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Weakly non-linear resonant hamiltonian PDEs and the problem of wave turbulence

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1 Introduction: wave turbulence (WT)

(One of) origines: Rudolf Peierls, in *Annalen der Physik* **3** (1929).

Modern state of affairs see in

[ZLF] Zakharov, Lvov, Falkovich, *Kolmogorov Spectra of Turbulence*, Springer 1992.

[Naz] S. Nazarenko, *Wave Turbulence*, Springer 2011.

The method of WT applies to various equations. E.g., to NLS:

Deterministic setting. Consider NLS equation with small nonlinearity:

$$\text{(NLS)} \qquad \dot{u} - i\Delta u + \varepsilon^2\, i|u|^2 u = 0, \qquad x \in \mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d), \quad 0 < \varepsilon \ll 1$$

(ε is the amplitude of small oscillations). Take the exponential basis $\{e^{i\mathbf{k}\cdot x}, \mathbf{k} \in \mathbb{Z}^d/L =: \mathbb{Z}^d_L\}$. The exponents are eigenfunctions of the Laplacian:

$$-\Delta e^{i\mathbf{k}\cdot x} = \lambda_{\mathbf{k}} e^{i\mathbf{k}\cdot x}; \quad \lambda_{\mathbf{k}} = |\mathbf{k}|^2\,, \quad \mathbf{k} \in \mathbb{Z}^d_L\,.$$

So there is plenty of exact resonances in the spectrum of the linear system, corresponding to $\varepsilon = 0$. – This is a prerequisite for WT.

The WT studies what happens to solutions of (NLS) when $t \gg 1, \varepsilon \ll 1, L \gg 1$.

It is not quite clear in what order we send $\varepsilon \to 0$ and $L \to \infty$, and how large is t in terms of ε and L.

So it may be better to talk not about the limit of WT, but about WT limits:

Physicists, working on the WT, insist that "some stochasticity" is crucial for the phenomena which they study to occur (cf. the title of the paper above). Often they simply postulate that the randomness "self-appear" in the equations in the regime of WT. Instead of doing that, we will follow the approach of V. Zakharov and V.L'vov,

V. Zakharov, V.L'vov, in *Radiophys. Quant. Electronics (1975)*,

also see in

Cardy, Falkovich, Gawedzki "Non-Equilibrium Stat. Phys. and Turbulence", CUP 2008,

and introduce the randomness to the equation by means of a small random force. The force brings to the system extra energy, and to dissipate it some damping has to be added. Accordingly we will consider small solutions of NLS equation with small random force and small damping :

(ZL) $\quad \dot{u} - i\Delta u + \varepsilon^2 \rho\, i|u|^2 u = -\nu(-\Delta+1)^p u + \sqrt{\nu}\,\langle \text{rand. force}\rangle, \quad x \in \mathbb{T}_L^d,$

where $\varepsilon, \nu \ll 1$, $p \in \mathbb{N}$ (e.g., $p = 1$) and ρ is a constant or order one; its role will become clear later.

Here ν – inverse time-scale of the forced oscillations; ε – amplitude of small oscillations. Some relation between ν and ε should be imposed, corresponding to the physics of the problem which we model with this equation.

As before, we use the exponential basis $\{e^{i\mathbf{k}\cdot x}, \mathbf{k} \in \mathbb{Z}^d/L =: \mathbb{Z}_L^d\}$, and assume that

Random Force is

$$\sum b_{\mathbf{k}} \frac{d}{dt}\boldsymbol{\beta}^{\mathbf{k}}(t) e^{i\mathbf{k}\cdot x}, \quad b_{\mathbf{k}} > 0 \text{ and } b_{\mathbf{k}} \to 0 \text{ fast}, \quad \mathbf{k} \in \mathbb{Z}_L^d\,,$$

where $\{\boldsymbol{\beta}^{\mathbf{k}}(t)\}$ – independent standard complex Wiener processes.

$\diamond$ We have to impose some restrictions on p and d to make (ZL) well posed. E.g., $p = 1, \quad d \le 3$ (if $p > 1$, then d may be bigger than 3).

Fact: As $t \to \infty$, solutions of (ZL) converge in distribution to a stationary measure $\mu_{\varepsilon,\nu,L}$ of the equation (which is a "statistical equilibrium of the equation"):

$$(*) \qquad \mathcal{D}u(t) \rightharpoonup \mu_{\varepsilon,\nu,L} \qquad \text{as } t \to \infty$$

($\mu_{\varepsilon,\nu,L}$ is a measure in the function space $\{u(x)\}$). Moreover, if $\mathcal{D}(u(0)) = \mu_{\varepsilon,\nu,L}$, then $\mathcal{D}u(t) \equiv \mu_{\varepsilon,\nu,L}$. Such solutions are called *stationary solutions*..

For the time being, to guarantee $(*)$ we have to strengthen our restrictions on (ZL) (in terms of the imposed relation between p and d).

