17}$$

if $\sigma \in \Sigma_n$ is a reindexation of the vertices preserving the topology of $\mathbb{T}$, i.e. such that

$$(i \twoheadrightarrow j \text{ in } \mathbb{T}) \Rightarrow (i \twoheadrightarrow j \text{ in } \sigma^{-1}(\mathbb{T})). \tag{5.18}$$

To say things shortly, *skeleton integrals* are convenient when using Fourier coordinates, since they avoid awkward boundary terms such as those generated by usual integrals, $\int_0^x e^{\mathrm{i}y\xi} dy = \frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi} - \frac{1}{\mathrm{i}\xi}$, which create terms with different homogeneity degree in ξ by iterated integrations. *Measure splitting* gives the *relative scales* of the Fourier coordinates; orders of magnitude of the corresponding integrals may be obtained separately in each sector $|\xi_{\sigma(1)}| \le \ldots \le |\xi_{\sigma(n)}|$. It turns out that these are easiest to get after a permutation of the integrations (applying Fubini's theorem) such that *innermost (or rightmost)integrals bear highest Fourier frequencies.* This is the essence of *Fourier normal ordering.*

Proposition 5.7 (permutation graph) *Let $\mathfrak{T}_n \in$ **Sh** be a trunk tree with n vertices, and $\sigma \in \Sigma_n$ a permutation of $\{1, \ldots, n\}$. Then there exists a unique element $\mathbb{T}^\sigma \in$ **H** called* permutation graph *such that*

$$I_x^{ts}(\mathfrak{T}_n) = I_x^{ts}(\mathbb{T}^\sigma). \tag{5.19}$$

Let us give an example. Let $\mathfrak{T}_n = \vdots^{\,a_3}_{\,a_2\, a_1}$ and $\sigma : (1,2,3) \to (2,3,1)$. Then

$$\begin{aligned} I_x^{ts}(\mathfrak{T}_n) &= \int_s^t dx_{a_1}(t_3) \int_s^{t_3} dx_{a_2}(t_1) \int_s^{t_1} dx_{a_3}(t_2) \\ &= \int_s^t dx_{a_2}(t_1) \int_s^{t_1} dx_{a_3}(t_2) \int_{t_1}^t dx_{a_1}(t_3) \\ &= \int_s^t dx_{a_2}(t_1) \int_s^{t_1} dx_{a_3}(t_2) \int_s^t dx_{a_1}(t_3) \\ &\quad - \int_s^t dx_{a_2}(t_1) \int_s^{t_1} dx_{a_3}(t_2) \int_s^{t_1} dx_{a_1}(t_3) \\ &= I_x^{ts}(\,\vdots^{a_3}_{a_2} \bullet_{a_1}) - I_x^{ts}(\,{}^{a_3}\mathrm{V}^{a_1}_{a_2}), \end{aligned}$$

so $\mathbb{T}^\sigma = \vdots^{a_3}_{a_2} \bullet_{a_1} - {}^{a_3}\mathrm{V}^{a_1}_{a_2}$. Note that all permutation graphs $\mathbb{T}^\sigma$ with σ fixed are obtained from the same sum of heap-ordered forests (also denoted by $\mathbb{T}^\sigma$, by abuse of notation) by including the decorations of $\mathfrak{T}_n$ permuted by σ.

As an elementary Corollary of Definition 5.6 and Proposition 5.7, one obtains:

Corollary 5.8 (Fourier normal ordering for smooth paths) *Let x be a smooth path and $\mathfrak{T}_n \in$ **Sh** a trunk tree with n vertices and decoration ℓ, then*

$$I_x^{ts}(\mathfrak{T}_n) = \sum_{\sigma \in \Sigma_n} I_{\mu^\sigma}^{ts}(\mathbb{T}^\sigma). \tag{5.20}$$

5.3 Fourier normal ordering and regularization

Definition 5.9 *(i) For every heap-ordered* $\mathbb{T}$ *with* n *vertices, and* $t \in \mathbb{R}$, *let* $\phi^t_{\mathbb{T}} : \mathcal{P}^{\mathbb{T}} Meas(\mathbb{R}^n) \to \mathbb{R}, \mu \mapsto \phi^t_{\mathbb{T}}(\mu)$, *also written* $\phi^t_\mu(\mathbb{T})$ *be a family of linear forms such that:*

(a) $\phi^t_{dx(i)}(\mathbb{T}_1) - \phi^s_{dx(i)}(\bullet_i) = I^{ts}_x(\mathbb{T}_1) = x_t(i) - x_s(i)$ *if* $\mathbb{T}_1$ *is the trivial heap-ordered tree with one vertex;*

(b) if $\mathbb{T}_i$, $i = 1, 2$ *are heap-ordered trees with* n_i *vertices, and* $\mu_i \in \mathcal{P}^{\mathbb{T}_i} Meas(\mathbb{R}^{n_i})$, $i = 1, 2$, *the following multiplicative property holds,*

$$\phi^t_{\mu_1}(\mathbb{T}_1)\phi^t_{\mu_2}(\mathbb{T}_2) = \phi^t_{\mu_1 \otimes \mu_2}(\mathbb{T}_1 \wedge \mathbb{T}_2), \tag{5.21}$$

where $\mathbb{T}_1 \wedge \mathbb{T}_2$ *is the non-decorated product* $\mathbb{T}_1.\mathbb{T}_2$ *with labels of* $\mathbb{T}_2$ *shifted by* n_1 [1]*;*

*(c) (*naturality property*) the following invariance condition under reindexation of the vertices holds, see preceding two Remarks,*

$$\phi^t_\mu(\mathbb{T}) = \phi^t_{\mu \circ \sigma}(\sigma^{-1}.\mathbb{F}) \tag{5.22}$$

if σ *– which acts by permuting the vertices of* $\mathbb{T}$ *– is such that*

$$(i \twoheadrightarrow j \text{ in } \mathbb{T}) \Rightarrow (i \twoheadrightarrow j \text{ in } \sigma^{-1}(\mathbb{T})). \tag{5.23}$$

(ii) Let, for $x = (x(1), \ldots, x(d))$, $\chi^t_x : \mathbf{Sh} \to \mathbb{R}$ *be the linear form on* $\mathbf{Sh}$ *defined by*

$$\chi^t_x(\mathfrak{T}_n) := \sum_{\sigma \in \Sigma_n} \phi^t_{\mu^\sigma_{(x,\ell)}}(\mathbb{T}^\sigma), \quad \mathfrak{T}_n = (\ell(1) \ldots \ell(n)) \tag{5.24}$$

as in Proposition 5.7.

The main result is the following.

Proposition 5.10 (rough path construction by Fourier normal ordering) *For every path* x *such that* χ^t_x *is well-defined,* χ^t_x *is a character of* $\mathbf{Sh}$.

Consequently, the following formula for $\mathfrak{T}_n \in \mathbf{Sh}$, $n \geq 1$, *with* n *vertices and decoration* ℓ,

$$J^{ts}_x(\ell(1), \ldots, \ell(n)) := \chi^t_x * (\chi^s_x \circ S)(\mathfrak{T}_n) \tag{5.25}$$

defines a rough path over x.

Furthermore, the following equivalent definition holds,

$$J^{ts}_x(\mathfrak{T}_n) := \sum_{\sigma \in \Sigma_n} \left(\phi^t * (\phi^s \circ \bar{S})\right)_{\mu^\sigma_{(x,\ell)}}(\mathbb{T}^\sigma), \tag{5.26}$$

[1] The product $\mathbb{T}_1 \wedge \mathbb{T}_2$ defines actually the product of the Hopf algebra of heap-ordered trees [15]

where the convolution in the right equation is defined by reference to the (heap-ordered) tree coproduct, namely, one sets

$$(\phi^t * (\phi^s \circ \bar{S}))_\nu(\mathbb{T}) = \sum_{\boldsymbol{v}\models V(\mathbb{T})} \phi^t_{\otimes_{v\in V(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T})}\nu_v}(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T})\phi^s_{\otimes_{v\in V(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T})}\nu_v}(\bar{S}(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T})) \qquad (5.27)$$

for a tensor measure $\nu = \nu_1 \otimes \ldots \otimes \nu_n$, and by multilinear extension

$$\left(\phi^t * (\phi^s \circ \bar{S})\right)_\nu(\mathbb{T}) = (2\pi)^{-n/2} \int \mathcal{F}\nu(\xi_1,\ldots,\xi_n) d\xi_1 \ldots d\xi_n \;\cdot$$
$$\cdot \sum_{\boldsymbol{v}\models V(\mathbb{T})} \phi^t_{\otimes_{v\in V(Roo_{\boldsymbol{v}}(\mathbb{T}))} e^{\mathrm{i}t_v\xi_v}dt_v}(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T})\phi^s_{\otimes_{v\in V(Lea_{\boldsymbol{v}}(\mathbb{T}))} e^{\mathrm{i}t_v\xi_v}dt_v}(\bar{S}(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T})), \quad \mathbb{T}\in\mathcal{F}_{ho}(n). \qquad (5.28)$$

for an arbitrary measure $\nu \in Meas(\mathbb{R}^n)$, $\nu = (2\pi)^{-n/2}\int d\boldsymbol{\xi}\mathcal{F}\nu(\boldsymbol{\xi}) \otimes_{j=1}^n e^{\mathrm{i}t_j\xi_j}dt_j$.

Quite naturally, we shall call formula (5.25), resp. (5.26) the *shuffle convolution*, resp. *tree convolution definition* of J.

Assuming x is smooth, then defining ϕ^t as the skeleton integral $\mathrm{Sk\,I}^t$ yields trivially by recombination $\chi^t_x = \mathrm{Sk\,I}^t$ too, and then $J^{ts}_x = I^{ts}_x$ is the canonical rough path over x. Proposition 5.10 shows that the same recombination algorithm yields a rough path over x whenever ϕ^t satisfies conditions (a), (b) and (c) of Definition 5.9. It is actually clear from Definition 5.9 that *any* rough path over x is obtained in this way.

The enormous advantage now with respect to the original problem is that one may construct as many linear forms ϕ^t as one wishes by assigning some arbitrary value to $\phi^t_\mu(\mathbb{T})$, $\mathbb{T}$ ranging over all (heap-ordered) trees with ≥ 2 vertices, and extending to forests by multiplication following condition (b). Hence the *algebraic* problem has been completely settled.

It is now natural to try and define ϕ^t as some regularized skeleton integral in such a way that J^{ts}_x satisfies the Hölder continuity property (i) in Definition 1.4. We shall do so in the next sections by *renormalizing* skeleton integrals.

For the sequel, we shall start from the *tree convolution definition* (5.26) of J, which will be used in the following guise. Assume $\nu = \mu^\sigma_{(x,\ell)}$ and $\mathbb{T} = \mathbb{T}_1 \wedge \ldots \wedge \mathbb{T}_p$ is the (heap-ordered) product of p trees. Set $\hat{\nu}_{\mathbb{T}'}(\boldsymbol{\xi}) = \otimes_{v\in V(\mathbb{T}')}\mathcal{F}(x'(\ell\circ\sigma(v)))(\xi_v)e^{\mathrm{i}t_v\xi_v}dt_v$ for $\mathbb{T}'$ subtree of $\mathbb{T}$ and $\boldsymbol{\xi} = (\xi_v)_{v\in V(\mathbb{T}')}$. Then, by the multiplicative property (b) for ϕ^s and ϕ^t, see Definition 5.9,

$$(\phi^t * (\phi^s \circ \bar{S}))_\nu(\mathbb{T}) = (2\pi)^{-n/2} \int d\xi_1 \ldots d\xi_n \mathbf{1}_{|\xi_1|\leq\ldots\leq|\xi_n|} \prod_{q=1}^p (\phi^t * (\phi^s \circ \bar{S}))_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v\in V(\mathbb{T}_q)})}(\mathbb{T}_q). \quad (5.29)$$

Now the inductive definition of the antipode implies

$$
\begin{aligned}
&(\phi^t * (\phi^s \circ \bar{S}))_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v\in V(\mathbb{T}_q)})}(\mathbb{T}_q) \\
&= \phi^t_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v\in V(\mathbb{T}_q)})}(\mathbb{T}_q) + \phi^s_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v\in V(\mathbb{T}_q)})}(\bar{S}(\mathbb{T}_q)) \\
&\quad + \sum_{\boldsymbol{v}\in V(\mathbb{T}_q)} \phi^t_{\hat{\nu}_{\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q}((\xi_v)_{v\in V(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q)})}(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q)\phi^s_{\hat{\nu}_{\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T}_q}((\xi_v)_{v\in V(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T}_q)})}(\bar{S}(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T}_q)) \\
&= (\phi^t - \phi^s)_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v\in V(\mathbb{T}_q)})}(\mathbb{T}_q) \\
&\quad + \sum_{\boldsymbol{v}\models V(\mathbb{T}_q),\boldsymbol{v}\neq\emptyset} (\phi^t - \phi^s)_{\hat{\nu}_{\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q}((\xi_v)_{v\in V(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q)})}(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q)\phi^s_{\hat{\nu}_{\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T}_q}((\xi_v)_{v\in V(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T}_q)})}(\bar{S}(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T}_q))
\end{aligned}
\tag{5.30}
$$

Finally, applying iteratively the inductive definition of the antipode leads to an expression of $\bar{S}(\mathrm{Lea}_{\boldsymbol{v}}\mathbb{T}_q)$ in terms of a sum of forests obtained by multiple cuts an in [10]. Applying once again the multiplicative property to ϕ^s yields $(\phi^t * (\phi^s \circ \bar{S}))_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v\in V(\mathbb{T}_q)})}(\mathbb{T}_q)$ as a sum of terms of the form

$$
\begin{aligned}
&\Phi^{ts}(\mathbb{T}_q;\boldsymbol{\xi};\boldsymbol{v},(\mathbb{T}'_j)) := \\
&\qquad (\phi^t - \phi^s)_{\hat{\nu}_{\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q}((\xi_v)_{v\in V(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q)})}(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q)\prod_{j=1}^{J}\phi^s_{\hat{\nu}_{\mathbb{T}'_j}((\xi_v)_{v\in V(\mathbb{T}'_j)})}(\mathbb{T}'_j),
\end{aligned}
\tag{5.31}
$$

with $V(\mathbb{T}_q) = V(\mathrm{Roo}_{\boldsymbol{v}}\mathbb{T}_q) \cup \uplus_{j=1}^{J} V(\mathbb{T}'_j)$.

Chapter 6

A quick introduction to Feynman diagrams, renormalization and multi-scale analysis

Consider as in the introduction to the previous Chapter a generalized Ginzburg-Landau type functional $\mathcal{H}(\phi) = \frac{1}{2}(C^{-1}\phi, \phi) + \lambda \int V(\phi(x))dx$, where C^{-1} is some positive operator, and V a potential, which we assume to be polynomial to make explicit computations. The associated Gibbs measure is $\mathbb{P}(d\phi) := \frac{1}{Z} e^{-\lambda \int V(\phi(x))dx}$, where Z is a normalization factor called *partition function*, and (denoting by $C(x, y)$ the kernel of the inverse operator C, called *propagator*) $d\mu(\phi)$ is the Gaussian measure with covariance $\mathbb{E}\phi(x)\phi(y) = C(x, y)$. If x belongs to a finite lattice, then the measure $\mathbb{P}(d\phi)$ may be rewritten as $e^{-\mathcal{H}(\phi)}\mathcal{D}\phi$, where $\mathcal{D}\phi$ is a finite-dimensional Lebesgue measure; this is merely a notation when x has a continuous range, although many physicists indulge in it. In this case, $\mathbb{P}(d\phi)$ is anyway ill-defined except in trivial examples; the reason is twofold: (i) the Hamiltonian $\mathcal{H}$ is translation-invariant, so the measure $\mathbb{P}(d\phi)$ is not normalizable; (ii) in most examples coming from physics, ϕ is distribution-valued, so $V(\phi(x)$ is not defined. So a careful analysis requires taking a double limit in a volume cut-off V and an ultraviolet cut-off scale ρ: the field ϕ is replaced by $\phi^{\to\rho}$ and the integration over x restricted to V. In general the limiting measure (when it exists) is not equivalent to the free measure. A careful use of a singular perturbation theory, close in spirit to block-spinning and resting on the analysis of Feynman diagrams, allows in the best cases to show that the theory is a regular perturbation of a scale-dependent Gaussian theory written in terms of a finite number of constants. The renormalizability of the theory means that perturbing around the scale-dependent *renormalized* Gaussian theory – instead of perturbing around the initial free theory – yields *finite* Feynman diagrams. Proving this requires subtle and difficult combinatorial and analytic procedure that go under the name of *constructive field theory* (see Chapter 8). Here we shall simply sketch the overall outcome of this procedure on Feynman diagrams, which is a recursive subtraction of diverging counterterms which make them finite. By a slight abuse of language, although this simplified algorithm has only a perturbative value, it is called *Feynman diagram renormalization* or simply *renormalization*. In practice it often (but not always) contains all the physics, i.e. it gives a correct semi-quantitative understanding of the theory.

Let us now explain the basics of perturbative quantum field theory.

6.1 Feynman diagrams

The general idea is to expand formally the exponential of the Lagrangian in order to compute polynomial moments, $\frac{1}{Z}\mathbb{E}\left[\phi(x_1)\dots\phi(x_n)e^{-\int V(\phi(x))dx}\right]$, also called *n-point functions* and denoted by $\langle\phi(x_1)\dots\phi_n(x_n)\rangle_\lambda$, as $\frac{1}{Z}\sum_{n\geq 0}\frac{(-1)^n}{n!}\mathbb{E}\left[\phi(x_1)\dots\phi(x_n)\left(\int V(\phi(x)dx)^n\right]\right.$. We do not bother about the volume and ultra-violet cut-off here. Let us first explain the expansion for quantum field theory on a finite lattice. A good reference for perturbative expansions in quantum field theory is e.g. [27]. In the following Proposition, $\Phi(X)$ stands for a sum of terms $\sum_i V(X_i)$, where X_i replaces $\phi(x)$.

Proposition 6.1 *1. (Wick's formula) Let $X = (X_1,\dots,X_{2n})$ be a (centered) Gaussian vector. Then*

$$\mathbb{E}[X_1\dots X_{2n}] = \sum_{(i_1 i_2)\dots(i_{2n-1}i_{2n})} \mathbb{E}[X_{i_1}X_{i_2}]\dots\mathbb{E}[X_{i_{2n-1}}X_{i_{2n}}], \tag{6.1}$$

where the indices range over all pairings of the indices $1,\dots,2n$. Each term in the sum is represented as a graph with $2n$ points connected two by two.

2. (connected moments) Let $\langle\,\cdot\,\rangle := \frac{\mathbb{E}[\,\cdot\,e^{\Phi(X)}]}{\mathbb{E}[e^{\Phi(X)}]}$ be a weighted measure, where $X = (X_1, X_2,\dots)$ is a (centered) Gaussian vector, and $\Phi(X)$ is a polynomial in $X_1, X_2,\dots$. Then the connected expectation $\langle X_1\dots X_n\rangle_c$ (c for connected) is (formally at least) the sum of all connected graphs obtained by (i) expanding the exponential; (ii) applying Wick's formula and drawing links between the paired points; (iii) identifying all points coming from the same vertex, *i.e. from the same monomial $V(X_i)$ in $\Phi(X)$ descended from the exponential.*

Connected expectations exclude in particular vacuum contributions, i.e. terms of the form $\mathbb{E}[e^{\Phi(X)}]\mathbb{E}[X_1\dots X_n] = Z\mathbb{E}[X_1\dots X_n]$. Discarding these contributions can be shown to provide automatically the normalizing factor $\frac{1}{Z}$. Then usual expectations $\langle X_1\dots X_n\rangle$ are obtained by taking all possible splittings of $\{1,\dots,n\}$ into disjoint subsets $I_1\uplus\dots\uplus I_p$ and summing over the products of connected expectations $\sum_p\sum_{I_1,\dots,I_p}\langle\prod_{i\in I_1}X_i\rangle_c\dots\langle\prod_{i\in I_p}X_i\rangle_c$. In practice the last operation is trivial for two-point functions $\langle X_{i_1}X_{i_2}\rangle$ if by parity (which is often the case in quantum field theory) the one-point functions $\langle X_i\rangle_c$ vanish, so that $\langle X_{i_1}X_{i_2}\rangle = \langle X_{i_1}X_{i_2}\rangle_c$.

