Statistical Mechanics arising from Random Matrix Theory

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Tom Spencer
Institute for Advanced Study
Princeton, NJ

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Introduction - Duality

Many problems in mathematical physics are best understood using by using a "dual" representation.

2D Ising $\Rightarrow$ Free Fermions, Quadratic action in Grassmann

2D Ising spin-glass $\Rightarrow$ random Dirac

2D X-Y, $s_j \in \mathbb{S}^1 \Rightarrow$ Coulomb Gas, charges $\approx$ vortices

If P(N) is the number of partitions of the integer N then

$$P(N) = \frac{1}{2\pi i} \oint \frac{1}{\prod_m (1 - z^m)} z^{-N-1} dz \approx \frac{1}{4\sqrt{3}N} e^{\pi\sqrt{2N/3}}$$

Spectral properties of Random Matrices are dual to (SUSY) Statistical Mechanics

Gaussian Band Matrices: $H = H^*$, $\Lambda = \Lambda_L$ finite box,

$$\langle H_{ij} \rangle = 0 \quad \langle H_{ij} \, \bar{H}_{i'j'} \rangle = J_{i,j} \; \delta_{i,i'} \, \delta_{j,j'} \quad i,j \in \Lambda \cap Z^d$$

Where

$$J_{i,j} = [-W^2 \Delta + 1]^{-1}(i,j) \approx \frac{e^{-|i-j|/W}}{W} \quad in \; 1D$$

In 1D, if W=L we have classical GUE matrix ensemble.

For $d \geq 1, L \gg 1$, Want to Understand:

- Fine distribution of eigenvalues (density of states)
- Behavior of eigenstates (localized or extended)
- Universality of eigenvalue spacing distribution, Wigner-Dyson
- Time evolution, quantum transport,
- Phase transition, Multifractal behavior of eigenstates

This information in contained in the Green's function:

$$G_\Lambda(E + i\epsilon; j, k) = [E + i\epsilon - H_\Lambda]^{-1}(j, k)$$

Averages of **G** can be written in terms of SUSY Statistical Mechanics.

Example $\langle Det\ G \rangle$, H Band Matrix

Let $\phi(j) \in \mathbb{R}$, $j \in \Lambda \cap \mathbb{Z}^d$,

$$\langle Det\ G_\Lambda \rangle = C_\Lambda \int e^{-[\phi \cdot (-W^2 \Delta + 1)\, \phi]} \prod_{j \in \Lambda} \frac{d\phi(j)}{E + i\epsilon - \phi(j)}$$

Derivation uses: $\det^{-1} M = \int e^{-\bar{z} \cdot M z} \prod_j dz_j\, d\bar{z}_j / \pi \quad z_j \in \mathbb{C}$

Saddle Point in Mean field: $\phi = E/2 - i\sqrt{1 - (E/2)^2}$

Deform contour of integration. Re Hessian > 0 .

Density of states $\rho(E)$

To calculate $\rho_\Lambda(E) = \frac{1}{\pi} Im\langle G_\Lambda(E + i\epsilon)\rangle,\ as\, \epsilon \downarrow 0$
Need ratio of determinants - Grassmann variables, SUSY

Theorem: For Gaussian Random Band matrices in 3D and $|E| \leq 1.8\, and\ W \gg 1$, density of states is **smooth** in E :

$$\rho(E) = \frac{1}{\pi} Im\langle G_\Lambda(E + i\epsilon)(0,0)\rangle$$

as $\epsilon \downarrow 0\, and\, \Lambda \uparrow \mathbb{Z}^3$. Aymptotic exapnsion in $1/W$

Constantinescu, Felder, Gawedzki, Kupiainen,
and Disertori, Pinson, Sp.

Hyperbolic Symmetry $\langle |\det G_\Lambda(E+i\epsilon)|^2\rangle$

Hyperbolic sigma model: Spins take values in hyperbolic space $U(1,1)/U(1)\times U(1)$

$$h_j=(x_j,y_j,z_j),\quad h\cdot h'\equiv xx'+yy'-zz',\quad h_j\cdot h_j=-1$$

$$\langle |\det G_\Lambda(E+i\epsilon)|^2\rangle \approx \int e^{-\beta\sum_{j\sim k} h_j\cdot h_k-\epsilon\sum z_j}\prod_{j\in\Lambda} d\mu(h_j)$$

where $\beta\approx W^2\rho^2(E)$

Theorem (Sp-Zirnbauer) In 3D this model has no phase transition. It is effectively convex for all β. Like a massless Free Field.

In Horospherical coordinates, $t_j, s_j \in \mathbb{R}$

$$A_\Lambda(\beta,\epsilon) = \beta \sum_{j\sim j'} \{ \, cosh(t_j - t_{j'}) + \frac{1}{2}(s_j - s_{j'})^2 \, e^{t_j + t_{j'}} \}$$

$$+\epsilon \sum_j [cosh t_j + s_j^2 e^{t_j}]$$

Define $D_{\beta,\epsilon}(t)$ by the quadratic form:

$$[s; D_{\beta,\epsilon}(t)s] = \sum_{j\sim k} \beta (s_j - s_k)^2 e^{t_j + t_k} + \epsilon \sum_j e^{t_j} s_j^2$$

Note $e^{t_j + t_k}$ is the local conductance across the edge $j \sim k$.