In the Z-L setting people usually study not solutions $u(t)$, but the stationary measures $\mu_{\varepsilon,\nu,L}$. The big advantage is that in this case we should not worry about the time-scale and the meaning of "$t \gg 1$".

Similar to what people (starting Peiels) do in the deterministic case, to study eq. (ZL), Zakharov - L'vov do the following:

◇ Write the equation in the Fourier presentation. That is, Z-L decompose $u(t,x)$ as $u=\sum_{\mathbf{k}\in\mathbb{Z}_L^d} u_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot x}$, and write (ZL) as

$$\dot{u}_{\mathbf{k}} + i\lambda_{\mathbf{k}} u_{\mathbf{k}} = -\varepsilon^2 \rho\, i \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3} - \nu(\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + \sqrt{\nu}\, b_{\mathbf{k}} \dot{\beta}^{\mathbf{k}}(t)\,,$$

$\mathbf{k}\in\mathbb{Z}_L^d$. Note that the nonlinear term $i\sum u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3}$ is hamiltonian, with the Hamiltonian function

$$\mathcal{H}^4 = \frac{1}{4} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4} u_{\mathbf{k}_1} u_{\mathbf{k}_2} \bar{u}_{\mathbf{k}_3} \bar{u}_{\mathbf{k}_4}.$$

◇ They study solutions and the stationary measure $\mu_{\varepsilon,\nu,L}$ under the limit

$$\varepsilon,\ \nu \to 0, \quad L\to\infty.$$

Same remark as before has to be made concerning the limits $\varepsilon,\nu\to 0$ and $L\to\infty$: it is unclear in which order they should be taken.

As it is customary in WT, Zakharov-L'vov study solutions when $\varepsilon,\nu\to 0,\ L\to\infty$, by replacing "everywhere" sums $\sum_{\mathbf{k}\in\mathbb{Z}_L^d}$ by integrals $\int_{\mathbf{k}\in\mathbb{R}^d}$.

DEFINITION. The energy spectrum for a solution $u(t,x)$ is the function

$$\mathbb{Z}_L^d \ni \mathbf{k} \mapsto E_{\mathbf{k}} = \tfrac{1}{2}\langle |u_{\mathbf{k}}(t)|^2 \rangle \,,$$

where $\langle \cdot \rangle$ stands for a suitable averaging (in ensemble and, maybe, in something else).

One of the main goals for Z-L (as well as for the whole WT community) is to study the energy spectrum and prove that it has the *Kolmogorov-Zakharov form*:

(KZ spectrum) $$E_{\mathbf{k}} \sim |\mathbf{k}|^{-\varkappa}, \qquad \varkappa > 0 \,,$$

provided that $|\mathbf{k}|$ "belongs to the inertial range". I.e., in some sense, $1 \ll |\mathbf{k}| \ll \infty$.

In my research I tie the small parameters ε and ν by choosing

$$\varepsilon^2 = \nu$$

– this is within the bounds, usually imposed in physics (cf. [Naz]). Then I pass to the slow time $\tau = \nu t$:

$$\text{(ZL)} \qquad u_\tau - i\nu^{-1}\Delta u + i\rho|u|^2 u = -(-\Delta + 1)^p u + \langle \text{rand. force} \rangle', \quad x \in \mathbb{T}_L^d.$$

This is the equation I will discuss, mostly following my work with Alberto Maiocchi, who is now a post-doc in Paris.

We suggest to study the WT limits (at least, some of them) by splitting the limiting process $\nu \to 0,\ L \to \infty$ in two steps:

Step I): prove that when $\nu \to 0$, main characteristics of solutions $u^{\nu,L}$ have limits of order one, described by certain well posed *effective equation.*

Step II): Show that main characteristics of solutions for the effective equation have non-trivial limits of order one when $L \to \infty$, if $\rho = \rho(L)$ is a suitable function of L.

Step I has been done rigorously, and I will discuss it below.

I stress that it is unclear if the results of Step I along can justify the predictions of WT since it is often believed in the WT that (KZ spectrum) cannot hold when the period L is fixed and finite.

Next I will show that a heuristic argument *a-la* WT with a suitable choice of the function $\rho(L)$ allows to justify Step II and leads under the limit $L \to \infty$ to a Kolmogorov-Zakharov type kinetic equation and a (KZ spectrum).

It may be that in physics this is called the *Litvak-Hasselman approach.*

2 Averaging for PDEs without resonances

(or **"weak turbulence without the weak turbulence"**). In my works

$[KP1]$ SK $\&$ A.Piatnitski, "Khasminsii - Whitham averaging for randomly perturbed KdV", JMPA (2008);

$[K2]$ SK, "Damped-driven KdV and effective equations...", GAFA (2010);

$[K3]$ SK, "Weakly nonlinear stochastic CGL..", Ann. Inst. Fourier - PR, 2013.