For a theory involving continuous fields $\phi_i:\mathbb{R}^D\to\mathbb{R}, i=1,2,\dots$, lines carry two indices i,j and are evaluated as a propagator $\mathbb{E}\phi_i(x)\phi_j(y)$ between two vertices x, y. An n-point function $\langle\phi_{i_1}(x_1)\dots\phi_{i_n}(x_n)\rangle$ is *formally* equal to the sum of all Feynman diagrams with n external points $x_1,\dots,x_n$. In practice the propagator $C(x,y)$ is in general translation-invariant, so computations are usually simpler after a Fourier transform; Fourier momenta are carried by the lines of the diagrams, and the sum of the momenta of the lines starting from a given point is zero (*momentum conservation constraint*). The Fourier-transformed n-point functions $\langle\hat{\phi}_1(\xi_1)\ldots\hat{\phi}_n(\xi_n)\rangle$ contain implicitly a delta-function $\delta(\xi_1+\ldots+\xi_n)$. This yields the following rules for the evaluation of Feynman diagrams:

Lemma 6.2 (evaluation of Feynman diagrams) *(i) (in x-space). Assign a factor $-\lambda$ to every internal point (i.e. vertex) and a propagator $\mathbb{E}\phi_i(x)\phi_j(y)$ to every line. Integrate over all internal points. Finally, multiply by a numerical factor which is the symmetry factor of the graph.*

(ii) (in Fourier space). Assign a factor $-\lambda$ to every internal point (i.e. vertex) and a propagator $\mathbb{E}\hat{\phi}_i(\xi)\hat{\phi}_j(\xi)$ to every line. Integrate over all independent internal momenta, taking into account the momentum conservation constraint at each vertex. Finally, multiply by a numerical factor which is the symmetry factor of the graph.

Example of Euclidean ϕ^4_4-theory. This is the standard textbook example in general (its Lorentzian counterpart is supposed to describe the self-interaction of the yet undiscovered Higgs boson, at the basis of our present understanding of the masses of particles in the standard model [46]). Euclidean ϕ^4_4-theory describes a scalar field $\phi:\mathbb{R}^4\to\mathbb{R}$ with interaction $V(\phi)(x)=\phi^4(x)$. Expand the exponential $e^{-\lambda\int V(\phi)(x)dx}=1-\lambda\int V(\phi(x))dx+\frac{\lambda^2}{2!}\left(\int\phi^4(x_1)dx_1\right)\left(\int\phi^4(x_2)dx_2\right)+\ldots$ and consider the term of order λ^2. Its contribution to the two-point function $\langle\phi(x)\phi(y)\rangle$, computed by means of Wick's theorem, contains a single connected Feynman diagram, see Fig. **??**,

$$\int d^4x_1\int d^4x_2\mathbb{E}[\phi(x)\phi(x_1)]\left(\mathbb{E}[\phi(x_1)\phi(x_2)]\right)^3\mathbb{E}[\phi(x_2)\phi(x)]. \tag{6.2}$$

Its Fourier transform contributes to $\langle|\hat{\phi}(\xi)|^2\rangle$ the integral

$$\frac{1}{(\xi^2+m^2)^2}\int d^4\xi_1\int d^4\xi_2\frac{1}{(\xi_1^2+m^2)(\xi_2^2+m^2)((\xi-\xi_1-\xi_2)^2+m^2)}. \tag{6.3}$$

Similarly, the lowest-order connected diagram contributing to the four-point function $\langle\phi(y_1)\phi(y_2)\phi(y_3)\phi(y_4)$ or $\langle\hat{\phi}(\xi_1)\hat{\phi}(\xi_2)\hat{\phi}(\xi_3)\hat{\phi}(\xi_4)\rangle$ $(\xi_1+\xi_2+\xi_3+\xi_4=0)$, see Fig. **??**, evaluates as

$$\int d^4x_1\int d^4x_2\mathbb{E}[\phi(y_1)\phi(x_1)]\mathbb{E}[\phi(y_2)\phi(x_1)]\mathbb{E}[\phi(y_3)\phi(x_2)]\mathbb{E}[\phi(y_4)\phi(x_2)]\left(\mathbb{E}[\phi(x_1)\phi(x_2)]\right)^2 \tag{6.4}$$

or

$$\left(\prod_{j=1}^4\frac{1}{\xi_j^2+m^2}\right)\int d^4\xi\frac{1}{(\xi^2+m^2)((\xi_1+\xi_2-\xi)^2+m^2)}. \tag{6.5}$$

Both these diagrams are ultra-violet divergent, as follows from a simple homogeneity argument (or *power-counting argument*, in the terminology of quantum field theory): assuming the momenta of all lines, internal or external, of the Fourier-transformed two-point, resp. four-point function are of order 2^j, i.e. $\log|\xi_i|,\log|\xi|=j+O(1)$ for some $j\geq 0$, the restricted integral on this momentum scale is of order $O(2^{2j})$, resp. $O(1)$. Summing over $0 \le j \le \rho$ for $\rho \to \infty$ yields a power-divergence $O(2^{2\rho}) = O(|\xi|^2)$, resp. a logarithmic divergence $O(\rho) = O(\log|\xi|)$. Call $\omega(G)$ the degree of divergence of a diagram, here $\omega(G) = 2$, resp. 0. For a general diagram G with $I(G)$ internal lines, $N(G)$ external lines and $V(G)$ vertices,, the degree of divergence is the sum of the overall degree of homogeneity of the integrand, $-2I(G)$, and of 4 times the number of independent momenta over which one integrates, namely $I(G) - |V(G)| + 1$; all together one obtains, taking into account the relation $V(G) = \frac{I(G)}{2} + \frac{N(G)}{4}$ (count four half-lines per vertex, except for external lines which are only connected to one vertex) $\omega(G) = 4 - N(G)$. By symmetry non-vanishing diagrams have necessarily an even number of external legs, while diagrams without external legs (so-called vacuum diagrams) are suppressed by the normalization factor $\frac{1}{Z}$. So, naively, it seems that divergent diagrams have either 2 or 4 external legs. However, a diagram with a large number of external legs may become divergent if one inserts a diagram such as Fig. **??** on a propagator. Hence a diagram G with $\omega(G) < 0$ is called simply *superficially convergent*

A fundamental theorem due to S. Weinberg states that *completely convergent diagrams*, i.e. diagrams G such that $\omega(G) < 0$ *and* $\omega(g) < 0$ for all sub-diagrams $g \subset G$, have a finite amplitude.

We shall now see how to give a sense to diagrams which are not completely convergent.

6.2 Renormalization of Feynman diagrams

It required some time to the founders of quantum field theory to understand how to make sense out of these divergent integrals. As in the case of D. Kreimer's toy model (which we mention here only for pedagogical reasons, since it was written 40 years after Bogolioubov et al. had settled the question for Feynman diagrams), the idea is to use a recursive algorithm. An important notion here is that of *graph without subdivergences* (or *primitive graph* in more algebraic terms). A primitive graph is a divergent graph such that all its subgraphs are convergent (in the case of ϕ^4_4-theory, this is a graph with two or four external legs) . By a straightforward generalization of Weinberg's theorem, primitive graphs are made convergent by removing just one counterterm. The operation is called *regularization.* There are many different possibilities (including dimensional regularization). Here we shall choose to remove the evaluation of the graph at zero momentum. This means the following in the case of ϕ^4_4-theory:

– for a graph G with four external legs ($\omega(G) = 0$), replace its Fourier amplitude $\hat{A}_G(\xi_1, \ldots, \xi_4)$ with $\hat{A}_G(\xi_1, \ldots, \xi_4) - \tau\hat{A}_G(\xi_1, \ldots, \xi_4)$, where $\tau\hat{A}_G(\xi_1, \ldots, \xi_4) := \hat{A}_G(0, 0, 0, 0)$;

– for a graph G with two external legs ($\omega(G) = 2$), replace its Fourier amplitude $\hat{A}_G(\xi, -\xi)$, expressed as a function of ξ^2, with $\hat{A}_G(\xi) - \tau\hat{A}_G(\xi)$, where $\tau\hat{A}_G(\xi) := \hat{A}_G(0) + \xi^2\hat{A}_G(\xi^2)$.

The letter τ stands for 'Taylor expansion': we used a Taylor expansion at $\xi = 0$ of order 0, resp. 2 for graphs with four external legs, resp. two external legs. In general, Taylor expanding to order n amounts to subtracting n to $\omega(G)$ if G is primitive. Hence, with this definition of τ, primitive diagrams have been made convergent.

Letting $C(G) := -\tau(A_G)$ be more generally the *counterterm associated to G*, defined as the

Taylor expansion at $\xi = 0$ to order 0 or 2 if G has 4, resp. 2 external legs, the analogue of the above formula of Kreimer is:

Theorem 6.1 (BPHZ renormalization scheme) *Let*

(i) (preparation)

$$\bar{\mathcal{R}}G := A_G + \sum_{g \subset G, g \neq G} C(g) A_{G/g} \tag{6.6}$$

and

(ii) (regularization) $\mathcal{R}G := \bar{\mathcal{R}}G - \tau(\bar{\mathcal{R}}G)$,

where g ranges among all (connected or non-connected) divergent subdiagrams of G, and G/g is the diagram obtained from G by contracting all internal lines of g to a point. The $\mathcal{R}G$ (called: renormalized amplitude of the graph G*) is finite.*

The original proof [22] relies on the following equivalent *forest formula*.

Lemma 6.3 (forests of diverging subgraphs) *A forest $\mathcal{F}$ of diverging subdiagrams of G is a set of diverging subdiagrams $g_1, \ldots, g_n$, $n = 1, 2, \ldots$ such that for every $i \neq j$, either $g_i \subsetneq g_j$ or $g_j \subsetneq g_i$.*

Such a forest may be represented as a decorated rooted tree, whose skeleton is induced by diagram inclusion, and vertices are decorated by contracted subdiagrams, in the following way. The smallest subdiagrams for inclusion decorate the leaves. Let g' be one of these subdiagrams. If $g \subsetneq g'$, $g \in \mathcal{F}$ and there does not exist $g'' \in \mathcal{F}$ such that $g \subsetneq g'' \subsetneq g'$, then one draws an upward line joining the contracted graph g'/g to g, and so on till the roots of the forest which are decorated by the multiple contractions of the largest subdiagrams in $\mathcal{F}$.

The forest formula states that

$$\mathcal{R}G := \sum_{\mathcal{F}} \left[\prod_{g \in \mathcal{F}} (-\tau_g) \right] A_G, \tag{6.7}$$

where $\mathcal{F}$ ranges among the set of all forests of diverging subdiagrams of G, and $\tau_{g_n} \cdots \tau_{g_1} A_G$ is obtained from A_G by successively Taylor expanding $g_1, \ldots, g_n$.

We shall not reproduce here the original proof, based on a clever decomposition of the space of independent momentum variables into sectors and an induction on dimension, but rather use a more straigthforward and illuminating proof based on a multi-scale expansion, which (contrary to the previous one) may be reproduced with little changes in *constructive* field theory.

Rewrite each propagator C associated to a line ℓ into a sum over scales, $\sum_j C^j$. Choosing a particular momentum attribution $\mu = (j_\ell)_\ell$, i.e. a scale j_ℓ for each line. Summing the contribution A_G^μ of such *multi-scale diagrams* over all possible momentum attributions μ yields back the original diagram amplitude $A(G)$.

Let $G^j \subset G$ be the subdiagram containing all lines ℓ with momentum $j_\ell \geq j$, and $(G_k^j)_k$ be its connected components. The set of connected subdiagrams $(G_k^j)_{j,k}$, called *local subdiagrams*, makes up a tree of subgraphs of G called *Gallavotti-Nicolò tree*; once again, one ends up with a forest of subgraphs. Note however the difference with the case of non multi-scale diagrams: the set of *all* subdiagrams of a given diagram does not have a forest structure since there may be overlapping usbdiagrams g, g' such that $g \not\subset g'$ and $g' \not\subset g$. The contribution $A^\mu(G)$ may be estimated as follows. Each propagator ℓ brings a factor $M^{j_\ell \beta_\ell}$, where β_ℓ is the ultra-violet degree of homogeneity of the propagator, $\beta_{ell} = -2$ for a ϕ^4-propagator $\frac{1}{\xi^2+m^2}$. Rewrite $M^{j_\ell \beta_\ell}$ as $\prod_{(j,k);\ell \in L(G_k^j)} M^{\beta_\ell}$. Similary, the integration over the independent momenta yields $\prod_{(j,k)} M^{|L(G_k^j)|-|V(G_k^j)|+D}$ in dimension D. All together one has bounded A_G^μ by $\prod_{(j,k)} M^{\omega(G_k^j)}$. This proves Weinberg's theorem for a completely convergent diagram since the sum over all momenta attribution is obviously finite if the exponents $\omega(G_k^j)$ are all strictly negative. For a general diagram, replace the forest formula $\sum_{\mathcal{F}} \left[\prod_{g\in\mathcal{F}}(-\tau_g)\right] A_G$ for the renormalized amplitude with the recursive subtraction of evaluations at zero momentum of the local subdiagrams, $\left[\prod_{(j,k)}(1-\tau_{G_k^j})\right] A_G^\mu$. The net effect of a Taylor expansion of order $r-1$, $r > \omega(G_k^j)$ ($r = 0$ or 2 for diverging subdiagrams in the case of ϕ_4^4-theory) for G_k^j is to multiply the bound $M^{\omega(G_k^j)}$ by a finite sum of factors of the type $M^{-r(j_\ell - j_{\ell'})}$, where ℓ, resp. ℓ' is an internal, resp. external line of G_k^j. This is equivalent to replacing $M^{\omega(G_k^j)}$ with $M^{\omega^*(G_k^j)}$, where $\omega^*(G_k^j) = \omega(G_k^j) - r < 0$. Hence the associated renormalized amplitude is convergent.

It remains to understand how to rewrite the forest formula in terms of these multi-scale renormalization operators $\left[\prod_{(j,k)}(1-\tau_{G_k^j})\right] A_G^\mu$. This requires some complicated combinatorics that may be found in [48] or [58], see Appendix B for a short presentation. It turns out that multi-scale diagrams need some extra 'preparation' in order to retrieve the renormalized amplitude $\mathcal{R}A(G)$, and that this preparation does not change the above estimates.

6.3 A multi-scale proof of convergence

Chapter 7

Renormalized rough paths

The preceding section reduces the problem of lifting paths to rough paths to the search for adequate *tree data*. The word adequate is rather loosy. In any case it means that tree data should have the correct Hölder regularity, *and* that the associated rough path should also have the correct Hölder regularity. The philosophy is that these tree data should also be obtained by some appropriately regularized integration procedure, in the hope maybe that this regularization could be implemented directly on the path as a limit of some approximation scheme that would regularize fine details of the path, in such a way as to smoothen up its iterated integrals. This is close in spirit to what renormalization does in quantum field theory: the renormalization of Feynman integrals is associated to counterterms in the Hamiltonian; in more probabilistic terms, renormalization is associated to a change of measure. Such a procedure may be implemented in the case of rough paths (see next sections), but it is way more complicated than what we propose to do in this section. Here we shall content ourselves with reinterpreting iterated integrals as Feynman integrals and making them finite by the so-called BPHZ renormalization formula. The algorithm is the same as in the case of Kreimer's toy model; in fact the starting point of the series of articles of Connes and Kreimers around 2000 what precisely that the BPHZ renormalization procedure could be understood in terms of an underlying Hopf algebraic structure of the set of Feynman diagrams, which was to be seen essentially as a 'decorated' version of the Hopf algebraic structure of rooted trees (the 'decoration' of each vertex corresponding to the choice of some overall diverging Feynman diagram without subdivergences). The remarkable fact is then that, in our case, renormalization yields the correct Hölder regularity. Although we do not expect this renormalization of iterated integrals to have direct interpretation in terms of an interacting quantum-field theoretic model, we expect our construction to give a fine understanding of the underlying singularities. As a by-product, it suggests a new axiomatic definition of rough paths, slightly more restrictive than general rough paths, which we propose to call *Fourier normal ordered rough paths*. One of the interests of this new definition is that one has a straightforward classification of Fourier normal ordered rough paths in terms of tree data, whereas the above classification of rough paths is only formal. As we shall see later, Fourier normal ordered rough paths are much better behaved than a general rough path, allowing for many essential results which could not be obtained so far with general rough paths, such as a global existence theorem for sde's, a generalized Malliavin calculus, and so forth.

7.1 Feynman diagram reformulation

Let $\mathbb{T}$ be a forest. We shall show in this section how to compute tree skeleton integrals $\mathrm{Sk}\,\mathrm{I}_B(\mathbb{T})$ of fractional Brownian motion by means of Feynman diagrams of a particular type. Quite generally, the associated physical theory contains particles of 2 types, corresponding to two Gaussian fields, σ, resp. ϕ, whose propagators are represented by simple, resp. double lines. Vertices are of type $(\phi\sigma^n)_{n\geq 2}$, namely, at each vertex meet $n \geq 2$ simple lines and exactly 1 double line. More specifically, we shall only need to consider *tree Feynman diagrams* in an unusual sense, namely, Feynman diagrams such that the subset of *simple* lines contains no loops.

We shall also speak for convenience of Feynman *half-diagrams*, which are Feynman diagrams in the above sense, except that it also possibly admits – besides *true* external ϕ-legs – *uncontracted ϕ-legs*, which are assumed to be cut in the middle (this implies special evaluation rules as we shall see). Gluing a Feynman half-diagram $G^{\frac{1}{2}}$ with its image in a mirror along the middle of its external double lines yields a *symmetric Feynman diagram* $G = (G^{\frac{1}{2}})^2$.

A tree (or more generally a forest) $\mathbb{T}$ determines a unique tree Feynman half-diagram $G^{\frac{1}{2}}(\mathbb{T})$ (called: *uncontracted tree Feynman half-diagram associated to* $\mathbb{T}$), whose underlying tree structure of simple lines is that of $\mathbb{T}$. One always assigns *zero momentum* to the simple external lines attached to the *leaves* of $\mathbb{T}$. All other tree Feynman half-diagrams are obtained from some $G^{\frac{1}{2}}(\mathbb{T})$ by pairwise contracting some of the external double lines, and denoted accordingly, see Fig. 7.1 and Fig. 7.3. If $G^{\frac{1}{2}}$ is a tree Feynman half-diagram (in the same sense as for Feynman diagram) then G is called a *symmetric tree Feynman diagram*. By definition, a symmetric Feynman diagram has no external double line.

Let us now define Feynman rules. If G is a diagram or half-diagram, the set of vertices, resp. internal lines shall be denoted by $V(G)$, resp. $L(G)$, as for trees. The set of external lines is denoted by $L_{ext}(G)$.