If $t \equiv 0$ then $D = -\beta\Delta + \epsilon$

Effective action - integrate over s

$$A_\Lambda(t) = \beta \sum_{j\sim k} cosh(t_j - t_k) + \frac{1}{2} \ln det[D_{\beta,\epsilon}(t)] - \sum_j t_j$$

Key observation:

$\ln det[D_{\beta,\epsilon}(t)]$ **is convex in t** $\Rightarrow$ **no transition**

Pair Correlation:

$$\langle y_0 y_k \rangle = \langle s_0 e^{t_0} s_k e^{t_k} \rangle = \langle D_{\beta,\epsilon}(t)^{-1}(0,k) e^{t_0+t_k} \rangle$$

Brascamp-Lieb gives uniform bounds on conductance in 3D since $Hess\, A \geq -\beta\Delta$.

3D SUSY Hyperbolic model

To get the SUSY hyperbolic sigma model, flip the red signs.

Convexity fails, however the partition function $Z(\beta, \epsilon) \equiv 1$

Theorem (Disertori, Sp, Zirnbauer) In 3 Dimensions the SUSY hyperbolic sigma model has a phase transition.
If $\beta > 0$ is small: exponential decay

$$\langle y_0 y_k \rangle \approx \epsilon^{-1} e^{-|k|/\ell} \quad \ell = \textit{localization length}$$

If $\beta \gg 1$ diffusion, $\langle e^{\pm p\, t_j} \rangle$ bounded hence

$$\langle y_0 y_k \rangle = \langle D_{\beta,\epsilon}(t)^{-1}(j,k) e^{t_j + t_k} \rangle \approx [-\bar{\beta}\Delta + \epsilon]^{-1}(j,k)$$

Equivalence of SUSY and History dependent Walk Sabot and Tarres

W. Werner: Vertex Reinforced Jump Process (VRJP) with jump rate β: Let $L_j(t)$ denote the amount of time walk has spent at j up to time t. At time t the walk jumps to a nearest neighbor site j at rate $\beta(1 + L_j(t))$. What happens to walk after long time?

1D strips , exponentially localized (Disertori-Merkl-Rolles).

For β small it is localized, $d \geq 1$

In 3D for large β it is transient.

Transition for 3D Edge Reinforced Random Walk (Disertori-Sabot-Tarres)

Conjectures for Hyperbolic SUSY models

In 2D exponential localization holds for all β.

VRJP and ERRW interpretation: If the walk starts at 0 and $\epsilon_j = \delta_{0,j}$ then $\langle e^{t_j/2} \rangle \approx e^{-|j|/\ell}$, $\ell \approx e^{\beta}$ Conductance $\Rightarrow 0$

Remark: $\langle e^{t_j} \rangle = 1$. In fact $\langle e^{qt_j} \rangle = \langle e^{(1-q)t_j} \rangle$

Theorem (Merkl-Rolles): for 2D ERRW $\langle e^{t_j/2} \rangle \leq C|j|^{-1/\beta}$.

In 3D there is a multi-fractal transition at β_c. This a reflects multi-fractal nature of the wavefunction at the mobility edge E_c, $\langle e^{\alpha t_0} \rangle (\beta_c, \epsilon)$ as $\epsilon \downarrow 0$

Universality of Mean Field Theory for $D \geq 3$?

Consider **O(n) invariant** interacting spins:
eg. X-Y model O(2), Heisenberg O(3)

Let $s_j \in \mathbb{R}^n$, $j \in \Lambda_L \cap Z^d$, periodic box of side L, $h \in R^n$

Conjecture:

$$\langle e^{\sum_{j\in\Lambda} h\cdot s_j/|\Lambda|}\rangle(\beta) \ as \ L \Rightarrow \infty$$

$$= \int_{S^{n-1}} e^{h\cdot S_0 M(\beta)} d\mu(S_0) \times [1 + O(\frac{1}{\beta L^{d-2}})]$$

$M(\beta) =$ Magnetization, $d\mu(S_0)$ is uniform measure on S^{n-1} .

The leading term is like a **Law of Large Numbers** and the correction is **CLT.**

Theorem (J. Fröhlich and T.S.) Holds for **O(2) symmetric systems**

Main ideas: Pure states are parmetrized by S^1, Fröhlich-Pfister, $+$ IR bounds.

Explanation of Conjectured Universality of Wigner-Dyson statistics in 3D:

Apply SUSY statistical mechanics, (Kravstov and Mirlin (1994))

$$s_j \in U(1,1|2)/U(1|1) \times U(1|1), \quad M(\beta) \approx \rho(E), \, DOS$$

Conclusion: Wigner Dyson should describe RBM and Random Schrödinger in 3 Dim in the extended region

T. Sherbina: Example with SU(2) symmetry:

Let H be an $N \times N$ Gaussian matrix such that

$$\langle H_{ij} \bar{H}_{i'j'} \rangle = \frac{e^{-|i-j|/W}}{W} \delta_{i,i'} \delta_{j,j'} \quad 1 \leq i, j \leq N$$

Define

$$F_N(E, u) = \frac{\langle \det(H - E + u/N) \det(H - E - u/N) \rangle}{\langle \det(H - E)^2 \rangle}$$

If the width $W^2 \gg N \gg 1$ then

$$F_N(E, u) \Rightarrow \int_{S^2} e^{iu\rho(E) S_0^{(3)}} d\mu = \frac{\sin(2u\rho(E))}{2u\rho(E)}$$

Main Idea

In 1D, F(E,u) is mapped onto classical Heisenberg spin chain of length N, $\beta = \rho(E)^2 W^2, \quad s_j \in S^2$

In one dimension the chain is ordered if $N \ll \beta \approx W^2$

For $d \geq 3$ it is ordered for $\beta \geq 1$. Hence we expect the same Mean field result to hold for fixed W and large N.

Happy Birthday Antti