I studied the long-time behaviour of solutions for perturbed hamiltonian PDE without strong resonances, for $L = 1$. Namely, in [KP1,K2] I consider equations like

$$\dot{u} - iu_{xx} + i|u|^2 u = \nu(u_{xx} - u) + \sqrt{\nu}\,\langle \text{rand. force}\rangle, \quad x \in S^1,$$

and in [K3] – equations like

$$(*) \quad \dot{u} + i(-\Delta + V(x))u + i\nu|u|^2 u = -\nu(-\Delta + 1)^p u + \sqrt{\nu}\,\langle \text{rand. force}\rangle, \quad x \in \mathbb{T}^d,$$

where $V(x)$ is such that there are no resonances in the spectrum of $-\Delta + V(x)$.

The key idea was suggested in [K2] – describe the long-time behaviour of the actions in the perturbed equation using certain auxiliary Effective Equation, which is a well posed quasilinear SPDE with a non-local nonlinearity.

For equations $(*)$ without resonances, the Effective Equation is linear and does not depend on the Hamiltonian term $\nu i|u|^2u$. In this case the limit $\nu \to 0$, $L \to \infty$ DOES NOT lead to the predictions of the WT.

REMARK. If the nonlinearity is not hamiltonian, then the effective equation is nonlinear.

3 Averaging for PDEs with resonances

Now the new results.

[KM1] SK & A. Maiocchi, Preprint, arXiv 1311.6793

[KM2] SK & A. Maiocchi, Preprint, arXiv 1311.6794.

We apply the method of [KP1-K3] to the equation of Zakharov-L'vov with $\varepsilon^2 = \nu$, written using the slow time $\tau = \nu t$:

$$\text{(ZL)} \qquad u_\tau - i\nu^{-1}\Delta u + i\rho|u|^2 u = -(-\Delta+1)^p u + \langle \text{rand. force}\rangle', \quad p \in \mathbb{N}\,.$$

We write $u(\tau,x) = \sum_{\mathbf{k}\in\mathbb{Z}_L^d} u_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot x}$, and re-write the equation in Fourier:

$$\frac{d}{d\tau}u_{\mathbf{k}} + i\lambda_{\mathbf{k}}\nu^{-1}u_{\mathbf{k}} = -i\rho \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} u_{\mathbf{k}_1}u_{\mathbf{k}_2}\bar{u}_{\mathbf{k}_3} - (\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau)$$

where $\mathbf{k} \in \mathbb{Z}_L^d$. We wish to control the asymptotic behaviour of the actions $\frac{1}{2}|u_{\mathbf{k}}|^2(\tau)$ and other characteristics of solutions via suitable effective equation.

The Effective Equation for (ZL) may be derived through the *interaction representation*, i.e. by transition to the fast rotating variables $a_{\mathbf{k}}$:

$$a_{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau} u_{\mathbf{k}}(\tau), \qquad \mathbf{k} \in \mathbb{Z}_L^d$$

(this substitution simply is *the variation of constant*). Note that

(*) $$|a_{\mathbf{k}}(\tau)| \equiv |u_{\mathbf{k}}(\tau)|.$$

In the a-variables the (ZL) equation reeds

$$\frac{d}{d\tau} a_{\mathbf{k}} = -\,(\lambda_{\mathbf{k}}+1)^p a_{\mathbf{k}} + b_{\mathbf{k}}\, e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau} \frac{d}{d\tau}\beta^{\mathbf{k}}(\tau)$$
$$- i\rho \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \bar{a}_{\mathbf{k}_3} \exp\left(-i\nu^{-1}\tau(\lambda_{\mathbf{k}_1}+\lambda_{\mathbf{k}_2}-\lambda_{\mathbf{k}_3}-\lambda_{\mathbf{k}})\right) .$$

The terms, constituting the nonlinearity, oscillate fast as ν goes to zero, unless the sum of the eigenvalues in the second line vanishes. So only the terms for which this sum equals zero contribute to the limiting dynamics as $\nu \to 0$. These terms are called *resonant.*

The processes

$$e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau}\frac{d}{d\tau}\beta^{\mathbf{k}}(\tau)\,,\quad \mathbf{k}\in\mathbb{Z}_L^d,$$

also are standard independent complex white noises. Accordingly, the effective equation should be the following damped/driven hamiltonian system

(Eff.Eq.) $$\frac{d}{d\tau}v_{\mathbf{k}} = -(\lambda_{\mathbf{k}}+1)^p v_{\mathbf{k}} - R_{\mathbf{k}}(v) + b_{\mathbf{k}}\frac{d}{d\tau}\tilde{\beta}^{\mathbf{k}}(\tau),\quad \mathbf{k}\in\mathbb{Z}_L^d\,,$$

where $R_{\mathbf{k}}(v)$ is the resonant part of the hamiltonian nonlinearity:

$$R_{\mathbf{k}}(v) = i\rho \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}\\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}|^2}} v_{\mathbf{k}_1}v_{\mathbf{k}_2}\bar{v}_{\mathbf{k}_3}\,,$$

and $\{\tilde{\beta}^{\mathbf{k}}(\tau)\}$ is the new set of standard independent white noises.