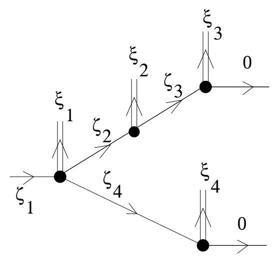
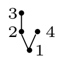


Figure 7.1: Feynman half-diagram $G^{\frac{1}{2}}(\mathbb{T})$ associated to $\mathbb{T} =$ 

Definition 7.1 (Feynman rules) *Let G be a Feynman diagram or half-diagram consisting of simple lines, double lines and vertices v connecting one double line with $k_v - 1 = 2, 3, \ldots$ double*

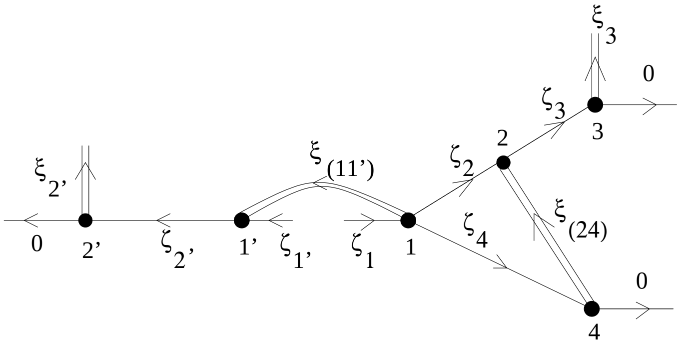


Figure 7.2: Feynman half-diagram $G^{\frac{1}{2}}(\mathbb{T}'.\mathbb{T};(11'),(24))$ associated to $\mathbb{T}'.\mathbb{T}$ with $\mathbb{T}' = {}^{2'}_{1'}$. By momentum conservation, $\xi_{2'} + \xi_3 = \zeta_{1'} + \zeta_1$.

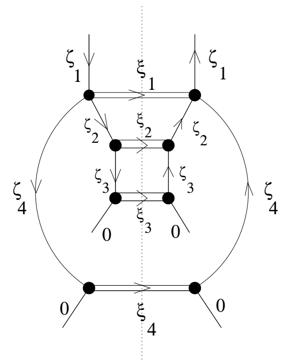


Figure 7.3: Full Feynman diagram $G(\mathbb{T}) = (G^{\frac{1}{2}}(\mathbb{T}))^2$ associated to Fig. 7.1.

lines, and a certain number of external lines. Each line is oriented and decorated by a real-valued momentum, conventionally denoted by ζ_i, resp. ξ_i or $\xi_{(ij)}$ for some index i or pair contraction (ij) for simple, resp. double lines; reversing the orientation of the line is equivalent to changing the sign of the momentum. The momentum preservation relation holds, namely, the sum of all momenta at any vertex is zero. We denote by $I_\sigma(G)$, resp. $I_\phi(G)$, the number of internal simple, resp. double lines, so that simple, resp. double lines may be thought as propagators of some field denoted by σ, resp. ϕ. We also let $I(G) := I_\sigma(G) + I_\phi(G)$ be the total number of internal lines.

(i) Feynman half-diagrams

Let $G^{\frac{1}{2}}$ be a half-diagram. Associate ζ_i^{-1} to each internal simple line with momentum ζ_i, $|\xi_i|^{\frac{1}{2}-\alpha}$ to each external double line with momentum ξ_i, and $|\xi_{(ij)}|^{1-2\alpha}$ to each contracted double line with momentum $\xi_{(ij)}$. The result is a function of the momenta of the external lines, $\boldsymbol{\zeta}_{ext}$ and $\boldsymbol{\xi}_{ext}$, denoted by $A_{G^{\frac{1}{2}}}(\boldsymbol{\zeta}_{ext}, \boldsymbol{\xi}_{ext})$. We shall denote by ζ_{ext}, resp. ξ_{ext} the sum of the momenta of the external simple, resp. double lines.

In the particular case when $G^{\frac{1}{2}} = G^{\frac{1}{2}}(\mathbb{T};(i_1 i_2),\ldots,(i_{2p-1}i_{2p}))$, $p \geq 0$, comes from a tree,

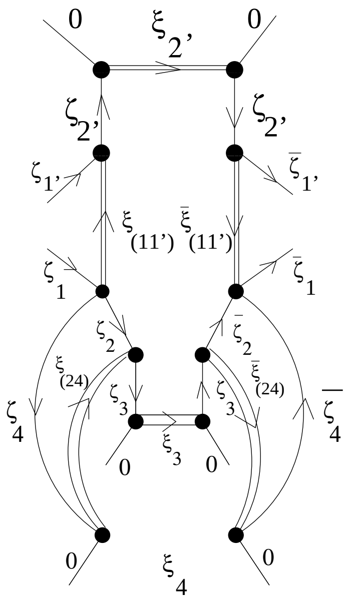


Figure 7.4: Full Feynman diagram $G(\mathbb{T}'.\mathbb{T};(11'),(24)) = (G^{\frac{1}{2}}(\mathbb{T}'.\mathbb{T};(11'),(24)))^2$ associated to Fig. 7.2.

one has $\boldsymbol{\xi}_{ext} = \{\xi_v \mid \xi_v \text{ uncontracted}\}$*, and*

$$A_{G^{\frac{1}{2}}}(\boldsymbol{\zeta}_{ext},\boldsymbol{\xi}_{ext}) = \delta(\zeta_{ext}+\xi_{ext})$$

$$\prod_{v\in V(\mathbb{T})\ \mid\ \xi_v \text{ uncontracted}} |\xi_v|^{\frac{1}{2}-\alpha}. \int \prod_{v\in V(\mathbb{T})\setminus\{\text{roots}\}} \frac{1}{\zeta_v} \cdot \prod_{q=1}^{p} |\xi_{(i_{2q-1}i_{2q})}|^{1-2\alpha} d\xi_{(i_{2q-1}i_{2q})}, \tag{7.1}$$

all external ζ*-momenta attached to the leaves of* $\mathbb{T}$ *vanishing, as explained before.*

(ii) Feynman diagrams

Associate ζ_i^{-1} *to each internal simple line with momentum* ζ_i*, and* $|\xi_i|^{1-2\alpha}$*, resp.* $|\xi_{(ij)}|^{1-2\alpha}$ *to each internal double line with momentum* ξ_i*, resp.* $\xi_{(ij)}$*.*

The resulting amplitude of the amputated diagram (i.e., shorn of its external legs), function of the momenta of the external lines, $\boldsymbol{\zeta}_{ext}$ *and* $\boldsymbol{\xi}_{ext}$*, is denoted by* $A_G(\boldsymbol{\zeta}_{ext},\boldsymbol{\xi}_{ext})$*.*

In the particular case when $G = (G^{\frac{1}{2}})^2$ *is a symmetric tree Feynman diagram, denoting by* $\bar{\zeta}_i$*,* $\bar{\xi}_{(ij)}$ *the momenta of the mirror lines when distinct from the original ones, and by* ζ_{ext}*, resp.* $\bar{\zeta}_{ext}$ *the sum of the external* ζ*-, resp.* $\bar{\zeta}$*-momenta, one has*

$$A_G(\boldsymbol{\zeta}_{ext};\bar{\boldsymbol{\zeta}}_{ext}) = \int \prod_{v\in V(\mathbb{T})\ \mid\ \xi_v \text{ uncontracted}} d\xi_v \left(A_{G^{\frac{1}{2}}}(\boldsymbol{\zeta}_{ext},\boldsymbol{\xi}_{ext})\right)\left(A_{G^{\frac{1}{2}}}(\bar{\boldsymbol{\zeta}}_{ext},\boldsymbol{\xi}_{ext})\right), \tag{7.2}$$

where $\boldsymbol{\xi}_{ext} = \{\xi_v;\ \xi_v \text{ uncontracted}\}$ *as in (i).*

Remark. The relation between the ζ- and ξ-coordinates for a half- or full diagram coming from a tree is simply $\zeta_v = \xi_v + \sum_{w\to v} \xi_w$ or conversely, $\xi_v = \zeta_v - \sum_{w\to v} \zeta_w$, where $\{w : w \to v\}$ are the children of v, and one has set $\xi_i = -\xi_j = \xi_{(ij)}$ for contracted double lines.

Let G be a connected Feynman diagram. It contains $I(G) - |V(G)| + 1$ independent momenta, since there is one momentum constraint at each vertex, which gives altogether $|V(G)| - 1$ independent constraints because the global translation invariance has already been taken into account by demanding that the sum of the external momenta be zero. Remove one internal line at each vertex, so that all remaining momenta are independent. The set of all lines which have been removed, together with the vertices at the end of the lines, constitute a subdiagram of G with no loops, hence a sub-forest. For such a choice of lines, $L'(G)$, say, we let $(z_\ell)_{\ell\in L(G)\setminus L'(G)}$, $z = \zeta$ or ξ, be the set of remaining, independent momenta. Each $z_{\ell'}$, $\ell' \in L'(G)$, may be written uniquely as some linear combination $z_{\ell'} = z_{\ell'}\left((z_\ell)_{\ell\in L_{ext}(G)\cup(L(G)\setminus L'(G))}\right)$, which yields an explicit formula for A_G,

$$A_G(\boldsymbol{z}_{ext}) = \delta(z_{ext}) \int \prod_{\ell\in L(G)\setminus L'(G)} dz_\ell \,.\, \prod_{\ell\in L_\phi(G)} |\xi_\ell|^{1-2\alpha} \prod_{\ell\in L_\sigma(G)} \zeta_\ell^{-2}, \tag{7.3}$$

where z_{ext} is the sum of the external momenta.

The relation with iterated integrals of fractional Brownian motion is the following.

Lemma 7.2 *1. Let $\mathbb{T}$ be a tree with n vertices and root indexed by 1. Then*

$$\overline{\mathrm{Sk\,I}}^t_B(\mathbb{T}) = (2\pi c_\alpha)^{-n/2} \int \prod_{v\in V(\mathbb{T})} dW_{\xi_v}(\ell(v))$$
$$\int \frac{e^{\mathrm{i}t\zeta_1}}{[\mathrm{i}\zeta_1]} d\zeta_1 A_{G^{\frac12}(\mathbb{T})}(\boldsymbol{\zeta}_{ext} = (\zeta_1, 0), \boldsymbol{\xi}_{ext} = (\xi_v)_{v\in V(\mathbb{T})}), \tag{7.4}$$

with

$$A_{G^{\frac12}(\mathbb{T})}(\boldsymbol{\zeta}_{ext} = (\zeta_1, 0), \boldsymbol{\xi}_{ext} = (\xi_v)_{v\in V(\mathbb{T})}) = \delta(\zeta_1 + \xi_{ext}) \prod_{v\in V(\mathbb{T})} |\xi_v|^{\frac12-\alpha} \cdot \prod_{v\in V(\mathbb{T})\setminus\{1\}} \frac{1}{\zeta_v}. \tag{7.5}$$

Hence, provided *the decorations $(\ell(v))_{v\in V(\mathbb{T})}$ are all distinct,*

$$\mathrm{Var}(\overline{\mathrm{Sk\,I}}^t_B(\mathbb{T}) - \overline{\mathrm{Sk\,I}}^s_B(\mathbb{T})) = (2\pi c_\alpha)^{-n} \int \frac{d\zeta_1}{\zeta_1^2} |e^{\mathrm{i}t\zeta_1} - e^{\mathrm{i}s\zeta_1}|^2 A_{G(\mathbb{T})}(\boldsymbol{\zeta}_{ext} = (\zeta_1, 0)), \tag{7.6}$$

with

$$A_{G(\mathbb{T})}(\boldsymbol{\zeta}_{ext} = (\zeta_1, 0)) = \int \prod_{v\in V(\mathbb{T})} d\xi_v \left| A_{G^{\frac12}}((\zeta_1, 0), \boldsymbol{\xi}) \right|^2 \tag{7.7}$$

$$= \int \prod_{v\in V(\mathbb{T})} d\xi_v \delta(\zeta_1 + \sum_{v\in V(\mathbb{T})} \xi_v) \prod_{v\in V(\mathbb{T})} |\xi_v|^{1-2\alpha} \cdot \prod_{v\in V(\mathbb{T})\setminus\{1\}} \frac{1}{\zeta_v^2}. \tag{7.8}$$

2. *Let more generally* $\mathbb{T}=\mathbb{T}_1\ldots\mathbb{T}_q$ *and* $\mathbb{T}'=\mathbb{T}'_1\ldots\mathbb{T}'_{q'}$, $q,q'\geq 1$, *with roots* $r_1,\ldots,r_q,r'_1,\ldots,r'_{q'}$. *Consider some multiple contraction* $(i_1i_2),\ldots,(i_{2p-1}i_{2p})$ *of* $\prod_{m=1}^q(\overline{\mathrm{Sk\,I}}^t_B(\mathbb{T}_m)-\overline{\mathrm{Sk\,I}}^s_B(\mathbb{T}_m))\prod_{m'=1}^{q'}\overline{\mathrm{Sk}}$ *connecting the vertices of* $\mathbb{T}$ *and* $\mathbb{T}'$*, which we write for short* $\delta\overline{\mathrm{Sk\,I}}^{ts}_B\overline{\mathrm{Sk\,I}}^s_B(\mathbb{T},\mathbb{T}';(i_1i_2),\ldots,(i_{2p-1}i_{2p}))$*, and let* $G^{\frac12}:=G^{\frac12}(\mathbb{T}.\mathbb{T}';(i_1i_2),\ldots,(i_{2p-1}i_{2p}))$ *be the corresponding (connected) Feynman half-diagram. Then*

$$\delta\overline{\mathrm{Sk\,I}}^{ts}_B\overline{\mathrm{Sk\,I}}^s(\mathbb{T},\mathbb{T}';(i_1i_2),\ldots,(i_{2p-1}i_{2p}))=(2\pi c_\alpha)^{-\frac12(|V(\mathbb{T})|+|V(\mathbb{T}')|)}$$
$$\int\prod_{v\in V(\mathbb{T}.\mathbb{T}')\ |\ \xi_v\ \text{uncontracted}}dW_{\xi_v}(\ell(v))\prod_{m=1}^q\frac{e^{\mathrm{i}t\zeta_{r_m}}-e^{\mathrm{i}s\zeta_{r_m}}}{[\mathrm{i}\zeta_{r_m}]}d\zeta_{r_m}$$
$$\prod_{m'=1}^{q'}\frac{e^{\mathrm{i}s\zeta_{r'_{m'}}}}{[\mathrm{i}\zeta_{r'_{m'}}]}d\zeta_{r'_{m'}}A_{G^{\frac12}}(\boldsymbol{\zeta}_{ext}=((\zeta_{r_m}),(\zeta_{r'_{m'}}),0),\boldsymbol{\xi}_{ext}) \tag{7.9}$$

where $\boldsymbol{\xi}_{ext}:=\{(\xi_v)_{v\in V(\mathbb{T})}\ |\ \xi_v\ \text{uncontracted}\}$.

Assume furthermore all non-contracted indices $\ell(i)$, $i\neq i_1,\ldots,i_{2p}$ *are distinct. Then*

$$\mathrm{Var}\delta\overline{\mathrm{Sk\,I}}^{ts}_B\overline{\mathrm{Sk\,I}}^s(\mathbb{T},\mathbb{T}';(i_1i_2),\ldots,(i_{2p-1}i_{2p}))=(2\pi c_\alpha)^{-(|V(\mathbb{T})|+|V(\mathbb{T}')|)}$$
$$\int\prod_{m=1}^q d\zeta_{r_m}\left[\frac{e^{\mathrm{i}t\zeta_{r_m}}-e^{\mathrm{i}s\zeta_{r_m}}}{\mathrm{i}\zeta_{r_m}}\right]$$
$$\int\prod_{m'=1}^{q'}\frac{d\zeta_{r'_{m'}}}{[\mathrm{i}\zeta_{r'_{m'}}]}A_{(G^{\frac12})^2}(\boldsymbol{\zeta}_{ext}=((\zeta_{r_m}),(\zeta_{r'_{m'}}),0);\bar{\boldsymbol{\zeta}}_{ext}=((\bar\zeta_{r_m}),(\bar\zeta_{r'_{m'}}),0)). \tag{7.10}$$

Example. Let $\mathbb{T}$, $\mathbb{T}'$ be as in Fig. 7.4. Then:

$$\delta\overline{\mathrm{Sk\,I}}^{ts}_B\overline{\mathrm{Sk\,I}}^s_B(\mathbb{T},\mathbb{T}';(11'),(24))$$
$$=(2\pi c_\alpha)^{-3}\int dW_{\xi_{2'}}dW_{\xi_3}\left[\frac{e^{\mathrm{i}t\zeta_1}-e^{\mathrm{i}s\zeta_1}}{[\mathrm{i}\zeta_1]}d\zeta_1\right]\left[\frac{e^{\mathrm{i}s\zeta_{1'}}}{[\mathrm{i}\zeta_{1'}]}d\zeta_{1'}\right]A_{G^{\frac12}}(\zeta_1,\zeta_{1'},\xi_{2'},\xi_3) \tag{7.11}$$

with

$$A_{G^{\frac12}}(\zeta_1,\zeta_{1'},\xi_{2'},\xi_3)=\delta(\zeta_1+\zeta_{1'}+\xi_{2'}+\xi_3)|\xi_{2'}\xi_3|^{\frac12-\alpha}\frac{|\xi_{(11')}\xi_{(24)}|^{1-2\alpha}d\xi_{(11')}d\xi_{(24)}}{[\mathrm{i}(\xi_{(24)}+\xi_3)][\mathrm{i}\xi_3][\mathrm{i}\xi_{(24)}][\mathrm{i}\xi_{2'}]}. \tag{7.12}$$

As for its variance,

$$
\begin{aligned}
&\mathrm{Var}\delta\overline{\mathrm{Sk\,I}}^{ts}_B\overline{\mathrm{Sk\,I}}^{s}(\mathbb{T},\mathbb{T}';(11'),(24)) \\
&= (2\pi c_\alpha)^{-6}\int \frac{d\zeta_1 d\zeta_{1'} d\bar\zeta_1 d\bar\zeta_{1'}}{[\mathrm{i}\zeta_{1'}][\mathrm{i}\bar\zeta_{1'}]}\left[\frac{e^{\mathrm{i}t\zeta_1}-e^{\mathrm{i}s\zeta_1}}{\mathrm{i}\zeta_1}\right]\left[\frac{e^{\mathrm{i}t\bar\zeta_1}-e^{\mathrm{i}s\bar\zeta_1}}{\mathrm{i}\bar\zeta_1}\right] A_{(G^{\frac12})^2}(\zeta_1,\zeta_{1'};\bar\zeta_1,\bar\zeta_{1'})
\end{aligned}
\tag{7.13}
$$

with

$$
\begin{aligned}
A_{(G^{\frac12})^2}(\zeta_1,\zeta_{1'};\bar\zeta_1,\bar\zeta_{1'}) &= \int d\xi_{2'} d\xi_3 d\xi_{(24)} d\xi_{(11')} d\bar\xi_{(24)} d\bar\xi_{(11')} \delta(\zeta_1+\zeta_{1'}+\xi_{2'}+\xi_3) \\
\delta(\bar\zeta_1+\bar\zeta_{1'}+\xi_2+\xi_3)|\xi_{2'}\xi_3|^{1-2\alpha} & \frac{|\xi_{(24)}\xi_{(11')}\bar\xi_{(24)}\bar\xi_{(11')}|^{1-2\alpha}}{(\xi_3\xi_{2'})^2(\xi_{(24)}+\xi_3)(\bar\xi_{(24)}+\xi_3)\xi_{(24)}\bar\xi_{(24)}}.
\end{aligned}
\tag{7.14}
$$

7.2 Definition of renormalization scheme

We present here the general features of the BPHZ renormalization scheme, together with its multi-scale formulation which will allow us to prove Hölder regularity. It relies

(i) on the choice of a set of graphs called *diverging graphs*. In general (see below) it is simply the subset of Feynman graphs G such that $\omega(G) > 0$, where ω is the *overall degree of divergence* (or simply degree of homogeneity) of the graph. However we shall be led to enlarge slighly this set in our model. The set of diverging graphs shall be defined in section 4.

