An introduction to rough path theory
At the crossroads between stochastic calculus and renormalization theory

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Abstract: Stochastic differential equations (sde's) driven by white noise, i.e. by a noise which is decorrelated in time, arise in many areas of physics and engineering, and have been widely studied since the time of Itô. Solutions are known to be semi-martingales, to which the standard tools of stochastic calculus – in particular, the competing Itô and Stratonovich stochastic integrations – apply. Considering now stochastic partial differential equations, or sde's driven by more irregular processes, one needs larger and larger excursions outside the realm of classical stochastic calculus in order to define solutions. Rough path theory, introduced by T. Lyons in 1998, has been developed as a general, geometric framework for defining integration along irregular paths, allowing e.g. (far from straightforward) extensions of such concepts as hypoellipticity or ergodicity from the case of classical diffusion equations to the case of sde's driven by irregular, colored noises. The general idea is that an integration procedure relies on a coherent choice of a finite number of iterated integrals of the noise, called *rough path*.

Difficulties of a new type appear when the regularity index of the driving process goes below the threshold $\alpha = 1/4$; for such wildly oscillating processes, iterated integrals depend strongly on the highest frequency Fourier components, leading to divergences which have a natural reinterpretation as Feynman diagram ultra-violet divergences. Renormalization and constructive field theory, blended with probabilistic and combinatorial tools, give a satisfactory answer to these problems.

The purpose of this short course is to give an up-to-date review of the main features and applications of rough path theory, and to give a glimpse of the wide variety of concepts and tools encountered in the field, ranging from control theory and sub-Riemannian geometry to Gaussian processes, Hopf algebras, and finally quantum field theory. The first part is meant to be a self-contained introduction to the subject, relying mostly on elementary computations and some Lie group theoretic arguments. The second part will concentrate on quantum field theoretic developments.