It is easy to see that the resonant part of the hamiltonian nonlinearity $R(v)$ is the hamiltonian vector field $R = i\rho\nabla\mathcal{H}^4_{\text{res}}$, where $\mathcal{H}^4_{\text{res}}$ is the resonant part of the Hamiltonian $\mathcal{H}^4$:

$$\mathcal{H}^4_{\text{res}} = \frac{1}{4} \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4 \\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}_4|^2}} v_{\mathbf{k}_1} v_{\mathbf{k}_2} \bar{v}_{\mathbf{k}_3} \bar{v}_{\mathbf{k}_4}.$$

The (Eff.Eq) is well posed; I will discuss this later.

$$(ZL) \qquad \frac{d}{d\tau}u_{\mathbf{k}} + i\lambda_{\mathbf{k}}\nu^{-1}u_{\mathbf{k}} = -i\rho\nabla_{\mathbf{k}}\mathcal{H}^4(u) - (\lambda_{\mathbf{k}}+1)^p u_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau)$$

$$(EffEq) \qquad \frac{d}{d\tau}v_{\mathbf{k}} = -i\rho\nabla_{\mathbf{k}}\mathcal{H}^4_{\text{res}}(v) - (\lambda_{\mathbf{k}}+1)^p v_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k}\in\mathbb{Z}^d_L.$$

Actions of a solution $u^\nu(\tau)$ are

$$I^\nu_{\mathbf{k}}(\tau) = \tfrac{1}{2}|u^\nu_{\mathbf{k}}(\tau)|^2, \qquad \mathbf{k}\in\mathbb{Z}^d_L.$$

Theorem 1. Let $\{u^\nu_{\mathbf{k}}(\tau)\}$ and $\{v_{\mathbf{k}}(\tau)\}$ be solutions of (ZL) and (EffEq) with same initial data. Then, for each $\mathbf{k}$ and for $0\le\tau\le 1$,

$$\mathcal{D}I^\nu_{\mathbf{k}}(\tau) \rightharpoonup \mathcal{D}\tfrac{1}{2}|v_{\mathbf{k}}(\tau)|^2 \text{ as } \nu\to 0.$$

Does the effective equation control the angles $\varphi_{\mathbf{k}} = \arg u_{\mathbf{k}}$? No, instead it controls the phases of the a-variables, $a^\nu_{\mathbf{k}}(\tau) = e^{i\nu^{-1}\lambda_{\mathbf{k}}\tau}v^\nu_{\mathbf{k}}(\tau)$, which fast rotate compare to the angles $\varphi_{\mathbf{k}}$ (these phases are completely artificial objects).

Now consider a stationary measure μ^ν for (ZL).

Theorem 2. Assume that Eff. Eq. has a unique stationary measure m_0. Then

$$\mu^\nu \rightharpoonup m_0 \quad \text{as} \quad \nu \to 0.$$

So, if in addition the (ZL) equation has a unique stationary measure, then for ANY its solution $u^\nu(\tau)$ we have

$$\lim_{\nu\to 0} \lim_{\tau\to\infty} \mathcal{D}\big(u^\nu(\tau)\big) = m_0.$$

But when Eff. Eq. has a unique stat. measure?

Theorem 3. 1) Let $p \geq 1$. Then Eff. Eq. has a unique stationary measure if $d \leq 3$.

2) Take any d. Then Eff. Eq. has a unique stationary measure if $p \geq p_d$ for a suitable $p_d \geq 0$.

Properties of $\mathcal{H}^4_{\text{res}}$ and of Eff. Eq.:

$$(EffEq) \qquad \frac{d}{d\tau}v_{\mathbf{k}} = -i\rho\nabla_{\mathbf{k}}\mathcal{H}^4_{\text{res}}(v) - (\lambda_{\mathbf{k}}+1)^p v_{\mathbf{k}} + b_{\mathbf{k}}\frac{d}{d\tau}\boldsymbol{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k}\in\mathbb{Z}^d_L.$$

Lemma. 1) $\mathcal{H}^4_{\text{res}}(v)$ is 4-homogeneous, and $\mathcal{H}^4_{\text{res}}(v) \geq C|v|^4_{l_2}$.
2) $\mathcal{H}^4_{\text{res}}$ has two convex quadratic integrals of motion, $H_0 = \sum |v_{\mathbf{k}}|^2$ and $H_1 = \sum(|v_{\mathbf{k}}|^2|\mathbf{k}|^2)$, as well as the vector-integral of momenta $\overrightarrow{M} = \sum \mathbf{k}|v_{\mathbf{k}}|^2 \in \mathbb{R}^d$.
3) (EffEq) is well posed in sufficiently smooth Sobolev spaces.