(ii) on a choice of *regularization scheme*. Here we choose the Taylor evaluation at zero external momenta, denoted by τ. To be definite, if $A_g(z_{ext,1},\ldots,z_{ext,N_{ext}})$ is the amplitude of the graph g with N_{ext} external momenta, then $\tau_g A_g(z_{ext,1},\ldots,z_{ext,N_{ext}}) = A_g(0,\ldots,0)$.

Consider now a subdiagram $g^{\frac12}$ of a Feynman half-diagram $G^{\frac12}$, with external legs $\boldsymbol{z}_{ext} := \boldsymbol{z}'_{ext} \uplus \{\xi_v : \ \xi_v \text{ uncontracted}\}$. The uncontracted ϕ-legs (ξ_v) are not cosidered as *true*, free external legs since they are attached on the mirror and must eventually be integrated, see e.g. eq. (7.7). Hence one sets

$$
\tau_{g^{\frac12}} A_{g^{\frac12}}(\boldsymbol{z}_{ext}) := A_{g^{\frac12}}(\boldsymbol{z}'_{ext}=\boldsymbol{0},\{\xi_v : \ \xi_v \text{ uncontracted}\}). \tag{7.15}
$$

For this reason, it is more natural to write $\tau_g A_{g^{\frac12}}$ instead of $\tau_{g^{\frac12}} A_{g^{\frac12}}$, where in the symmetric graph $g := (g^{\frac12})^2$, the uncontracted ϕ-legs have now become *internal legs*.

7.3 Diverging graphs

Consider a given graph G. In order to decide whether to renormalize it or not, we compute its degree of divergence $\omega(G)$. It is simply obtained as the sum of the overall degree of homogeneity of the integrand, $(1-2\alpha)I_\phi(G) - I_\sigma(G)$, and of the number of $I(G) - |V(G)| + 1$ independent momenta, with respect to which the integrand is integrated; hence it is simply the overall homogeneity degree of the Feynman integral. Taking into account the relation $|V(G)| = 2I_\phi(G) + N_\phi(G)$ (obtained by counting one half double line per vertex, except for external double lines which are only connected to one vertex), yields

$$\omega(G) = 1 - \alpha|V(G)| - (1-\alpha)N_\phi(G). \tag{7.16}$$

Definition 7.3 (diverging graphs) *We call a Feynman graph G* diverging *if and only if it has external ϕ-legs.*

Clearly enough, with this definition, small graphs (i.e. with $\alpha|V(G)| < 1$) are diverging if and only if $\omega(G) > 0$ (which is the usual definition). It is natural to extend this notion to Feynman *half-diagrams* by letting $\omega(G^{\frac{1}{2}}) := 1)\alpha|V(G^{\frac{1}{2}})| - (1-\alpha)N_\phi(G^{\frac{1}{2}})$, where $N_\phi(G^{\frac{1}{2}})$ is the number of *true* external ϕ-legs. Then Feynman half-diagrams associated to skeleton integrals of order $n < \lfloor 1/\alpha \rfloor$ are diverging if and only if $\omega(G^{\frac{1}{2}}) > 0$.

7.4 The multiscale BPHZ algorithm

We denote hereafter by $\mathcal{F}^{div}(G^{\frac{1}{2}})$ the set of forests of diverging subgraphs of $G^{\frac{1}{2}}$. Equivalently,

$$\mathcal{F}^{div}(G) := \{g = (g^{\frac{1}{2}})^2 \;\; g^{\frac{1}{2}} \in \mathcal{F}^{div}(G^{\frac{1}{2}})\}, \tag{7.17}$$

if $G = (G^{\frac{1}{2}})^2$ is a symmetric graph, is the set of diverging *symmetric* subgraphs.

The following considerations are sufficiently general to be independent from the precise definition of diverging graphs.

Definition 7.4 (Bogolioubov's non-recursive definition of renormalization) *(i) Let*

$$\mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T})}((\zeta_1,0),\boldsymbol{\xi}) := \sum_{\mathbb{F}\in\mathcal{F}^{div}(G(\mathbb{T}))} \prod_{g\in\mathbb{F}} (-\tau_g) A_{G^{\frac{1}{2}}(\mathbb{T})}((\zeta_1,0),\boldsymbol{\xi})). \tag{7.18}$$

(ii) Define correspondingly, for $\nu := D(f)\mu = \mathcal{F}^{-1}(f\cdot(\mathcal{F}\mu))$, with $\mu = \otimes_{v\in V(\mathbb{T})} dB_{x_v}(\ell(v))$ and $f = f(\xi_1,\ldots,\xi_n)$ such that $\mathrm{supp}(f) \subset \mathbb{R}_+^{\mathbb{T}}$,

$$\phi_\nu^t(\mathbb{T}) := (2\pi c_\alpha)^{-n/2} \int \prod_{v\in V(\mathbb{T})} dW_{\xi_v}(\ell(v)) \frac{e^{\mathrm{i}t\zeta_1}}{[\mathrm{i}\zeta_1]} d\zeta_1$$
$$f(\boldsymbol{\xi})\mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T})}(\boldsymbol{\zeta}_{ext} = (\zeta_1,0), \boldsymbol{\xi}_{ext} = (\xi_v)_{v\in V(\mathbb{T})}), \tag{7.19}$$

so that, assuming all decorations $(\ell(v))_{v\in V(\mathbb{T})}$ are distinct,

$$\mathrm{Var}\left(\phi^t_\nu(\mathbb{T})-\phi^s_\nu(\mathbb{T})\right)=(2\pi c_\alpha)^{-n}\int\frac{d\zeta_1}{\zeta_1^2}|e^{\mathrm{i}t\zeta_1}-e^{\mathrm{i}s\zeta_1}|^2\mathrm{Var}\hat{\phi}^{\zeta_1}_\nu(\mathbb{T}),$$
$$\mathrm{Var}\hat{\phi}^{\zeta_1}_\nu(\mathbb{T})=D(f)\mathcal{R}A_{G(\mathbb{T})}(\boldsymbol{\zeta}_{ext}=(\zeta_1,0)), \tag{7.20}$$

where

$$D(f)\mathcal{R}A_{G(\mathbb{T})}(\zeta_1,0):=\int\prod_{v\in V(\mathbb{T})}d\xi_v f^2(\boldsymbol{\xi})\left|\mathcal{R}A_{G^{\frac12}}((\zeta_1,0),\boldsymbol{\xi})\right|^2. \tag{7.21}$$

Now come two essential remarks, based on the fact that divergent subgraphs have *no* external ϕ-leg by definition.

1. Since renormalization leaves ξ-momenta unchanged, one may consider the integration measure $f(\boldsymbol{\xi})\prod_{v\in V(\mathbb{T})}dW_{\xi_v}(\ell(v))$ in eq. (**??**) as a simple decoration of the vertices. In this sense $\phi^t_\nu(\mathbb{T})$ may be considered as a *renormalized skeleton integral*, denoted by $\mathcal{R}\overline{\mathrm{Sk\,I}}^t_\nu(\mathbb{T})$.

2. Consider some multiple contraction $\phi^t_\nu(\mathbb{T};(i_1i_2),\ldots,(i_{2p-1},i_{2p}))$ of $\phi^t_\nu(\mathbb{T})$. Then

$$\phi^t_\nu(\mathbb{T};(i_1i_2),\ldots,(i_{2p-1}i_{2p})):=(2\pi c_\alpha)^{-n/2}\int\prod_{v\in V(\mathbb{T})}dW_{\xi_v}(\ell(v))\frac{e^{\mathrm{i}t\zeta_1}}{[\mathrm{i}\zeta_1]}d\zeta_1$$
$$f(\boldsymbol{\xi})\mathcal{R}A_{G^{\frac12}(\mathbb{T};(i_1i_2),\ldots,(i_{2p-1}i_{2p}))}(\boldsymbol{\zeta}_{ext}=(\zeta_1,0),\boldsymbol{\xi}_{ext}=(\xi_v)_{v\in V(\mathbb{T})}). \tag{7.22}$$

In other words, contraction and renormalization *commute*. This remark extends in a straightforward way to contractions between different trees as in Lemma 7.2 (2). This allows us to extend the BPHZ construction to contracted graphs. Namely, consider the Feynman diagram $G=(G^{\frac12})^2$ obtained by gluing two identical Feynman half-diagrams *with the same external structure*, i.e. such that $\bar z=z$ whenever z is a *true* external leg. Then *all* (internal or external) momenta ζ or ξ are equal to their image $\bar\zeta$ or $\bar\xi$ in the mirror. Now one defines

$$\mathcal{R}A_G(z_{ext})=\int\prod_{\xi\ |\ \xi\ \text{uncontracted}}d\xi|\mathcal{R}A_{G^{\frac12}}(z_{ext},\xi)|^2 \tag{7.23}$$

where $\mathcal{R}A_{G^{\frac12}}(\ .\ )=\sum_{\mathbb{F}\in\mathcal{F}^{div}(G)}\prod_{g\in\mathbb{F}}(-\tau_g)A_{G^{\frac12}}$ is defined by the BPHZ formula as in eq. (7.18).

3. Let $G=(G^{\frac12})^2$ be as in 2. As already mentioned, $\mathcal{R}A_G(\ .\ )$ differs from the usual BPHZ renormalized graph amplitude since (due to the square in the right-hand side of eq. (7.23)) divergent subgraphs are in some sense renormalized *twice* from the point of view of power-counting.

Choose some constant $M > 1$. An *attribution of momenta* μ for a Feynman diagram G is a choice of M-adic scale for each momentum of G, i.e. a function $\mu : L(G) \cup L_{ext}(G) \to \mathbb{Z}$ and an associated restriction of the momentum $z_\ell = \zeta_\ell$ or ξ_ℓ, $\ell \in L(G) \cup L_{ext}(G)$ to the M-adic interval $[M^{\mu(\ell)}, M^{\mu(\ell)+1})$. Thus one may define, or a tree Feynman half-diagram $G^{\frac{1}{2}}$, compare with eq. (7.1),

$$A^{\mu}_{G^{\frac{1}{2}}}(\boldsymbol{\zeta}_{ext}, \boldsymbol{\xi}_{ext}) := \delta(\zeta_{ext} + \xi_{ext}) \prod_{v \in V(\mathbb{T}) \ | \ \xi_v \text{ uncontracted}} |\xi_v|^{\frac{1}{2}-\alpha} \cdot \int \prod_{q=1}^{p} |\xi_{(i_{2q-1}i_{2q})}|^{1-2\alpha} d\xi_{(i_{2q-1}i_{2q})} \prod_{v \in V(\mathbb{T}) \setminus \{\text{roots}\}} \frac{1}{\zeta_v} \prod_{v \in V(\mathbb{T})} \left(\mathbf{1}_{|\zeta_v| \in [M^{\mu(\zeta_v)}, M^{\mu(\zeta_v)+1})} \right) \left(\mathbf{1}_{|\xi_v| \in [M^{\mu(\xi_v)}, M^{\mu(\xi_v)+1})} \right) \tag{7.24}$$

where by definition $M^{\mu(\xi_{i_{2q-1}})} = M^{\mu(\xi_{i_{2q}})} = M^{\mu(\xi_{(i_{2q-1}i_{2q})})}$ for contracted lines,

and similarly for an arbitrary Feynman diagram G, compare with eq. (7.3),

$$A^{\mu}_{G}(\boldsymbol{z}_{ext}) := \delta(z_{ext}) \int \prod_{\ell \in L(G) \setminus L'(G)} dz_\ell \prod_{\ell \in L(G) \cup L_{ext}(G)} \mathbf{1}_{z_\ell \in [M^{\mu(\ell)}, M^{\mu(\ell)+1})} \prod_{\ell \in L_\phi(G)} |\xi_\ell|^{1-2\alpha} \prod_{\ell \in L_\sigma(G)} \zeta_\ell^{-2}. \tag{7.25}$$

Definition 7.5 (Gallavotti-Nicolò tree) *Let $G^j \subset G$, $j \in \mathbb{Z}$ be the subdiagram with set of lines $L(G^j) \cup L_{ext}(G^j) := \{\ell \in L(G) \cup L_{ext}(G);\ \mu(\ell) \geq j\}$, and $(G^j_k)_{k=1,2,\ldots}$ the connected components of G^j.*

Thet set of connected subgraphs $(G^j_k)_{j,k}$ makes up a tree of subgraphs of G, called Gallavotti-Nicolò tree.

Two instances of Gallavotti-Nicolò trees are represented on Fig. 7.5, 7.6. By shifting slightly the M-adic intervals, it is possible to manage to have both lines of highest momentum of any given vertex in the same interval.

Definition 7.6 *Let $\mathbb{F} \in \mathcal{F}^{div}(G)$ be a forest of diverging subgraphs of G.*

(i) Let $g \in G$ be a subgraph of G. Then g is compatible with $\mathbb{F}$ *if and only if $\mathbb{F} \cup \{g\}$ is a forest.*

(ii) Assume $g \in G$ is compatible with $\mathbb{F}$. We let $g^-_{\mathbb{F}}$ be the ancestor of g in the forest of graphs $\mathbb{F} \cup \{G\}$, and $g^{\uparrow}_{\mathbb{F}}$ be the union of its children, namely,

$$g^{\uparrow}_{\mathbb{F}} = \cup_{h \subsetneq g, h \in \mathbb{F}} h. \tag{7.26}$$

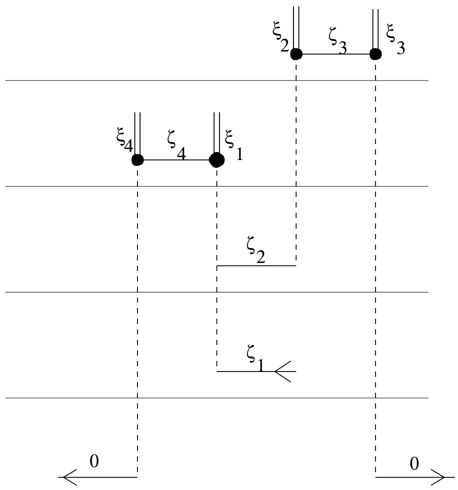


Figure 7.5: A Gallavotti-Nicolò tree (case 1).

(iii) Let μ be a momentum scale attribution. The dangerous forest $D^\mu(\mathbb{F}) \subset \mathbb{F}$ *associated to the forest $\mathbb{F}$ and the momentum scale attribution μ is the sub-forest defined by*

$$(g \in D^\mu(\mathbb{F})) \Longleftrightarrow \Big(\min\{i_\ell(\mu) \ : \ \ell \in L(g \setminus g_{\mathbb{F}}^{\uparrow})\} > \max\{i_\ell(\mu) \ : \ \ell \in L_{ext}(g) \cap L(g_{\mathbb{F}}^{-})\}\Big). \tag{7.27}$$

(iv) Call the sub-forest $ND^\mu(\mathbb{F}) := \mathbb{F} \setminus D^\mu(\mathbb{F}) \subset \mathbb{F}$ the non-dangerous *or* harmless forest *associated to $\mathbb{F}$ and μ.*

One can prove that $ND^\mu \circ ND^\mu = ND^\mu$. Hence

$$\mathcal{F}^{div}(G) = \cup_{\mathbb{F} \in \mathcal{F}^{div}(G) \ | \ T_\mu(\mathbb{F}) = \mathbb{F}} \{\mathbb{F}' \supset \mathbb{F} \ : \ ND_\mu(\mathbb{F}') = \mathbb{F}\}. \tag{7.28}$$

One obtains the following classification of forests:

Proposition 7.7 *Let*

(i) $Safe^\mu(G) \subset \mathcal{F}^{div}(G)$ be the set of forests of diverging graphs which are invariant under the projection operator ND^μ and thus harmless, namely, $Safe^\mu(G) := \{\mathbb{F} \in \mathcal{F}^{div}(G) \ : \ ND^\mu(\mathbb{F}) = \mathbb{F}\}$;

(ii) $Ext^\mu(\mathbb{F}) \subset \mathcal{F}^{div}(G)$, with $\mathbb{F} \in Safe^\mu(G)$, be the "maximal dangerous extension" of the harmless forest $\mathbb{F}$ within the ND^μ-equivalence class of $\mathbb{F}$, namely, $\mathbb{F} \uplus Ext^\mu(\mathbb{F})$ is the maximal forest such that $ND^\mu(\mathbb{F} \uplus Ext^\mu(\mathbb{F})) = \mathbb{F}$.

Then:

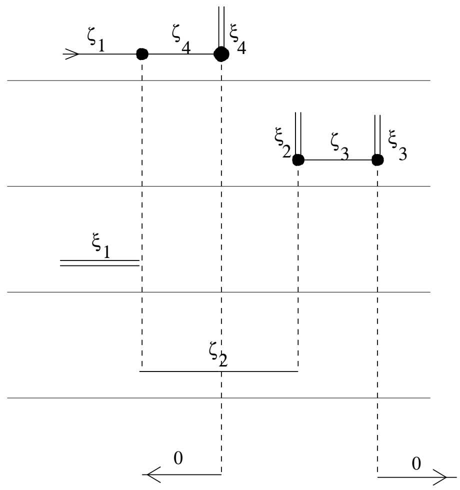


Figure 7.6: A Gallavotti-Nicolò tree (case 2).

(i)

$$\left(ND^{\mu}(\mathbb{F}')=\mathbb{F}\right)\Longleftrightarrow\left(\mathbb{F}\subset\mathbb{F}'\subset\mathbb{F}\uplus Ext^{\mu}(\mathbb{F})\right); \tag{7.29}$$

(ii) $Ext^{\mu}(\mathbb{F})$ *is the set of subgraphs* $g\in G$*, compatible with* $\mathbb{F}$*, such that* $g\in D^{\mu}(\mathbb{F}\cup\{g\})$*.*

Corollary 7.8

$$\mathcal{R}A_{G^{\frac{1}{2}}}=\sum_{\mathbb{F}\in\mathcal{F}^{div}(G)}\mathcal{R}A_{G^{\frac{1}{2}},\mathbb{F}} \tag{7.30}$$

where

$$\mathcal{R}A_{G^{\frac{1}{2}},\mathbb{F}}:=\sum_{\mu\ |\ \mathbb{F}\in Safe^{\mu}(G)}\prod_{g\in\mathbb{F}}(-\tau_g)\prod_{h\in Ext^{\mu}(\mathbb{F})}(1-\tau_h)A^{\mu}_{G^{\frac{1}{2}}}. \tag{7.31}$$

7.5 Main bound for Feynman diagrams

This section is devoted to the proof of the following theorem.

Theorem 7.1 *Let* $G:=G(\mathbb{T};(i_1i_2)\dots(i_{2p-1}i_{2p}))$ *be a symmetric tree Feynman diagram, with* $\boldsymbol{\zeta}_{ext}=(\zeta_{r_1},\dots,\zeta_{r_q},\bar\zeta_{r_1},\dots,\bar\zeta_{r_q})$*. Assume* $\zeta_{r_m}=\bar\zeta_{r_m}$*,* $m=1,\dots,q$*, so that each* ζ*-momentum and each contracted* ξ*-momentum is equal to the corresponding* $\bar\zeta$*- or* $\bar\xi$*-momentum on the other side of the mirror. Label* $\boldsymbol{\zeta}_{ext}$ *so that* $|\zeta_{r_1}|<\dots<|\zeta_{r_q}|$*. Choose some* ξ*-leg* ξ' *belonging to* G*.* Fix $j(\xi')$ *and sum over all scale attributions* μ *such that* $\mu(\xi')=j(\xi')$*. Call* $\mathcal{R}A^{j(\xi')}_G(\boldsymbol{\zeta}_{ext}):=\sum_\mu\mathcal{R}A^{\mu}_G(\boldsymbol{\zeta}_{ext})$ *the result. Then:*

1.

$$\mathrm{Var}\mathcal{R}A^{j(\xi')}_G(\boldsymbol{\zeta}_{ext})\lesssim M^{(1-2n\alpha)j(\xi')}\frac{\min(|\zeta_{r_1}|,M^{j(\xi')})}{\max(|\zeta_{r_q}|,M^{j(\xi')})}. \tag{7.32}$$

2.

$$\sum_{j(\xi')=-\infty}^{\infty} M^{-2n'\alpha j(\xi')}\mathrm{Var}\mathcal{R}A_G^{j(\xi')}(\boldsymbol{\zeta}_{ext}) \lesssim |\zeta_{r_1}|^{1-2(n+n')\alpha}\frac{|\zeta_{r_1}|}{|\zeta_{r_q}|}. \tag{7.33}$$

Remark. The factor $\frac{|\zeta_{r_1}|}{|\zeta_{r_q}|}$ in eq. (7.33) is obtained and shall be used as a product of *spring factors*, $\prod_{m=1}^{q-1}\left|\frac{\zeta_{r_m}}{\zeta_{r_{m+1}}}\right|$. The same remark holds for the factor in the right-hand side of eq. (7.32) obtained by inserting $|\xi'|$ or equivalently $M^{j(\xi')}$ at the right place in the ordered list $(|\zeta_{r_1}|,\dots,|\zeta_{r_q}|)$.

Using the Cauchy-Schwarz inequality, this result yields immediately

Corollary 7.9

Proof.

Choose inductively, starting from the highest momentum scale, a subset of lines $L'(G)\subset L(G)$ so that $(z_\ell)_{\ell\in L(G_k^j)\setminus L'(G_k^j)}$, where $L'(G_k^j):=L'(G)\cap L(G_k^j)$, make up a maximal set of independent momenta of the graph G_k^j shorn of its external legs.

Assume for a moment that all $(z_\ell)_{\ell\in L(G_k^j)}$ are of the same order, $M^{j'}$, say $(j'\geq j)$. Then the previous power-counting argument shows that $A_{G_k^j}$ is of order $M^{j\omega(G_k^j)}$. If $\omega(G_k^j)\geq 0$, then clearly the sum $\sum_{j'=j}^{\infty} M^{\omega(G_k^j)}$ diverges, so the sum over all momenta attributions diverges. On the other hand, if $\omega(G_k^j)<0$, then the sum over all momenta attributions may still diverge because of so-called sub-divergences due the higher subgraphs $G^{j'\to}$, $j'>j$; the graph is said to be *overall convergent.* Since the graph $G=G(\mathbb{T})$ is symmetric, $N_\phi(G_k^j)$ is necessarily even, so a graph with external double lines is always overall convergent.

Let us now see how renormalization will make all subgraphs convergent. Consider any of the local subgraphs G_k^j. Assume $\omega(G_k^j)>0$, so that G_k^j must be renormalized. We introduce some notations for the sake of clarity. To each $v\in V_{ext}(G_k^j)$, one associates the unique line $\ell'_v\in L'(G_k^j)\cap L_v(G_k^j)$, and lets $z_v^*:=\sum_{\ell\in L_v(G_k^j)\setminus L'_v(G_k^j)} z_\ell$ and $z_{v,ext}:=\sum_{\ell\in L_{v,ext}(G_k^j)} z_\ell$, so that $z_{\ell'_v}+z_v^*+z_{v,ext}=0$. Choose some arbitrary ordering of the external legs of G, $L_{ext}(G_k^j)=\{\ell_1,\dots,\ell_{|L_{ext}(G_k^j)|}\}$. The renormalization changes only the values of the external momenta, so it acts really on the product $A_{ext}(G_k^j):=\prod_{v\in V_{ext}(G_k^j)}(z_v^*+z_{v,ext})^{\beta_{\ell'_v}}$,

$$A_{ext}(G_k^j)\rightsquigarrow \mathcal{R}A_{ext}(G_k^j):=\prod_{v\in V_{ext}(G_k^j)}(z_v^*+z_{v,ext})^{\beta_{\ell'_v}}-\prod_{v\in V_{ext}(G_k^j)}(z_v^*)^{\beta_{\ell'_v}}$$

$$=\sum_{i=1}^{|L_{ext}(G_k^j)|}\int_0^1 z_{\ell_i}\partial_{z_{\ell_i}}\prod_{v\in V_{ext}(G_k^j)}(z_v^*+z_{v,ext}(s))^{\beta_{\ell'_v}}\,ds, \tag{7.34}$$

where $z_{v,ext}(s) = z_{v,ext}$ if $\ell = \ell'_{i'}$, $i' < i$, $sz_{v,ext}$ if $\ell = \ell_i$, and 0 otherwise.

Now only one or two factors in the product over $V_{ext}(G^j_k)$ depend on z_{ℓ_i}; the derivative $\partial_{z_{\ell_i}}$ acting on each of these, generically denoted by $(\xi^*_v + \xi_{v,ext}(s))^{\beta_{\ell'_v}}$, generates an extra multiplicative factor called *spring factor*, $\frac{z_{\ell_i}}{z^*_v + z_{v,ext}(s)}$, up to a constant, namely, $\beta_{\ell'_v}$ or $s\beta_{\ell'_v}$. This spring factor is at most $O(M^{\min\mu(G^j_k) - \max\mu(\partial G^j_k)})$, where $\min\mu(G^j_k)$ is the minimal scale index of all *internal* lines of G^j_k, and $\max\mu(\partial G^j_k)$ the maximal scale index of all *external* lines of G^j_k.

We shall now rewrite $\mathcal{R}A(G)$ by using the local graph decomposition of G. First, each factor M^{β_ℓ}, $\ell \in L(G)$ may be rewritten as $\prod_{(j,k)\ ;\ell\in L(G^j_k)} M^{\beta_\ell}$. Similarly, the integration over the independent momenta yield $\prod_{(j,k)} M^{|L(G^j_k)|-|V(G^j_k)|+1}$. Finally, the spring factors due to renormalization contribute a factor M^{-1} per scale until G^j_k absorbs one external line, so all together $\prod_{(j,k);\omega(G^j_k)>0} M^{-1}$. The initial naive power-counting may be reproduced here, leading to

$$\mathcal{R}A(G) \le \prod_{(j,k)} M^{\omega^*(G^j_k)}, \tag{7.35}$$

where $\omega^*(G^j_k) = \omega(G^j_k)$ if $\omega(G^j_k) < 0$, $\omega(G^j_k) - 2$ otherwise, *except* for the total graph G which is not renormalized (having only external legs of zero momentum), so that $\omega^*(G) = \omega(G) = 1 - 2n\alpha$. Note the following essential inequality,

$$\omega^*(G^j_k) \le -1, \quad G^j_k \ne G \tag{7.36}$$

by the remark following 7.3.

Fix the scales of μ, say, $j_1 < j_2 < \ldots, j_I = j_{max}$, with $j_{I_1} = j(\zeta_1)$. Then the renormalized amplitude is bounded by

$$\sum_{j_1 > -\infty} M^{j_1\omega^*(G)} \left(\sum_{j_2 \ge j_1} M^{(j_2-j_1)\omega^*(G^{j_2})} \left(\cdots \left(\sum_{j_I \ge j_{I-1}} M^{(j_I - j_{I-1})\omega^*(G^{j_I})} \right) \cdots \right)\right), \tag{7.37}$$

the scales j_{I_1} being fixed, and the scales $j_1, \ldots, j_{I_1-1}$ constrained to be below $j(\zeta_1)$. Since all $\omega^*(G^{j_I})$ *except* possibly $\omega^*(G) = \omega(G)$ are < 0, summing down to scale j_{I_1} leads to the following bound,

$$\sum_{j_1 > -\infty} M^{j_1\omega^*(G)} \left(\sum_{j_2 \ge j_1} M^{(j_2-j_1)\omega^*(G^{j_2})} \left(\cdots \left(\sum_{j_{I_1} \ge j_{I_1-1}} M^{(j_{I_1} - j_{I_1-1})\omega^*(G^{j_{I_1}})} \right) \cdots \right)\right). \tag{7.38}$$

However, j_{I_1} is fixed, hence this expression must be computed as

$$M^{j_{I_1}\omega^*(G^{j_{I_1}})} \cdot \sum_{j_1 \le j_{I_1}} M^{j_1(\omega^*(G) - \omega^*(G^{j_2}))} \sum_{j_2 = j_1}^{j_{I_1}} M^{j_2(\omega^*(G^{j_2}) - \omega^*(G^{j_3}))} \cdots$$

$$\sum_{j_{I_1-1}=j_{I_1-2}}^{j_{I_1}} M^{j_{I_1-1}(\omega^*(G^{j_{I_1-1}})-\omega^*(G^{j_{I_1}}))}, \tag{7.39}$$

or (integrating from the lowest to the highest scale instead)

$$M^{j_{I_1}\omega^*(G^{j_{I_1}})}\cdot \sum_{j_{I_1-1}<j_{I_1}} M^{j_{I_1-1}(\omega^*(G^{j_{I_1-1}})-\omega^*(G^{j_{I_1}}))} \ldots \sum_{j_2<j_3} M^{j_2(\omega^*(G^{j_2})-\omega^*(G^{j_3}))} \sum_{j_1<j_2} M^{j_1(\omega^*(G)-\omega^*(G^{j_2}))}. \tag{7.40}$$

This is convergent if and only if $\omega^*(G)-\omega^*(G^{j_2})$, $\big(\omega^*(G)-\omega^*(G^{j_2})\big)+\big(\omega^*(G^{j_2})-\omega^*(G^{j_3})\big) = \omega^*(G)-\omega^*(G^{j_3}),\ldots,$ $\omega^*(G)-\omega^*(G^{j(\zeta_1)})$ are >0. This holds true since $\omega^*(G)-\omega^*(G^j)\geq (1-2n\alpha)+1>0$ by eq. (7.36). Hence one gets in the end a bound of order $O(|\zeta_1|^{1-2n\alpha})$.

□

Examples.

1. Consider the first Gallavotti-Nicolò tree of Fig. 7.5. One may choose as integration variables $L(G)\setminus L'(G)=\{\zeta_2,\zeta_3,\zeta_4\}$, so that $\xi_2=\zeta_2-\zeta_3$, $\xi_3=\zeta_3$, $\xi_4=\zeta_4$, $\xi_1=\zeta_1-\zeta_2-\zeta_4$. Hence

$$A(G)=\int d\zeta_2 d\zeta_3 d\zeta_4 \left(|\zeta_2-\zeta_3|^{\frac12-\alpha}|\zeta_3|^{-\frac12-\alpha}\ \cdot\ |\zeta_1-\zeta_2-\zeta_4|^{\frac12-\alpha}|\zeta_4|^{-\frac12-\alpha}.\ \cdot\ \zeta_2^{-1}\right)^2. \tag{7.41}$$

The subdiagrams with lines (ξ_2,ζ_3,ξ_3), (ξ_4,ζ_4,ξ_1) are renormalized by subtracting their value at $\zeta_2=0$, and then the larger subdiagram $(\xi_4,\zeta_4,\xi_1,\zeta_2,\xi_2,\zeta_3,\xi_3)$ is further renormalized by subtracting its value at $\zeta_1=0$. Hence $|\zeta_2-\zeta_3|^{\frac12-\alpha}$ is replaced with $|\zeta_2-\zeta_3|^{\frac12-\alpha}-|\zeta_3|^{\frac12-\alpha}=O(\zeta_2\ \cdot|\zeta_3|^{-\frac12-\alpha})$, and $|\zeta_1-\zeta_2-\zeta_4|^{\frac12-\alpha}$ by

$$\left(|\zeta_1-\zeta_2-\zeta_4|^{\frac12-\alpha}-|\zeta_1-\zeta_4|^{\frac12-\alpha}\right)-\left(|\zeta_2+\zeta_4|^{\frac12-\alpha}-|\zeta_4|^{\frac12-\alpha}\right)=O(\zeta_1\zeta_2\ \cdot\ |\zeta_4|^{-3/2-\alpha}). \tag{7.42}$$

Integrating the square of the renormalized amplitude yields (going *down* the scales *above* ζ_1)

$$\begin{aligned}
&\zeta_1^2\left(\int_{|\zeta_1|}^{\infty}\zeta_2^2 d\zeta_2\left(\int_{|\zeta_2|}^{\infty}|\zeta_4|^{-4-4\alpha}d\zeta_4\left(\int_{|\zeta_4|}^{\infty}|\zeta_3|^{-2-4\alpha}d\zeta_3\right)\right)\right)\\
&\lesssim \zeta_1^2\int_{|\zeta_1|}^{\infty}\zeta_2^2 d\zeta_2\int_{|\zeta_2|}^{\infty}|\zeta_4|^{-5-8\alpha}d\zeta_4\\
&\lesssim \zeta_1^2\int_{|\zeta_1|}^{\infty}|\zeta_2|^{-2-8\alpha}d\zeta_2\\
&=O(|\zeta_1|^{1-8\alpha}).
\end{aligned} \tag{7.43}$$

Note that the exponents are sufficiently *negative* so that these ultra-violet integrals converge.

The computation of the integrals yields the same bound as

$$M^{j(\zeta_1)\omega^*(G)} \sum_{j(\zeta_2)>j(\zeta_1)} M^{(j(\zeta_2)-j(\zeta_1))\omega^*(G^{j(\zeta_2)})} \cdot \\ \cdot \sum_{j(\zeta_4)>j(\zeta_2)} M^{(j(\zeta_4)-j(\zeta_2))\omega^*(G^{j(\zeta_4)})} \sum_{j(\zeta_3)>j(\zeta_4)} M^{(j(\zeta_3)-j(\zeta_4))\omega^*(G^{j(\zeta_3)})}, \tag{7.44}$$

see eq. (7.37), since $\omega^*(G^{j(\zeta_3)}) = (1-4\alpha)-2 = -1-4\alpha$, $\omega^*(G^{j(\zeta_4)}) = -4-8\alpha$ (due to the fact that the subdiagram with lines (ξ_4, ζ_4, ξ_1) is renormalized twice), $\omega^*(G^{j(\zeta_2)}) = (1-8\alpha)-2 = -1-8\alpha$ and $\omega^*(G) = \omega(G) = 1-8\alpha$.

2. Consider now the second Gallavotti-Nicolò tree, see Fig. 7.6. One may choose as integration variables $L(G) \setminus L'(G) = \{\xi_1, \zeta_2, \zeta_3\}$, so that $\zeta_4 = \xi_4 = \zeta_1 - \zeta_2 - \xi_1$, $\xi_2 = \zeta_2 - \zeta_3$, $\xi_3 = \zeta_3$. Hence

$$A(G) = \int d\xi_1 d\zeta_2 d\zeta_3 \left(|\zeta_1 - \zeta_2 - \xi_1|^{-\frac{1}{2}-\alpha} \cdot |\zeta_3|^{-\frac{1}{2}-\alpha} |\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} \cdot |\xi_1|^{\frac{1}{2}-\alpha} \cdot \zeta_2^{-1} \right)^2 . \tag{7.45}$$

The subdiagram with lines $(\zeta_1, \zeta_4, \xi_4)$ has one external ϕ-leg, ξ_1, hence needs not be renormalized. On the other hand, the subdiagrams with lines $(\zeta_1, \zeta_4, \xi_4, \xi_1)$ and (ξ_2, ζ_3, ξ_3) must be renormalized by subtracting their values at $\zeta_2 = 0$. Hence $|\zeta_1 - \zeta_2 - \xi_1|^{\frac{1}{2}-\alpha}$ is replaced with $|\zeta_1 - \zeta_2 - \xi_1|^{-\frac{1}{2}-\alpha} - |\zeta_1 - \xi_1|^{-\frac{1}{2}-\alpha} = O(\zeta_2 \cdot |\zeta_1|^{-\frac{3}{2}-\alpha})$, and $|\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha}$ by $|\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} - |\zeta_3|^{\frac{1}{2}-\alpha} = O(\zeta_2 \cdot |\zeta_3|^{-\frac{1}{2}-\alpha})$.

Integrating the square of the renormalized amplitude yields (going *up* the scales *below* ζ_1)

$$|\zeta_1|^{-3-2\alpha} \left(\int_0^{\zeta_1} |\zeta_3|^{-2-4\alpha} d\zeta_3 \left(\int_0^{\zeta_3} |\xi_1|^{1-2\alpha} d\xi_1 \left(\int_0^{|\xi_1|} \zeta_2^2 d\zeta_2 \right) \right) \right) = O(|\zeta_1|^{1-8\alpha}). \tag{7.46}$$

Note that the exponents are sufficiently *positive* so that these infra-red integrals converge.

In order to make the connection with eq. (7.40), we replace $(|\zeta_1|^{-3/2-\alpha} \zeta_2 . |\xi_1|^{\frac{1}{2}-\alpha})^2 = (|\zeta_1|^{-1/2-\alpha} |\xi_1|^{1/2-\alpha} \cdot \frac{\zeta_2}{\zeta_1})^2$ with $(|\zeta_1|^{-1/2-\alpha} |\xi_1|^{1/2-\alpha} \cdot \frac{\zeta_2}{\xi_1})^2 = |\zeta_1|^{-1-2\alpha} |\xi_1|^{-1-2\alpha} \zeta_2^2$. The reduced spring factor $\frac{\zeta_2}{\xi_1}$ takes into account the difference between the *minimum scale* of the diagram with lines $(\zeta_1, \zeta_4, \xi_4, \xi_1)$ and its external leg ζ_2, corresponding to the lifetime of this diagram; it is the factor which is counted in the multi-scale estimates. The actual spring factor $\frac{\zeta_2}{\zeta_1}$, which is better, is due to the difference of scales between the scale where the vertex connecting ζ_1, ζ_4, ξ_1 and ζ_2 appears and the scale of the external leg ζ_2. With this slight modification, one gets

$$|\zeta_1|^{-1-2\alpha} \int_0^{\zeta_1} |\zeta_3|^{-2-4\alpha} d\zeta_3 \int_0^{\zeta_3} |\xi_1|^{-1-2\alpha} d\xi_1 \int_0^{|\xi_1|} \zeta_2^2 d\zeta_2. \tag{7.47}$$

This is equivalent to the bound given in eq. (7.40),

$$M^{j(\zeta_1)\omega^*(G^{j(\zeta_4)})} \sum_{j(\zeta_3)<j(\zeta_1)} M^{j(\zeta_3)(\omega^*(G^{j(\zeta_3)})-\omega^*(G^{j(\zeta_1)}))} \cdot \sum_{j(\xi_1)<j(\zeta_3)} M^{j(\xi_1)(\omega^*(G^{j(\xi_1)})-\omega^*(G^{j(\zeta_3)}))} \sum_{j(\zeta_2)<j(\xi_1)} M^{j(\zeta_2)(\omega^*(G)-\omega^*(G^{j(\xi_1)}))}, \tag{7.48}$$

since $\omega^*(G^{j(\zeta_4)}) = \omega^*(G^{j(\zeta_1)}) = -1-2\alpha$, $\omega^*(G^{j(\zeta_3)}) = -6\alpha-2$, $\omega^*(G^{j(\xi_1)}) = -8\alpha-2$ and $\omega^*(G) = \omega(G) = 1-8\alpha$.