- So (Eff. Eq.) is similar to the 2d NSE on a torus

$$u_\tau(\tau,x) = (u\cdot\nabla)u - \nabla p + \big(\delta\Delta - \delta_1\big)u + \langle\text{random force}\rangle, \quad x\in\mathbb{T}^2,$$

and the limit "$L\to\infty$ while $\rho=\rho(L)$ improperly scaled" for (Eff. Eq.) is similar to the limit, assumed in the Kraichnan theory of 2d turbulence. But I believe that the former is significantly simpler.

4 Limit $L \to \infty$ for the Eff. Eq.

(on the physical level of accuracy).

Consider the Effective Equation:

$$(EffEq) \qquad \begin{aligned} &\frac{d}{d\tau} v_{\mathbf{k}} = -\rho R_{\mathbf{k}}(v) - \gamma_{\mathbf{k}} v_{\mathbf{k}} + b_{\mathbf{k}} \frac{d}{d\tau} \boldsymbol{\beta}^{\mathbf{k}}(\tau), \quad \mathbf{k} \in \mathbb{Z}_L^d, \\ &R_{\mathbf{k}}(v) = i \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k} \\ |\mathbf{k}_1|^2+|\mathbf{k}_2|^2=|\mathbf{k}_3|^2+|\mathbf{k}|^2}} v_{\mathbf{k}_1} v_{\mathbf{k}_2} \bar{v}_{\mathbf{k}_3}. \end{aligned}$$

Here $\gamma_{\mathbf{k}} = (a|\mathbf{k}|^m + b)$. Consider the moments

$$M^{\mathbf{k}_1,\dots\mathbf{k}_{n_1}}_{\mathbf{k}_{n_1+1},\dots\mathbf{k}_{n_1+n_2}}(\tau) = \mathbf{E}(v_{\mathbf{k}_1} \dots v_{\mathbf{k}_{n_1}} \bar{v}_{\mathbf{k}_{n_1+1}} \dots \bar{v}_{\mathbf{k}_{n_1+n_2}}).$$

The goal is to study them as $L \to \infty$, while $\rho = \rho(L)$. Especially, to study the second moments $M^{\mathbf{k}}_{\mathbf{k}} = \mathbf{E}|v_{\mathbf{k}}|^2$.

Physical Assumptions. Following the works on WT, we postulate:

i) *Quasi-Gaussian approximation:*

$$M^{\mathbf{l}_1,\mathbf{l}_2}_{\mathbf{l}_3,\mathbf{l}_4} \sim M^{\mathbf{l}_1}_{\mathbf{l}_1} M^{\mathbf{l}_2}_{\mathbf{l}_2} (\delta^{\mathbf{l}_3}_{\mathbf{l}_1} + \delta^{\mathbf{l}_4}_{\mathbf{l}_1})(\delta^{\mathbf{l}_3}_{\mathbf{l}_2} + \delta^{\mathbf{l}_4}_{\mathbf{l}_2}) \,,$$

and similar for higher order moments.

ii) *Quasi stationary approximation for equations in the chain of moment equations.*

Denote

$$n_{\mathbf{k}} = L^d M^{\mathbf{k}}_{\mathbf{k}}/2 = L^d E_{\mathbf{k}} \,, \qquad b_{\mathbf{k}} = L^{-d/2}\, \tilde{b}_{\mathbf{k}} \,, \qquad \rho = L^{1/2}\, \tilde{\rho};$$

$n_{\mathbf{k}}$ - normalised energy of the wave-vector $\mathbf{k}$. See in

[ZLF] Zakharov, Lvov, Falkovich, “Kolmogorov Spectra of Turbulence”, Springer 1992.

Recall that $\gamma_{\mathbf{k}} = (a|\mathbf{k}|^m + b)$. Accepting the two hypotheses above, under the limit $L \to \infty$ we get for the vector of energies $\{n_{\mathbf{k}}(\tau)\}$ a kinetic equation:

Theorem 4. When $L \to \infty$, $\{n_{\mathbf{k}}\}$ becomes a function on $\mathbb{R}^d \ni \mathbf{k} \mapsto n_{\mathbf{k}}$, which satisfies

$$\frac{d}{d\tau} n_{\mathbf{k}} = -2\gamma_{\mathbf{k}} n_{\mathbf{k}} + \tilde{b}^2_{\mathbf{k}}$$

(kinetic)
$$+4\,\tilde{\rho}^{\,2} \int_{\Gamma_{\mathbf{k}}} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \frac{f_{\mathbf{k}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}_1} + \gamma_{\mathbf{k}_2} + \gamma_{\mathbf{k}_3}}$$
$$\times \left(n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}} n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_3}\right),$$

$\mathbf{k} \in \mathbb{R}^d$. Here $\Gamma_{\mathbf{k}}$ is the resonant surface,

$$\Gamma_{\mathbf{k}} = \{(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \in \mathbb{R}^{3d} : \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k} + \mathbf{k}_3,\ |\mathbf{k}_1|^2 + |\mathbf{k}_2|^2 = |\mathbf{k}|^2 + |\mathbf{k}_3|^2\},$$

and $f_{\mathbf{k}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is a bounded smooth function, constructed in terms of the normal frame to $\Gamma_{\mathbf{k}}$ at a point $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$.