7.6 Proof of Hölder regularity for renormalized skeleton integrals

We want to prove that, for any indices $(\ell(1),\ldots,\ell(n))$ and $n \le \lfloor 1/\alpha \rfloor$,

$$\mathrm{Var} J_B^{ts}(\ell(1),\ldots,\ell(n)) \lesssim |t-s|^{2n\alpha}. \tag{7.49}$$

Consider some multiple contraction $(i_1 i_2),\ldots,(i_{2p-1}i_{2p})$ – assuming that $\ell(i_1)=\ell(i_2)$, …, $\ell(i_{2p-1}) = \ell(i_{2p})$ – and the associated contracted integral $J_B^{ts}(\mathbb{T};(i_1i_2),\ldots,(i_{2p-1}i_{2p}))$, where $\mathbb{T} = (\ell(1)\ldots\ell(n))$. By arguments which may be found in [54], §4.1,

$$\mathrm{Var} : J_B^{ts}(\mathbb{T};(i_1i_2),\ldots,(i_{2p-1}i_{2p})) :\le n!\ \mathrm{Var} J_B^{ts}(\mathbb{T}';(i_1i_2),\ldots,(i_{2p-1}i_{2p})), \tag{7.50}$$

where $\mathbb{T}'$ has decorations $(\ell'(1)\ldots\ell'(n))$ such that $\ell'(i) \ne \ell'(j)$ if $i \ne j$ *except* if $\{i,j\} = \{i_{2m-1}, i_{2m}\}$ is a pair contraction, as in Lemma 7.2. Hence, by Wick's lemma, it suffices to prove that $\mathrm{Var} J_B^{ts}(\mathbb{T}';(i_1i_2),\ldots,(i_{2p-1}i_{2p})) \lesssim |t-s|^{2n\alpha}$.

By eq. (5.29) and (5.31), $J_B^{ts}(\mathbb{T}';(i_1i_2),\ldots,(i_{2p-1}i_{2p}))$ is a sum of terms of the form $\int d\boldsymbol{\xi} \left[\prod_{q=1}^p \mathcal{R}\Phi^{ts}\right]((\mathbb{T}_q),\boldsymbol{\xi},(\boldsymbol{v}_q),(\mathbb{T}'_{q,j});(i_1i_2),\ldots,(i_{2p-1}i_{2p}))]$, with (following the notations of Lemma 7.2)

$$\prod_{q=1}^p \mathcal{R}\Phi^{ts}(\ .\ ) = \left[\delta\mathcal{R}\overline{\mathrm{Sk\,I}}^{ts} \ \cdot\ \mathcal{R}\overline{\mathrm{Sk\,I}}^{s}\right]_{\hat\nu(\boldsymbol{\xi})} \left(\prod_{q=1}^p \mathrm{Roo}_{\boldsymbol{v}_q}\mathbb{T}_q, \prod_{q=1}^p \prod_j \mathbb{T}'_{q,j};(i_1i_2),\ldots,(i_{2p-1}i_{2p})\right). \tag{7.51}$$

The contractions induce links between some of the trees $(\mathbb{T}_q)_q$, $(\mathbb{T}'_{q,j})_{q,j}$. The resulting connected components may be represented by Feynman graphs of two types: (i) *"rooted" Feynman diagrams* $G_1,\ldots,G_I$ containing some (possibly many) root part $\mathrm{Roo}_{\boldsymbol{v}_q}\mathbb{T}_q$; (ii) *"unrooted Feynman diagrams* $G'_1,\ldots,G'_{I'}$ containing only leaf parts of type $\mathbb{T}'_{q,j}$. It turns out that the unconvenient vertex-decorating characteristic function $\mathbf{1}_{|\xi_1|\le\cdots\le|\xi_n|}$ may be replaced zith the follozing much simpler characteristic function f. Let G_1 be the graph of type (i) containing ξ_1. For every graph G'_i of type (ii), choose some ξ-leg ξ'_i belonging to G'_i and let $f_i := \mathbf{1}_{j(\xi'_i)\ge j(\xi_1)}$. Then set

$f = f^{j(\xi_1)} := \prod_{i=1}^{I'} f_i(\boldsymbol{\xi})$. The integral $\int d\boldsymbol{\xi} \mathbf{1}_{|\xi_1|\le\ldots\le|\xi_n|}$ in (5.29) is now replaced by a simple sum $\sum_{j=-\infty}^{+\infty}$, with $j = j(\xi_1)$, and $\hat{\nu}(\boldsymbol{\xi})$ by a measure depending only on the scale $j(\xi_1)$,

$$\nu^j := \sum_{j'_i \ge j, i=1,\ldots,I'} \int d\xi_1 \ldots d\xi_n \mathbf{1}_{j(\xi_1)=j} \mathbf{1}_{j(\xi'_i)=j'_i} \otimes_{k=1}^n \mathcal{F}(x'(\ell \circ \sigma(i)))(\xi_k). \tag{7.52}$$

Since $f(\boldsymbol{\xi}) \ge \mathbf{1}_{|\xi_1|\ge\ldots\ge|\xi_n|}$, the associated renormalized quantity

$$\sum_{j=-\infty}^{+\infty} \left[\prod_{q=1}^p \mathcal{R}\Phi^{ts} \right] ((\mathbb{T}_q), j; (\boldsymbol{v}_q), (\mathbb{T}'_{q,j}); (i_1 i_2), \ldots, (i_{2p-1} i_{2p})) \tag{7.53}$$

has a larger variance than the orginal one contributing to J_B^{ts}.

The purpose of this section is to prove the estimates

$$\text{Var} \left(\sum_{j=-\infty}^{+\infty} \left[\prod_{q=1}^p \mathcal{R}\Phi^{ts} \right] ((\mathbb{T}_q), j; (\boldsymbol{v}_q), (\mathbb{T}'_{q,j}); (i_1 i_2), \ldots, (i_{2p-1} i_{2p})) \right) \lesssim |t-s|^{2n\alpha}, \tag{7.54}$$

from which Theorem 8.1 follows. They are a simple consequence of Theorem 7.1 and of the following two lemmas.

Lemma 7.10 (bound for "rooted" diagrams) *Let $q \ge 1$ and $q' \ge 0$ such that $n := q + q' < \lfloor 1/\alpha \rfloor$. Rename $(\zeta_{r_1}, \ldots, \zeta_{r_q}, \zeta_{r'_1}, \ldots, \zeta_{r'_{q'}})$, resp. $(\bar{\zeta}_{r_1}, \ldots, \bar{\zeta}_{r_q}, \bar{\zeta}_{r'_1}, \ldots, \bar{\zeta}_{r'_{q'}})$, as $\zeta_1, \ldots, \zeta_n$, resp. $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n$, so that $|\zeta_1| < \ldots < |\zeta_n|$ and $|\tilde{\zeta}_1| < \ldots < |\tilde{\zeta}_n|$, and let $\zeta_{ext} := \sum_{m=1}^n \zeta_m$, $\tilde{\zeta}_{ext} := \sum_{m=1}^n \tilde{\zeta}_m$. Then*

$$I := \int \delta(\zeta_{ext} + \tilde{\zeta}_{ext}) \left(|\zeta_1|^{1-2n\alpha} \prod_{m=1}^{n-1} \left| \frac{\zeta_m}{\zeta_{m+1}} \right| \right)^{\frac{1}{2}} \left(|\tilde{\zeta}_1|^{1-2n\alpha} \prod_{m=1}^{n-1} \left| \frac{\tilde{\zeta}_m}{\tilde{\zeta}_{m+1}} \right| \right)^{\frac{1}{2}} \cdot$$
$$\left(\prod_{m=1}^q \left| \frac{e^{it\zeta_{r_m}} - e^{is\zeta_{r_m}}}{\zeta_{r_m}} \right| \prod_{m'=1}^{q'} \left| \frac{1}{\zeta_{r'_{m'}}} \right| \right) \left(\prod_{m=1}^q \left| \frac{e^{it\bar{\zeta}_{r_m}} - e^{is\bar{\zeta}_{r_m}}}{\bar{\zeta}_{r_m}} \right| \prod_{m'=1}^{q'} \left| \frac{1}{\bar{\zeta}_{r'_{m'}}} \right| \right) \lesssim |t-s|^{2n\alpha}. \tag{7.55}$$

Proof.

(i) Assume to begin with that $|\zeta_k| < \frac{1}{|t-s|} < |\zeta_{k+1}|$ and $|\tilde{\zeta}_{\tilde{k}}| < \frac{1}{|t-s|} < |\tilde{\zeta}_{\tilde{k}+1}|$. Integrate first over the variables *larger* than $\frac{1}{|t-s|}$. Let for instance $|\tilde{\zeta}_n| > |\zeta_n|$. One fixes $|\tilde{\zeta}_n|$ by momentum conservation. Leaving out the spring factors and bounding $\frac{1}{|\tilde{\zeta}_n|}$ by $\frac{1}{|\zeta_n \cdot \tilde{\zeta}_n|^{\frac{1}{2}}}$, one gets

$$\int_{|\tilde{\zeta}_{n-1}|>|\tilde{\zeta}_{n-2}|} \frac{d\tilde{\zeta}_{n-1}}{|\tilde{\zeta}_{n-1}|} \cdot \mathbf{1}_{|\tilde{\zeta}_n|>|\tilde{\zeta}_{n-1}|} \cdot |\tilde{\zeta}_n|^{-\frac{1}{2}} \lesssim |\tilde{\zeta}_{n-2}|^{-\frac{1}{2}} \tag{7.56}$$

and then

$$\int_{|\tilde\zeta_{\tilde k+1}|>\frac{1}{|t-s|}} \frac{d\tilde\zeta_{\tilde k+1}}{|\tilde\zeta_{\tilde k+1}|}\cdots\int_{|\tilde\zeta_{n-2}|>|\tilde\zeta_{n-3}|}\frac{d\tilde\zeta_{n-2}}{|\tilde\zeta_{n-2}|}\ \cdot\ |\tilde\zeta_{n-2}|^{-\frac12}=O(|t-s|^{\frac12}). \tag{7.57}$$

Similarly, using the spring factors this time,

$$|t-s|^{-\frac12}\int_{|\zeta_{k+1}|>\frac{1}{|t-s|}}\frac{d\zeta_{k+1}}{|\zeta_{k+1}|}\cdots\int_{|\zeta_n|>|\zeta_{n-1}|}\frac{d\zeta_n}{|\zeta_n|^{3/2}}\ \cdot\ |\zeta_n|^{-\frac12}=O(|t-s|^{\frac12}). \tag{7.58}$$

Next, integrating over the variables *smaller* than $\frac{1}{|t-s|}$, one gets at most (using simply $|\frac{e^{it\zeta}-e^{is\zeta}}{\zeta}|=O(\frac{1}{|\zeta|})$)

$$|t-s|^{\frac12}\cdot\int_{|\zeta_k|<\frac{1}{|t-s|}}\frac{d\zeta_k}{|\zeta_k|}\cdots\int_{|\zeta_2|<|\zeta_3|}\frac{d\zeta_2}{|\zeta_2|}\int_{|\zeta_1|<|\zeta_2|}\frac{d\zeta_1}{|\zeta_1|}\cdot|\zeta_1|^{(\frac12-n\alpha)+\frac12}=O(|t-s|^{n\alpha-\frac12}) \tag{7.59}$$

since $n\alpha<1$, and similarly for the tilded integrals.

(ii) Assume $\frac{1}{|t-s|}<|\zeta_1|,|\tilde\zeta_1|$. Integrating by the same method over $\tilde\zeta_n,\ldots,\tilde\zeta_2$ or $\zeta_n,\ldots,\zeta_1$ yields $\int_{|\tilde\zeta_1|>\frac{1}{|t-s|}}\frac{d\tilde\zeta_1}{|\tilde\zeta_1|^{3/2}}\ \cdot\ |\tilde\zeta_1|^{\frac12-n\alpha}=O(|t-s|^{n\alpha})$ and $\int_{|\zeta_1|>\frac{1}{|t-s|}}\frac{d\zeta_1}{|\zeta_1|}|\zeta_1|^{(\frac12-n\alpha)-\frac12}=O(|t-s|^{n\alpha})$.

(iii) Assume $\frac{1}{|t-s|}>|\zeta_n|,|\tilde\zeta_n|$. This time one must use the hypothesis that at least one of the ζ-variables, say, ζ_k, is accompanied by the factor $|\frac{e^{it\zeta_k}-e^{is\zeta_k}}{\zeta_k}|=O(|t-s|)$ instead of $O(\frac{1}{|\zeta_k|})$, and similarly for some $\tilde\zeta$-variable, say, $\tilde\zeta_{\tilde k}$. One computes, using the spring factors,

$$\int_{|\tilde\zeta_{\tilde k-1}|<|\tilde\zeta_{\tilde k}|}\frac{d\tilde\zeta_{\tilde k-1}}{|\tilde\zeta_{\tilde k-1}|}\cdots\int_{|\tilde\zeta_2|<|\tilde\zeta_3|}\frac{d\tilde\zeta_2}{|\tilde\zeta_2|}\int_{|\tilde\zeta_1|<|\tilde\zeta_2|}\frac{d\tilde\zeta_1}{|\tilde\zeta_1|}\ \cdot\ |\tilde\zeta_1|^{(\frac12-n\alpha)+\frac12}=O(|\tilde\zeta_{\tilde k}|^{1-n\alpha}) \tag{7.60}$$

and

$$\begin{aligned}|t-s|^{\frac12}\int_{|\tilde\zeta_{n-1}|<|\tilde\zeta_n|}\mathbf{1}_{|\tilde\zeta_n|>|\tilde\zeta_{n-1}|}\ \cdot\ |\tilde\zeta_n|^{-\frac12}\frac{d\tilde\zeta_{n-1}}{|\tilde\zeta_{n-1}|}\cdots\\ \int_{|\tilde\zeta_{\tilde k+1}|<|\tilde\zeta_{\tilde k+2}|}\frac{d\tilde\zeta_{\tilde k+1}}{|\tilde\zeta_{\tilde k+1}|}\int_{|\tilde\zeta_{\tilde k}|<|\tilde\zeta_{\tilde k+1}|}d\tilde\zeta_{\tilde k}|t-s|\ \cdot\ |\tilde\zeta_{\tilde k}|^{1-n\alpha}=O(|t-s|^{n\alpha}),\end{aligned} \tag{7.61}$$

and similarly for the untilded integrals (except that one integrates with respect to ζ_n in the end).

The mixed cases, when e.g $\frac{1}{|t-s|}$ is large with respect to the ζ-variables but small with respect to the $\tilde\zeta$-variables, are treated in the same way and left to the reader.

□

Lemma 7.11 (bound for "unrooted" diagrams) *(same notations as in Lemma 7.10). Assume $q = 0$, so that $n' := q' < \lfloor 1/\alpha \rfloor$. Then*

$$I^{j(\xi_1)} := \int \delta(\zeta_{ext} + \tilde{\zeta}_{ext}) \left(M^{(1-2n'\alpha)j(\xi_1)} \frac{min(|\zeta_1|, M^{j(\xi_1)})}{\max(|\zeta_n|, M^{j(\xi_1)})} \right)^{\frac{1}{2}} \left(M^{(1-2n'\alpha)j(\xi_1)} \prod_{m=1}^{q-1} \left| \frac{\tilde{\zeta}_m}{\tilde{\zeta}_{m+1}} \right| \right)^{\frac{1}{2}} \cdot \left(\prod_{m=1}^{q} \left| \frac{1}{\zeta_m} \right| \right) \left(\prod_{m=1}^{q} \left| \frac{1}{\tilde{\zeta}_m} \right| \right) \lesssim M^{-2n\alpha j(\xi_1)}. \tag{7.62}$$

Proof.

Integrate first down to scale $j(\xi_1)$. Let for instance $|\tilde{\zeta}_{n'}| > |\zeta_{n'}|$. One fixes $|\tilde{\zeta}_{n'}|$ by momentum conservation. Leaving out the spring factors and bounding $\frac{1}{|\tilde{\zeta}_{n'}|}$ by $\frac{1}{|\zeta_{n'} \cdot \tilde{\zeta}_{n'}|^{\frac{1}{2}}}$, one gets, exactly as in the proof of the preceding lemma, see eq. (7.56, 7.57),

$$\int_{|\tilde{\zeta}_{\tilde{k}+1}| > M^{j(\xi_1)}} \frac{d\tilde{\zeta}_{\tilde{k}+1}}{|\tilde{\zeta}_{\tilde{k}+1}|} \cdots \int_{|\tilde{\zeta}_{n'-2}| > |\tilde{\zeta}_{n'-3}|} \frac{d\tilde{\zeta}_{n'-2}}{|\tilde{\zeta}_{n'-2}|} \int_{|\tilde{\zeta}_{n'-1}| > |\tilde{\zeta}_{n'-2}|} \frac{d\tilde{\zeta}_{n'-1}}{|\tilde{\zeta}_{n'-1}|^{3/2}} = O(M^{-\frac{j(\xi_1)}{2}}) \tag{7.63}$$

and similarly for the untilded integrals.

Next, integrating up to scale $j(\xi_1)$, see eq. (**??**),

$$M^{-\frac{j(\xi_1)}{2}} \int_{|\zeta_k| < M^{j(\xi_1)}} \frac{d\zeta_k}{|\zeta_k|} \cdots \int_{|\zeta_2| < |\zeta_3|} \frac{d\zeta_2}{|\zeta_2|} \int_{|\zeta_1| < |\zeta_2|} \frac{d\zeta_1}{|\zeta_1|} \cdot |\zeta_1|^{\frac{1}{2}} = O(1) \tag{7.64}$$

and similarly for the tilded integrals.

□

We may now easily finish the proof. Lemma 7.11 yields an estimate for renormalized skeleton integrals associated to "unrooted" diagrams G, where the scale $j(\xi')$ of some ξ-leg ξ' has been fixed. Summing over all scales $j(\xi') \geq j(\xi_1)$ yields $O(M^{-2n'\alpha j(\xi_1)})$. Then the product of the factors $\prod_{i=1}^{I'} M^{-2n'_i \alpha j(\xi_1)}$ associated to all unrooted diagrams $(G'_i)_{i=1,\ldots,I'}$ is inserting in the left-hand side of eq. (7.33).

Chapter 8

First quantum field theoretic construction

Let us explain our strategy for $1/6 < \alpha < 1/4$. Roughly speaking, our rough path is obtained by making $B = (B(1), B(2))$ interact through a weak but singular quartic, non-local interaction, which plays the rôle of a squared kinetic momentum, or *bending energy*, and makes its Lévy area – and at the same time the iterated integrals of higher order – *finite*. Following the common use of quantum field theory, this is implemented by multiplying (probabilists would say: *penalizing*) the Gaussian measure by the exponential weight $e^{-\frac{1}{2}c'_\alpha \int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2}$ [1], with

$$\mathcal{L}_{int}(\phi_1, \phi_2)(t_1, t_2) = \lambda^2 \left\{ (\partial\mathcal{A}^+)(t_1)(\partial\mathcal{A}^+)(t_2) + (\partial\mathcal{A}^-)(t_1)(\partial\mathcal{A}^-)(t_2) \right\}, \tag{8.1}$$

where: λ (the coupling parameter) is a small, positive constant; ϕ_1, ϕ_2 are the (infra-red divergent) stationary fields associated to B_1, B_2, with covariance kernel as in eq. (9.4), and similarly, $\mathcal{A}^\pm$ are stationary *left- and right-turning fields*, built out of ϕ_1, ϕ_2 and representing the singular part of the Lévy area (see section 1 for details). As usual in quantum field theory, one considers first the truncated measure obtained by an "ultra-violet cut-off" and on a finite "volume" (or finite horizon, in the probabilistic terminology) $V = [-T, T]$, i.e. one multiplies the Fourier transforms of the fields ϕ_1, ϕ_2 by some cut-off function with compact support in $[-M^\rho, M^\rho]$ (for some fixed constant $M > 1$) and integrates over V; see Definition **??** for the precise procedure. Then $\partial\mathcal{A}^\pm$ are replaced by the *truncated quantities* $(\partial\mathcal{A}^\pm)^{\to\rho}$ built out of the *truncated fields* $\phi^{\to\rho}$. The truncated interacting Lagrangian reads

$$\begin{aligned}
&\frac{1}{2}c'_\alpha \int\int_{V\times V} |t_1 - t_2|^{-4\alpha} \mathcal{L}_{int}^{\to\rho}(\phi_1, \phi_2)(t_1, t_2) dt_1 dt_2 + \int_V \mathcal{L}_{bdry}^{\to\rho} \\
&\quad := \frac{1}{2}c'_\alpha \lambda^2 \int\int_{V\times V} |t_1 - t_2|^{-4\alpha} \left\{ (\partial\mathcal{A}^+)^{\to\rho}(t_1)(\partial\mathcal{A}^+)^{\to\rho}(t_2) \right. \\
&\qquad \left. + (\partial\mathcal{A}^-)^{\to\rho}(t_1)(\partial\mathcal{A}^-)^{\to\rho}(t_2) \right\} dt_1 dt_2 + \int_V \mathcal{L}_{bdry}^{\to\rho},
\end{aligned}$$

[1]The unessential constant c'_α is fixed e.g. by demanding that the Fourier transform of the kernel $c'_\alpha|t_1 - t_2|^{-4\alpha}$ is the function $|\xi|^{4\alpha-1}$.

(8.2)

where $\mathcal{L}_{bdry}^{\to\rho}$ is some singular "Fourier boundary term" multiplied by an evanescent factor $M^{-\kappa\rho}$ ($\kappa > 0$), which cures unwanted difficulties due to the ultra-violet cut-off [2]. When ρ and V are finite, the underlying Gaussian fields are smooth, which ensures the existence of the penalized measure. The assertion is that *the penalized measures converge weakly when* $\rho, |V| \to \infty$ *to some well-defined, unique measure*, while the *truncated iterated integrals* themselves converge in law to a *rough path over* B.

Note that the statistical weight is maximal when $\partial\mathcal{A}^+ = \partial\mathcal{A}^- = 0$, i.e. for sample paths which are "essentially" straight lines. Another way to motivate this interaction (following an image due to A. Lejay) is to understand that the divergence of the Lévy area is due to the accumulation in a small region of space of small loops [30]; the statistical weight is unfavorable to such an accumulation. On the other hand, the law of the quantities in the first-order Gaussian chaos, characterized by the n-point functions

$$\begin{aligned}&\langle B_{i_1}(x_1)\dots B_{i_n}(x_n)\rangle_\lambda \\ &\quad = \frac{1}{Z}\mathbb{E}\left[B_{i_1}(x_1)\dots B_{i_n}(x_n) e^{-\frac{1}{2}c'_\alpha \int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2}\right],\end{aligned} \tag{8.3}$$

$i_1,\dots,i_n = 1,2$, where

$$Z := \mathbb{E}\left[e^{-\frac{1}{2}c'_\alpha \int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2}\right] \tag{8.4}$$

is a normalization constant playing the rôle of a *partition function*, is insensitive to the interaction [3]. Thus we have built a rough path *over fBm.* This conveys the idea that the paths have been straightened by removing *in average* small bubbles of scale $M^{-\rho}$. In doing so, the paths of the limiting process when $\rho \to \infty$ are indistinguishable from those of B, *but* higher-order integrals have been corrected so as to become finite.

Starting from the above field-theoretic description, the proof of finiteness and Hölder regularity of the Lévy area for $\lambda > 0$ small enough follows, despite some specific features, the broad scheme of constructive field theory, see e.g. the monographies [1, 41, 48].

The main theorem may be stated as follows. As a rule, we denote in this section and the following one by $\mathbb{E}[...]$ the Gaussian expectation and by $\langle ... \rangle_{\lambda,V,\rho}$ the expectation with respect to the λ-weighted interaction measure with scale ρ ultraviolet cut-off restricted to a compact interval V, so that in particular $\mathbb{E}[...] = \langle ... \rangle_{0,\infty}$.

[2]The exact form of $\mathcal{L}_{bdry}^{\to\rho}$ requires detailed constructive explanations and will not be required here. It is to be found in the companion article [38].

[3]In the two preceding equations, $\mathbb{E}\left[\,\cdot\, e^{-\frac{1}{2}c'_\alpha \int\int \mathcal{L}_{int}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2}\right]$ stands for the limit of $\mathbb{E}\left[\,\cdot\, e^{-\frac{1}{2}c'_\alpha \int\int_{V\times V} \mathcal{L}_{int}^{\to\rho}(\phi_1,\phi_2)(t_1,t_2)|t_1-t_2|^{-4\alpha}dt_1dt_2 + \int_V \mathcal{L}_{bdry}^{\to\rho}}\right]$ when $\rho, |V| \to \infty$ as we explained above.

Theorem 8.1 *Assume $\alpha \in (\frac{1}{6}, \frac{1}{4})$. Consider for $\lambda > 0$ small enough the family of probability measures (also called: $(\phi, \partial\phi, \sigma)$-*model*)*

$$\mathbb{P}_{\lambda,V,\rho}(\phi_1,\phi_2) = e^{-\frac{1}{2}c'_\alpha \int\int dt_1 dt_2 |t_1-t_2|^{-4\alpha} \mathcal{L}_{int}^{\to\rho}(\phi_1,\phi_2)(t_1,t_2) - \int \mathcal{L}_{bdry}^{\to\rho}} d\mu^{\to\rho}(\phi_1) d\mu^{\to\rho}(\phi_2), \tag{8.5}$$

*where $d\mu^{\to\rho}(\phi_i) = d\mu(\phi_i^{\to\rho})$ is a Gaussian measure obtained by an ultra-violet cut-off at Fourier momentum $|\xi| \approx M^\rho$ ($M > 1$), see Definition **??**. Then $(\mathbb{P}_{\lambda,V,\rho})_{V,\rho}$ converges in law when $|V|, \rho \to \infty$ to some measure $\mathbb{P}_\lambda$, and the associated iterated integrals*

$$\int_s^t d\phi_{i_1}^{\to\rho}(t_1) \int_s^{t_1} d\phi_{i_2}^{\to\rho}(t_2), \ldots, \int_s^t d\phi_{i_1}^{\to\rho}(t_1) \int_s^{t_1} d\phi_{i_2}^{\to\rho}(t_2) \ldots \int_s^{t_{n-1}} d\phi_{i_n}^{\to\rho}(t_n), \ldots$$

converge in law to a rough path over B.

Let us return to our model. Using a straightforward extension of the above Proposition, one may represent $\langle \psi_1(x_1) \ldots \psi_n(x_n) \rangle_\lambda$, $\psi = \phi$ or σ as a sum over Feynman diagrams, $\sum_x A(\Gamma)$, where Γ ranges over a set of diagrams with n external legs, and $A(\Gamma) \in \mathbb{R}$ is the evaluation of the corresponding diagram (see examples below); connected expectations will then be obtained as a sum over connected Feynman diagrams. More precisely, one obtains formally a (diverging) power series in λ, $\sum_{n\geq 0} \lambda^n \sum_{\Gamma_n} A(\Gamma_n)$, where Γ_n ranges over the set of Feynman diagrams with n vertices. The Gaussian integration by parts formula [4] yields a so-called *Schwinger-Dyson identity*,

$$\begin{aligned}
\langle \partial\mathcal{A}^\pm(x) \partial\mathcal{A}^\pm(y) \rangle_\lambda &= -\frac{1}{\lambda^2 Z(\lambda)} \mathbb{E}\left[\frac{\delta}{\delta\sigma_+(y)} \frac{\delta}{\delta\sigma_+(x)} e^{-\int \mathcal{L}_{int}(\phi_1,\phi_2,\sigma_+)(t)dt} \right] \\
&= -\frac{1}{\lambda^2 Z(\lambda)} \mathbb{E}\left[(C_{\sigma_+}^{-1}\sigma_+)(y) \frac{\delta}{\delta\sigma_+(x)} e^{-\int \mathcal{L}_{int}(\phi_1,\phi_2,\sigma_+)(t)dt} \right] \\
&= -\frac{1}{\lambda^2} \left[-C_{\sigma_+}^{-1}(x,y) + \langle (C_{\sigma_+}^{-1}\sigma_+)(x)(C_{\sigma_+}^{-1}\sigma_+)(y) \rangle_\lambda \right],
\end{aligned} \tag{8.6}$$

with Fourier transform

$$\langle |\mathcal{F}(\partial\mathcal{A}^\pm)(\xi)|^2 \rangle_\lambda = \frac{1}{\lambda^2} |\xi|^{1-4\alpha} \left[1 - |\xi|^{1-4\alpha} \langle |(\mathcal{F}\sigma_+)(\xi)|^2 \rangle_\lambda \right]. \tag{8.7}$$

By parity, $\langle |\mathcal{F}(\partial\mathcal{A}^\pm)(\xi)|^2 \rangle_\lambda$ is a power series in λ^2.

Introduce an ultra-violet cut-off at scale ρ as in Definition **??**. For the simplicity of the exposition we shall actually use a brute-force ultraviolet cut-off at momentum M^ρ, i.e. cut off all Fourier components with momentum $|\xi| > M^\rho$. After Fourier transformation, $\int \mathcal{L}_{int}(\cdot; t)dt$

[4] an infinite-dimensional extension of the well-known formula for Gaussian vectors, $\mathbb{E}[\partial_{X_i} F(X_1, \ldots, X_n)] = \sum_j C^{-1}(i,j) \mathbb{E}[X_j F(X_1, \ldots, X_n)]$ if C is the covariance matrix of $(X_1, \ldots, X_n)$.

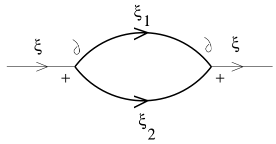


Figure 8.1: Bubble diagram with 2 vertices. By momentum conservation $\xi = \xi_1 + \xi_2$, which leaves out one free internal momentum.

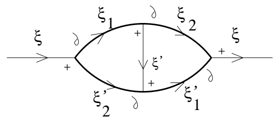


Figure 8.2: More complicated bubble diagram with 4 vertices. By momentum conservation $\xi = \xi_1 + \xi'_2 = \xi'_1 + \xi_2$ and $\xi_1 = \xi' + \xi_2$, which leaves out two independent internal momenta.

becomes $\mathrm{i}\lambda \int_{|\xi_1|<|\xi_2|} d\xi_1 d\xi_2 d\xi \delta_0(\xi_1+\xi_2+\xi)\mathcal{F}\sigma_+(\xi)\mathcal{F}(\partial\phi_1)(\xi_1)\mathcal{F}\phi_2(\xi_2)$, minus a similar term involving σ_-. The square of this expression contributes the following term of order $O(\lambda^2)$ to $\langle|\mathcal{F}\sigma_+(\xi)|^2\rangle_\lambda$,

$$(-\mathrm{i}\lambda)^2 \int_{|\xi_1|<|\xi-\xi_1|}^{M^\rho} d\xi_1 \left\{ \left(\mathbb{E}[|\mathcal{F}\sigma_+(\xi)|^2]\right)^2 \ \mathbb{E}[|\mathcal{F}(\partial\phi_1)(\xi_1)|^2] \ \mathbb{E}[|\mathcal{F}\phi_2(\xi-\xi_1)|^2] \right\}$$
$$= -\lambda^2|\xi|^{8\alpha-2} \int_{|\xi_1|<|\xi-\xi_1|}^{M^\rho} d\xi_1 |\xi_1|^{1-2\alpha}|\xi-\xi_1|^{-1-2\alpha} \sim_{\rho\to\infty} -K\lambda^2|\xi|^{8\alpha-2}(M^\rho)^{1-4\alpha}. \tag{8.8}$$

This is the evaluation of the *Feynman diagram represented* in Fig. 8.1, according to the following rules.

Definition 8.1 (Feynman rules) *A Feynman diagram in our theory is made up of (1)* bold lines *of type $i = 1,2$, with momenta $\xi_i, \xi'_i, \ldots$ evaluated as $\mathbb{E}|\mathcal{F}\phi_i(\xi_i)|^2 = \frac{1}{|\xi_i|^{1+2\alpha}}$; (2)* plain lines *of type $\pm$, with momenta $\xi, \xi', \ldots$, evaluated as $\mathbb{E}|\mathcal{F}\sigma_\pm(\xi)|^2 = \frac{1}{|\xi|^{1-4\alpha}}$; (3) vertices where two plain lines – one of each type – and a bold line meet, with a momentum conservation rule, $\xi = \pm\xi_1 \pm \xi_2$ (depending on the orientation of the lines). The definition of the interaction implies the presence of a further derivation – represented by the symbol ∂ on the Feynman diagram – on the ϕ_1-, resp. ϕ_2-field, and a momentum scale restriction $|\xi_1| < |\xi_2|$, resp. $|\xi_1| > |\xi_2|$, at vertices involving a σ_+-, resp. σ_--field. The derivation translates into a multiplication by $\mathrm{i}\xi_1$, resp. $\mathrm{i}\xi_2$ when evaluating the diagram.*

It is sometimes useful to consider the evaluation of the corresponding *amputated Feynman diagram*, from which the contribution of the external legs has been removed. Here for instance, the evaluation of the amputated Feynman diagram associated to Fig. 8.1 is $(|\xi|^{1-4\alpha})^2$ times the previous expressions, hence is equivalent to the ξ-independent expression $-K\lambda^2(M^\rho)^{1-4\alpha}$ when $\rho \to \infty$. It is a diverging *negative* quantity. (Using the Fourier truncation of Definition **??** only

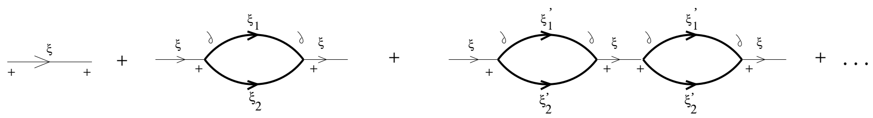


Figure 8.3: First three terms of the bubble series. The renormalized covariance of the σ-field is equal to the sum of the series.

changes the constant K.) However, resumming *formally* the bubble series as in Fig. 8.3 yields, starting from the right-hand side of eq. (7.32),

$$\frac{1}{\lambda^2}|\xi|^{1-4\alpha}\left[1-\sum_{n\geq 0}(-1)^n\left(\frac{1}{|\xi|^{1-4\alpha}}\cdot K\lambda^2(M^\rho)^{1-4\alpha})\right)\right]$$
$$= \frac{1}{\lambda^2}|\xi|^{1-4\alpha}\cdot\frac{K\lambda^2(M^\rho/|\xi|)^{1-4\alpha}}{1+K\lambda^2(M^\rho/|\xi|)^{1-4\alpha}}$$
$$\to_{\rho\to\infty}\frac{1}{\lambda^2}|\xi|^{1-4\alpha}. \tag{8.9}$$

On the other hand (see Fig. 8.3), the bare σ-covariance $\frac{1}{|\xi|^{1-4\alpha}}$ has been replaced with the renormalized covariance

$$\frac{1}{|\xi|^{1-4\alpha}}\cdot\frac{1}{1+K\lambda^2(M^\rho/|\xi|)^{1-4\alpha}}=\frac{1}{|\xi|^{1-4\alpha}+K\lambda^2(M^\rho)^{1-4\alpha}}, \tag{8.10}$$

which vanishes in the limit $\rho\to\infty$. The essential reason for this is of course that the oscillating signs $(-1)^n$ in the bubble series evaluation – due to the fact that the interaction Lagrangian $\mathcal{L}_{int}(\phi_1,\phi_2,\sigma)$ is purely *imaginary* – result by summing in a huge, virtually infinite denominator. Taking into account the possible insertion of σ_--lines between σ_+-lines amounts to a simple change of the constant K. In physical terms, the interaction in $\frac{1}{|\xi|^{1-4\alpha}}$ has been *screened* by a *huge mass term* $K\lambda^2 M^{\rho(1-4\alpha)}\to_{\rho\to\infty}+\infty$ (see section 3 for the definition of the mass). More complicated diagrams contributing to $\langle|(\mathcal{F}\sigma_+)(\xi)|^2\rangle_\lambda$, and involving internal σ-lines as in Fig. 8.2 also vanish when $\rho\to\infty$. Thus there remains simply:

$$\langle|\mathcal{F}\mathcal{A}^\pm(\xi)|^2|\rangle_\lambda=\frac{1}{\lambda^2}|\xi|^{-1-4\alpha}. \tag{8.11}$$

Hence $\mathbb{E}|\mathcal{A}^\pm(t)-\mathcal{A}^\pm(s)|^2\lesssim\frac{1}{\lambda^2}|t-s|^{4\alpha}$, as in eq. (9.13).

As for the mixed term $\langle\partial\mathcal{A}^\pm(x)\partial\mathcal{A}^\mp(y)\rangle_\lambda$, its Fourier transform is given by $\frac{1}{\lambda^2}|\xi|^{1-4\alpha}\left[-\frac{1}{1+K''\lambda^2(\Lambda/|\xi|)^{1-4\alpha}}\right]$, where $K''<K$ due to the constraints on the scales for bubbles of mixed type with one σ_+- and

one σ_--leg, which vanishes in the limit $\rho \to \infty$ (note the disappearance of the factor 1 compared to eq. (8.9), due to the fact that $\mathbb{E}\sigma^+(x)\sigma^-(y) = 0$). Thus the covariance of the two-component σ-field has been renormalized to $\frac{1}{|\xi|^{1-4\alpha}\mathrm{Id}+m^\rho}$, where m^ρ is a two-by-two positive "mass" matrix with eigenvalues $\approx \lambda^2 M^{\rho(1-4\alpha)}$.

Using eq. (9.11), one obtains:

$$\begin{aligned}(2\pi c_\alpha)^2\langle \mathcal{A}(s,t)^2\rangle_\lambda &= \langle \left|\mathcal{A}^+(t)-\mathcal{A}^+(s)\right|^2\rangle_\lambda + \langle \left|\mathcal{A}^-(t)-\mathcal{A}^-(s)\right|^2\rangle_\lambda \\ &\qquad + \mathbb{E}\left|\mathcal{A}^+_\partial(s,t)-\mathcal{A}^-_\partial(s,t)\right|^2 \\ &= \frac{4}{\lambda^2}\int (1-\cos(t-s)\xi)|\xi|^{-1-4\alpha}d\xi + \mathbb{E}\left|\mathcal{A}^+_\partial(s,t)-\mathcal{A}^-_\partial(s,t)\right|^2 \\ &= (\frac{4}{\lambda^2}K_1+K_2)|t-s|^{4\alpha} \qquad (8.12)\end{aligned}$$

for some constants K_1, K_2.