Because of the dissipation in the Eff. Eq., our kinetic equation is “better” then usually kinetic equations in WT are: the divisor in the integrand has no zeroes.

The KZ spectra.

Looking for solutions $n_{\mathbf{k}}$ of the kinetic equation which depend only on $|\mathbf{k}|$ and arguing *a-la* Zakharov, we find that the equation has the following time-independent homogeneous solutions:

i)

$$n_{\mathbf{k}} \sim |\mathbf{k}|^{-d+2/3}\,, \quad \text{or} \;\; n_{\mathbf{k}} \sim |\mathbf{k}|^{-d}.$$

ii)

$$n_{\mathbf{k}} \sim |\mathbf{k}|^{-\frac{m+3d-2}{3}}\,, \quad \text{or} \;\; n_{\mathbf{k}} \sim |\mathbf{k}|^{-\frac{m+3d}{3}}\,.$$

E.g., if $d = 2$, $m = 2$, then the possible exponents in the KZ spectral law $\langle |u_{\mathbf{k}}|^2 \rangle \sim |\mathbf{k}|^{-\varkappa}$ are

$$\varkappa = \frac{4}{3},\ \frac{5}{3},\ 2,\ \frac{8}{3}.$$

The law $\varkappa = 2$ is well observable in numerics.

5 Rossby waves

Consider the quasi-geostrophic equation on the β-plane:

$$\Delta\psi_\tau(\tau, x) + \rho J(\psi, \Delta\psi) + \beta\psi_x = 0, \quad x = (x_1, x_2) \in \mathbb{T}^2_{L,1} = (\mathbb{R}/L\mathbb{Z}) \times \mathbb{R}/\mathbb{Z}.$$

Here:

J- the Jacobian, i.e. $J(u, v) = u_x v_y - u_y v_x$,

β – intensity of the Coriolis force,

L – the length of the equator (horizontal length scale),

ρ – scale for velocity-field (so $\psi \sim 1$).

Rossby number: $Ro = \frac{\rho}{L\beta}$.

Now consider the damped-driven equation:

$$(\beta) \qquad \Delta\psi_\tau + \rho J(\psi, \Delta\psi) + \beta\psi_x = \varkappa\Delta^2\psi - \Delta\psi + \langle \text{random force} \rangle; \quad .$$

$\varkappa\Delta^2\psi$ – kinematic viscosity, $\varkappa \geq 0$;

$\Delta\psi$ – Ekman damping.

This is a well-posed equation with a unique stationary measure. Write $\psi(t, x)$ as Fourier series:

$$\psi(\tau, x) = \sum_{\mathbf{k}\in\mathbb{Z}^2_*} v_{\mathbf{k}}(\tau) e^{i\mathbf{k}_L\cdot x}, \quad \mathbf{k}_L = (k_1/L, k_2); \quad v_{\mathbf{k}} = \bar{v}_{-\mathbf{k}}.$$

Then:

$$\dot{v}_{\mathbf{k}} - i\beta\lambda^L_{\mathbf{k}} = \frac{\rho}{L|\mathbf{k}_L|^2} \sum_{\mathbf{l},\mathbf{n}\in\mathbb{Z}^2_L} |\mathbf{n}_L|^2 (\mathbf{l}\times\mathbf{n}) v_{\mathbf{l}} v_{\mathbf{n}} \delta^{\mathbf{ln}}_{\mathbf{k}} - \gamma_{\mathbf{k}} v_{\mathbf{k}} + b_{\mathbf{k}} \dot{\beta}^{\mathbf{k}}(\tau),$$

$$\lambda^L_{\mathbf{k}} = \frac{(k_1/L)}{|\mathbf{k}_L|^2}, \quad \gamma_k = \varkappa|\mathbf{k}_L|^2 + 1; \qquad \mathbf{k} \in \mathbb{Z}^2_*.$$

Lemma on 3-waves resonances. For typical L,

$$\lambda^L_{\mathbf{k}_1} + \lambda^L_{\mathbf{k}_2} - \lambda^L_{\mathbf{k}_3} = 0 \implies \lambda^L_{\mathbf{k}_j} = 0 \text{ for some } j.$$

Besides, $\lambda^L_{\mathbf{k}} = 0$ iff $\mathbf{k} = (0, k_2)$.

Note that if $|\lambda^L_{\mathbf{k}_1} + \lambda^L_{\mathbf{k}_2} - \lambda^L_{\mathbf{k}_3}| \neq 0$, then we do not have a good positive lower bound for this resonance!