Let us now consider briefly other correlations. For a general discussion we need the following easy power-counting lemma:

Lemma 8.2 (power-counting rules) *Let Γ be a Feyman diagram with N_σ external σ-lines, N_ϕ external ϕ-lines, and $N_{\partial\phi}$ external $\partial\phi$-lines. Then the overall degree of homogeneity (in powers of ξ) of the evaluation of the corresponding amputated diagram – also called:* overall degree of divergence *– is $1-2\alpha N_\sigma+\alpha N_\phi+(\alpha-1)N_{\partial\phi}$.*

Proof. Let: I_σ, resp. I_ϕ, be the number of *internal* lines of type σ, resp. ϕ or $\partial\phi$; $I = I_\sigma + I_\phi$ be the total number of internal lines; and $L = I - V + 1$ be the number of loops, equal to the number of independent momenta (one per internal line, minus one per vertex due to momentum conservation, plus one due to overall momentum conservation). Since one σ- and two ϕ-lines meet at each vertex, one also has the relations $2I_\sigma + N_\sigma = V$, and $2I_\phi + N_\phi + N_{\partial\phi} = 2V$. Now the amputated diagram is homogeneous to $|\xi|^{-(1-4\alpha)I_\sigma-(1+2\alpha)I_\phi+L+V-N_{\partial\phi}}$ (counting one derivative per vertex, and minus one derivative per *external* $\partial\phi$-leg which is not taken into account in the evaluation). Putting all these relations together yields the result. □

If a diagram is overall divergent, i.e. if its overall degree of divergence is positive, then the diagram diverges (except if by chance the coefficient of the term of highest degree in ξ vanishes). On the other hand, the fact that a diagram is overall convergent (i.e. its overall degree of divergence is negative) does not imply that it is convergent, since it may contain overall divergent *sub*-diagrams. One must hence study the behaviour of all possible diagrams, with arbitrary external leg structure.

The above simple power-counting argument shows that the overall degree of divergence of a connected diagram with $2n$ external σ-legs is $1-4n\alpha$. For $n \geq 2$, this is $\leq 1-8\alpha < 0$ since $\alpha > \frac{1}{8}$ by hypothesis, so such diagrams are overall convergent. By the above arguments, there remain only the connected diagrams in the limit $\Lambda \to \infty$, see Fig. 8.4, whose evaluation is independent of λ.

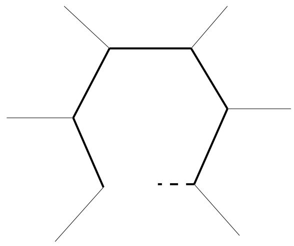


Figure 8.4: Higher connected moments of the Lévy area.

General considerations following from the multi-scale expansions (one may refer to [58] for a good, accessible presentation, or to [53] for an application to the Gaussian renormalization of iterated integrals evoked in subsection 2.3) show that it is enough to consider the behaviour of diagrams whose internal legs have higher (or even: much higher) momentum scale than external legs, the so-called *dangerous diagrams.* Then the momentum scale constraint on the vertices coming from Fourier normal ordering implies that the external legs of dangerous diagrams may be either of type σ or of type $\partial\phi$, but not of type ϕ. Consider now any diagram whose external structure contains external $\partial\phi$-legs. By parity it has at least two such external legs, and the previous power-counting rules show that such a diagram is always overall convergent.

Finally, the law of the field ϕ is left unchanged by the interaction. Namely, all non-trivial diagrams contributing e.g. to $\langle\phi_1(x)\phi_2(x)\rangle_\lambda$ involve internal σ-lines which (as previously "shown") vanish in the limit $\rho \to \infty$.

On the whole, this is the content of Theorem 8.1.

The art of constructive field theory is to make the previous speculations rigorous. It relies on the following considerations, corresponding to the weak points (not to say flaws!) in the above arguments:

1. While going from eq. (8.8) to (8.9), we have replaced the amputated bubble diagram evaluation by its asymptotics when $\rho \to \infty$, namely, $-K\lambda^2 M^{\rho(1-4\alpha)}$, which is simply equal to its *evaluation at zero external momentum* ξ, also called *local part.* Thus we have actually not resummed the whole bubble series, but only the corresponding local parts, and observed that this was equivalent to adding a mass term of the form $K'\lambda^2 M^{\rho(1-4\alpha)} \int |\sigma(x)|^2 dx$ to the Lagrangian.

2. The bubble series is really a terribly *diverging geometric series.* Renormalization must actually be performed scale by scale. Considering only bubble diagrams with momentum in the dyadic slice $M^{\rho-1} < |\xi| < M^\rho$ leads on the other hand to a converging geometric series for λ small enough since the term between parentheses in eq. (8.9), $K\lambda^2 \left(\frac{M^\rho}{|\xi|}\right)^{1-4\alpha}$, is then < 1. This is equivalent to integrating out the highest field components (σ^ρ, ϕ^ρ), as explained in section 3. One obtains thus a running mass coefficient m^ρ of order $\lambda^2 M^{(1-4\alpha)\rho}$. The procedure must then be iterated by going down the scales step by step. Since renor-

malization *reduces* the covariance of the σ-field, the bound on λ ensuring convergence does not become worse and worse after each step.

3. We neglected more complicated bubble diagrams as in Fig. 8.2. Although these have the same order as the simple bubble diagram of Fig. 8.1, as follows from the above power-counting rules, taking into consideration *all possible* bubble diagrams lead to a terribly diverging power series in λ due to the rapidly increasing number of such diagrams in terms of the number of vertices, with a coefficient roughly of order $n!$ in front of λ^n. This divergence is actually due to the accumulation of vertices in a small region of space of size $O(M^{-j})$, where j is the momentum scale under consideration. *Multi-scale cluster expansions* in constructive field theory, by considering only *partial* series expansions, avoid this dangerous accumulation process.

4. By splitting each vertex $\int \mathcal{L}_{int}^{\to\rho}(\cdot;x)dx$ into its different scales, there may appear fields $\phi_1^{j_1}, \phi_2^{j_2}, \sigma^j$ with different scales $j_1 \neq j_2 \neq j$. Taking this into account in a coherent way in the previous partial series expansions lead to complicated combinatorial expressions encoded by so-called *polymers*, which are the main object in use in constructive field theory.

5. In the previous vertex splitting, the field with lowest momentum scale (j_1, j_2 or j, depending on the case) is called *low-momentum field*. Even though the cluster expansion in each momentum scale prevents an accumulation of vertices in the same region of space, the compound effect of *all* cluster expansions at *all* scales produces unavoidably accumulations of fields with very low momentum in very large regions of space, which is a dangerous problem called *domination problem*. This accounts for the addition of the extra boundary term $\mathcal{L}_{bdry}^{\to\rho}$ in the interaction Lagrangian. Writing out this term and explaining its precise form would however take us too far away.

Chapter 9

Appendix: a Fourier analysis of the Lévy area for fBm ($\alpha < 1/4$)

The quantity we want to define in the case of fractional Brownian motion is the following.

Definition 9.1 (Lévy area) *The Lévy area of a two-dimensional path* $\Gamma : \mathbb{R} \to \mathbb{R}^2$ *between* s *and* t *is the area between the straight line connecting* $(\Gamma_1(s), \Gamma_2(s))$ *to* $(\Gamma_1(t), \Gamma_2(t))$ *and the curve* $\{(\Gamma_1(u), \Gamma_2(u)); s \le u \le t\}$*. It is given by the following antisymmetric quantity,*

$$\mathcal{LA}_\Gamma(s,t) := \int_s^t d\Gamma_1(t_1) \int_s^{t_1} d\Gamma_2(t_2) - \int_s^t d\Gamma_2(t_2) \int_s^{t_2} d\Gamma_1(t_1). \tag{9.1}$$

The purpose of this section is to show by using Fourier analysis why the Lévy area of fBm diverges when $\alpha \le 1/4$. This is hopefully understandable to physicists, and also profitable to probabilists who are aware of other proofs of this fact, originally proved in [11], because Fourier analysis is essential in the analysis of Feynman graphs which shall be needed in section 4. We follow here the computations made in [54] or [53].

Definition 9.2 (Harmonizable representation of fBm) *Let* $W(\xi), \xi \in \mathbb{R}$ *be a complex Brownian motion* [1] *such that* $W(-\xi) = -\overline{W(\xi)}$*, and*

$$B_t := (2\pi c_\alpha)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}t\xi} - 1}{\mathrm{i}\xi} |\xi|^{\frac{1}{2}-\alpha} dW(\xi), \quad t \in \mathbb{R}. \tag{9.2}$$

The field $B_t, t \in \mathbb{R}$ is called *fractional Brownian motion* [2] . Its paths are almost surely α^- Hölder, i.e. $(\alpha - \varepsilon)$-Hölder for every $\varepsilon > 0$. It has dependent but identically distributed (or in

[1] Formally, $\langle W'(\xi_1) W'(\xi_2) \rangle = 0$ and $\langle W'(\xi_1) \overline{W'(\xi_2)} \rangle = \delta(\xi_1 - \xi_2)$ if $\xi_1, \xi_2 > 0$.
[2] The constant c_α is conventionally chosen so that $\mathbb{E}(B_t - B_s)^2 = |t-s|^{2\alpha}$.

other words, stationary) increments $B_t - B_s$. In order to gain translation invariance, we shall rather use the closely related *stationary process*

$$\phi(t) := \int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi} |\xi|^{\frac{1}{2}-\alpha} dW(\xi), \quad t \in \mathbb{R} \tag{9.3}$$

– with covariance

$$\langle \phi(x)\phi(y) \rangle = \int e^{\mathrm{i}\xi(x-y)} \frac{1}{|\xi|^{1+2\alpha}} d\xi \tag{9.4}$$

– which is *infrared divergent*, i.e. *divergent around* $\xi = 0$. However, the increments $\phi(t) - \phi(s) = B_t - B_s$ are well-defined for any $(s,t) \in \mathbb{R}^2$.

In order to understand the analytic properties of the Lévy area of fBm, we shall resort to a Fourier transform. One obtains, using the harmonizable representation of fBm,

$$\begin{aligned} \mathcal{A}(s,t) &:= \int_s^t dB_1(t_1) \int_s^{t_1} dB_2(t_2) \\ &= \frac{1}{2\pi c_\alpha} \int \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \int_s^t dt_1 \int_s^{t_1} dt_2 \ \cdot \ e^{\mathrm{i}(t_1\xi_1 + t_2\xi_2)}. \end{aligned} \tag{9.5}$$

The Lévy area $\mathcal{LA}(s,t) := \mathcal{LA}_B(s,t)$ is obtained from this twice iterated integral by anti-symmetrization. Note that $\mathcal{LA}(s,t)$ is homogeneous of degree 2α in $|t-s|$ since $B(ct) - B(cs)$, $c > 0$ has same law as $c^\alpha (B(t) - B(s))$ by self-similarity.

Expanding the right-hand side yields an expression which is not homogeneous in ξ. Hence it is preferable to define instead the following stationary quantity called *skeleton integral*, which depends only on *one* variable,

$$\begin{aligned} \mathcal{A}(t) &:= \int^t dB_1(t_1) \int^{t_1} dB_2(t_2) \\ &= \frac{1}{2\pi c_\alpha} \int \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \int^t dt_1 \int^{t_1} dt_2 \ \cdot \ e^{\mathrm{i}(t_1\xi_1 + t_2\xi_2)} \\ &= \frac{1}{2\pi c_\alpha} \int \frac{dW_1(\xi_1) dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2} |\xi_2|^{\alpha-1/2}} \cdot \frac{e^{\mathrm{i}t(\xi_1+\xi_2)}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_2]}, \end{aligned} \tag{9.6}$$

where by definition $\int^t e^{\mathrm{i}u\xi} du = \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi}$. From $\mathcal{A}(t)$ and the one-dimensional skeleton integral

$$\phi_i(t) = (2\pi c_\alpha)^{-\frac{1}{2}} \int^t dB_i(u) = \int \frac{dW_i(\xi)}{|\xi|^{\alpha-1/2}} \ \cdot \ \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi}, \tag{9.7}$$

which is the above-defined infra-red divergent stationary process associated to B, one easily retrieves $\mathcal{A}(s,t)$ since

$$\begin{aligned} \mathcal{A}(s,t) &= \int_s^t dB_1(t_1) \left(\int^{t_1} dB_2(t_2) - \int^s dB_2(t_2) \right) \\ &= \mathcal{A}(t) - \mathcal{A}(s) + \mathcal{A}_\partial(s,t), \end{aligned} \tag{9.8}$$

where $(2\pi c_\alpha)^{\frac{1}{2}}\mathcal{A}_\partial(s,t) := (B_1(t)-B_1(s))\phi_2(s)$ (called *boundary term*) is a *product of first-order integrals.*

One may easily estimate these quantities in each sector $|\xi_1| \gtrless |\xi_2|$. In practice, it turns out that estimates are easiest to get *after* a permutation of the integrals (applying Fubini's theorem) such that (for twice or multiple iterated integrals equally well) *innermost (or rightmost) integrals bear highest Fourier frequencies*; this is the essence of *Fourier normal ordering* [55, 15, 56]. This gives a somewhat different decomposition with respect to (9.8) since $\int_s^t dB_1(t_1)\int_s^{t_1} dB_2(t_2)$ is rewritten as $-\int_s^t dB_2(t_2)\int_t^{t_2} dB_1(t_1)$ in the "negative" sector $|\xi_1| > |\xi_2|$. After some elementary computations, one gets the following.

Lemma 9.3 *Let*

$$\begin{aligned} \mathcal{A}^+(t) &:= 2\pi c_\alpha \int^t dt_1 \int^{t_1} dt_2 \mathcal{F}^{-1}\left((\xi_1,\xi_2)\mapsto \mathbf{1}_{|\xi_1|<|\xi_2|}(\mathcal{F}B_1')(\xi_1)(\mathcal{F}B_2')(\xi_2)\right)(t_1,t_2) \\ &= \int_{|\xi_1|<|\xi_2|} \frac{dW_1(\xi_1)dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2}|\xi_2|^{\alpha-1/2}} \cdot \frac{e^{\mathrm{i}t(\xi_1+\xi_2)}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_2]} \end{aligned} \tag{9.9}$$

and

$$\begin{aligned} \mathcal{A}^-(t) &:= 2\pi c_\alpha \int^t dt_2 \int^{t_2} dt_1 \mathcal{F}^{-1}\left((\xi_1,\xi_2)\mapsto \mathbf{1}_{|\xi_2|<|\xi_1|}(\mathcal{F}B_1')(\xi_1)(\mathcal{F}B_2')(\xi_2)\right)(t_1,t_2) \\ &= \int_{|\xi_2|<|\xi_1|} \frac{dW_1(\xi_1)dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2}|\xi_2|^{\alpha-1/2}} \cdot \frac{e^{\mathrm{i}t(\xi_1+\xi_2)}}{[\mathrm{i}(\xi_1+\xi_2)][\mathrm{i}\xi_1]}. \end{aligned} \tag{9.10}$$

Then

$$\mathcal{A}(s,t) = \frac{1}{2\pi c_\alpha}\left\{(\mathcal{A}^+(t)-\mathcal{A}^+(s)) - (\mathcal{A}^-(t)-\mathcal{A}^-(s)) + (\mathcal{A}_\partial^+(s,t)-\mathcal{A}_\partial^-(s,t))\right\}, \tag{9.11}$$

the boundary term $\mathcal{A}_\partial^+ - \mathcal{A}_\partial^-$ *being given by*

$$\begin{aligned} \mathcal{A}_\partial^+(s,t) - \mathcal{A}_\partial^-(s,t) = \Big\{ &-\int_{|\xi_1|<|\xi_2|} \frac{(e^{\mathrm{i}t\xi_1}-e^{\mathrm{i}s\xi_1})e^{\mathrm{i}s\xi_2}}{[\mathrm{i}\xi_1][\mathrm{i}\xi_2]} \\ &+\int_{|\xi_2|<|\xi_1|} \frac{(e^{\mathrm{i}t\xi_2}-e^{\mathrm{i}s\xi_2})e^{\mathrm{i}t\xi_1}}{[\mathrm{i}\xi_1][\mathrm{i}\xi_2]}\Big\} \cdot \frac{dW_1(\xi_1)dW_2(\xi_2)}{|\xi_1|^{\alpha-1/2}|\xi_2|^{\alpha-1/2}}. \end{aligned} \tag{9.12}$$

Two lines of computations show immediately that

$$\begin{aligned} \mathrm{Var}\mathcal{A}_\partial^\pm(s,t) &\lesssim \int |e^{\mathrm{i}t\xi}-e^{\mathrm{i}s\xi}|^2|\xi|^{-1-4\alpha}d\xi \\ &\lesssim \int_{|\xi|>\frac{1}{|t-s|}} \frac{d\xi}{|\xi|^{1+4\alpha}} + \int_{|\xi|<\frac{1}{|t-s|}} \frac{|t-s|^2|\xi|^2}{|\xi|^{1+4\alpha}}d\xi \\ &\lesssim |t-s|^{4\alpha}, \end{aligned} \tag{9.13}$$

so that (essentially by the Kolmogorov-Centsov lemma, see section 2) the Hölder regularity indices of B_1 and B_2 add in the case of the boundary term, to produce a quantity which is $2\alpha^-$-Hölder. (Note that the artificial infrared divergence at $\xi_1 = 0$ disappears when Taylor expanding $e^{it\xi_1} - e^{is\xi_1}$). On the other hand, letting $\xi := \xi_1 + \xi_2$ and introducing an ultra-violet cut-off at $|\xi_2| = \Lambda \gg 1$, one may see for instance $\mathcal{A}^+(t)$ as an inverse random Fourier transform of the integral $\xi \mapsto \int_{|\xi-\xi_2|<|\xi_2|}^{\Lambda} \frac{dW_2(\xi_2)}{\xi_2} \frac{1}{|\xi-\xi_2|^{\alpha-1/2}|\xi_2|^{\alpha-1/2}}$, whose variance diverges like $\int^{\Lambda} \frac{d\xi_2}{\xi_2^{4\alpha}} = O(\Lambda^{1-4\alpha})$ or $O(\ln\Lambda)$ in the ultra-violet limit $\Lambda \to \infty$ as soon as $\alpha \le 1/4$. Note that the ultraviolet divergence is in the region $|\xi_1|, |\xi_2| \gg |\xi|$.

It is apparent that the central rôle in this decomposition is played by the Fourier projection operator $D(\mathbf{1}_{|\xi_1|<|\xi_2|}) = \mathcal{F}^{-1}\left(\mathbf{1}_{|\xi_1|<|\xi_2|} \cdot \mathcal{F}(\ .\ )\right)$. Since $\mathcal{A}^{\pm}_{\partial}$ are obtained by Fourier projecting $(B_1(t)-B_1(s))\phi_2(s)$, or $(B_2(t)-B_2(s))\phi_1(t)$, which are perfectly well-defined products of continuous fields [3], it was clear from the onset that these would be regular terms. Hence singularities come only from the *one-time quantity* $\mathcal{A}^{\pm}(t)$, which *does not* split into a product of first-order integrals. This quantity is called the *singular part of the Lévy area* in section ???.

[3] apart from the spurious infra-red divergence (see above)

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