Effective equation for the limit $\beta \to 0$ in eq. (β) is obtained by the same procedure as for NLS: ... Based on the lemma above, we have

Theorem 5. Eff. Equation is an infinite system of equations which decomposes to a collection of triads of complex equations with catalytic interaction between them:

$$dv_{\mathbf{k}} = \Big(2\rho L \frac{k_1 k_2}{k_1^2 + L^2 k_2^2}(3k_2^2 - k_1^2 L^{-2})\, v_{\overline{\mathbf{k}}}\, v_{(0,2k_2)} - \gamma_{\mathbf{k}} v_{\mathbf{k}}\Big) d\tau + b_{\mathbf{k}}\, d\beta^{\mathbf{k}}(\tau),$$

$$dv_{\overline{\mathbf{k}}} = \ldots,$$

$$dv_{(0,2k_2)} = -\gamma_{(0,2k_2)} v_{(0,2k_2)}\, d\tau + b_{(0,2k_2)} d\,\beta^{(0,2k_2)}(\tau)\,.$$

Here $\overline{\mathbf{k}} = (k_1, -k_2)$, $\mathbf{k} \in \mathbb{Z}^2_*$.

In particular, in Eff. Equation there is no cascade of energy!

Theorem 6. Eff. Equation has a unique stationary measure. It describes the limiting behaviour of the stationary measure for (β) as $\beta \to \infty$ (this is a limit $Ro \to \infty$.)

Moreover:

1) the Eff. Equation is soluble explicitly. But the formulas are very clumsy.

2) If $\rho = L$, then the limit $L \to \infty$ in (Eff) is straightforward.

3) If $\rho =$ Const, then the limit $L \to \infty$ is a small regular perturbation of the Ornstein-Uhlenbeck process.

4) If $L =$ Const, then limit $\rho \to \infty$ for (Eff) follows from classical stochastic averaging.

Many happy returns, Antti!

Figure 1 below corresponds to $L = 10, \rho = 1$, while ν decays from 0.1 to 0.005, and the dotted line represents the KZ law $E_{\mathbf{k}} = \mathbf{k}^{-2}$ (e.g. $\varkappa = 2$):

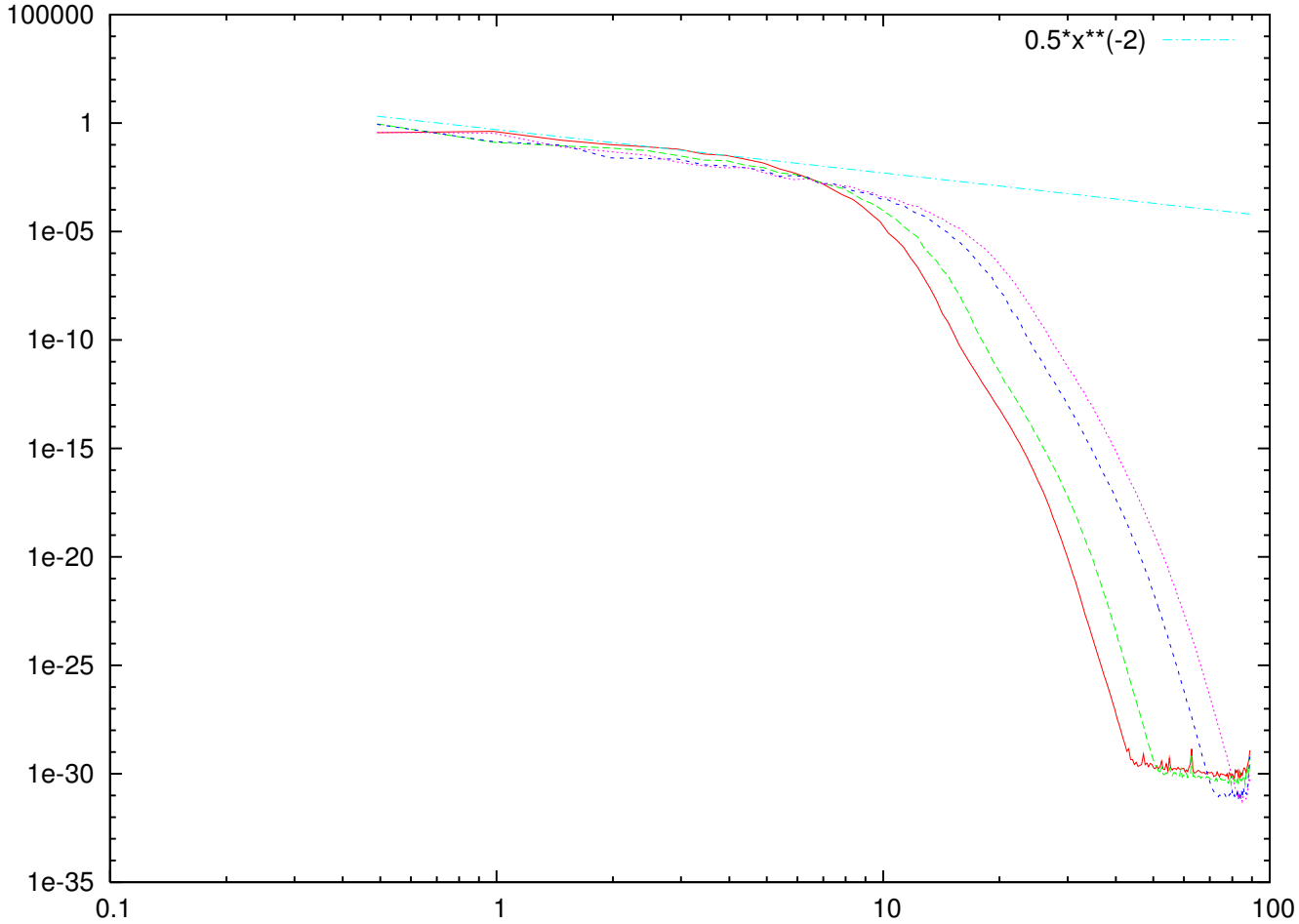
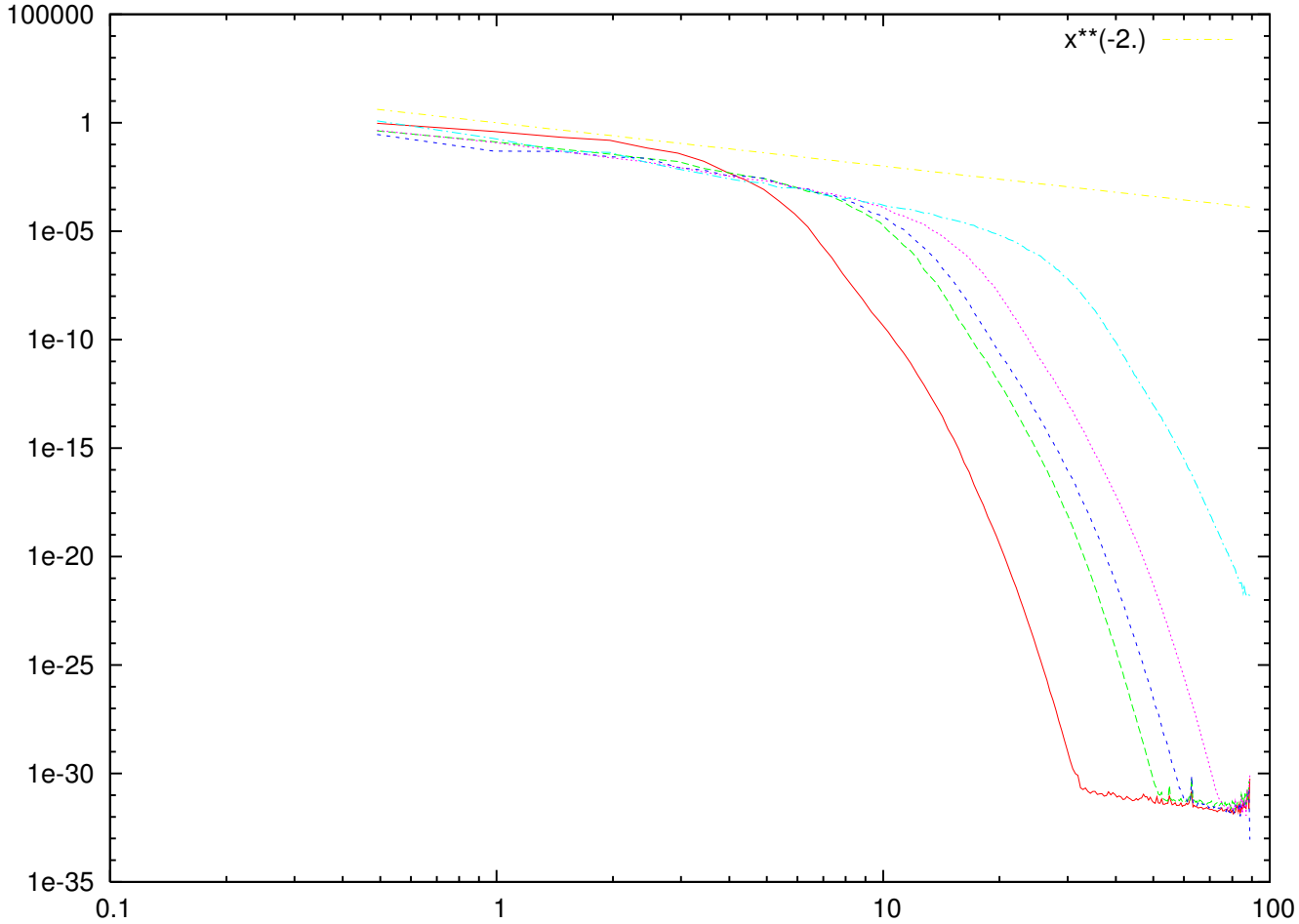
Figure 2 corresponds to $\nu = 0.1,\ L = 10;$ while ρ changes from 0.1 to 10. Again, the dotted line represents the KZ law $E_{\mathbf{k}} = \mathbf{k}^{-2}$: