. . . Chaos is the place which serves to contain all things; for if this had not subsisted neither earth nor water nor the rest of the elements, nor the Universe as a whole, could have been constructed. . . .

Sextus Empiricus, *Against the Physics, against the Ethicists*, R. G. Bury, p. 217, Harvard University Press, 1997.

ON THE CRITERION OF STOCHASTIC STRUCTURE FORMATION IN RANDOM MEDIA

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It is shown that, in parametrically excited stochastic dynamic systems described by partial differential equations, spatial structures (clusters) can appear with probability one, i.e., almost in every system realization, due to rare events happened with probability approaching to zero. The problems of such type arise in hydrodynamics, magnetohydrodynamics, physics of plasma, astrophysics, and radiophysics. The general theory is illustrated by the examples of specific physical problems.

Introduction

Parametrically excited dynamical systems are encountered in all branches of physics. Dynamical systems can be described by ordinary and partial differential equations.

Two features are characteristic of such parametric excitation in dynamic systems described by partial differential equations:

1. On the one hand, at the initial stages of dynamic system evolution, such a parametric excitation is accompanied by the increase of all statistical characteristics of the problem solution (such as moment and correlation functions of any order) with time;

2. On the other hand, *separate field realizations* can show the stochastic nonstationary phenomenon of *clustering* in phase and physical spaces.

Clustering of a field is identified as the emergence of compact areas with large values of this field against the residual background of areas where these values are fairly low. Their spatial pattern is permanently changing. *Naturally, statistical averaging completely destroys all data on clustering.*

Clustering is a particular case of *stochastic structure formation* in stochastic dynamic systems. The notion of clustering by itself is related to the spatial behavior of a dynamic system *in separate realizations*! Consideration of clustering in terms of traditional statistical characteristics such as moment and correlation functions of arbitrary order is meaningless! Clustering either exists or not exists.

In itself, the physical phenomenon of structure formation in stochastic parametrically excited dynamic systems is well known in physics [1-3].

The examples are the *Anderson localization* for wave eigenfunctions of the stationary one-dimensional Schrödinger equation with a random potential and the *dynamic localization* of wave field intensity in a wave problem of propagation in randomly layered medium (Helmholtz stochastic equation). Clearly, both examples are characterized by exponential growth of the moments of wave field intensity with the distance in medium from the source.

Moreover, in a number of cases, *clustering of both passive scalar tracer* (density field) and *vector tracer* (magnetic field energy) can occur in problems on turbulent transfer in the scope of *kinematic approximation*!

The basic stochastic equations for the density field $\rho(\boldsymbol{r}, t)$ and the nondivergent magnetic field $\boldsymbol{H}(\boldsymbol{r}, t)$ at the kinematic stage are the scalar continuity equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{u}(\boldsymbol{r}, t)\right)\rho(\boldsymbol{r}, t) = \mu_\rho \Delta \rho(\boldsymbol{r}, t), \quad \rho(\boldsymbol{r}, 0) = \rho_0(\boldsymbol{r}) \tag{1}$$

and the vector induction equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{u}(\boldsymbol{r}, t)\right)\boldsymbol{H}(\boldsymbol{r}, t) = \left(\boldsymbol{H}(\boldsymbol{r}, t) \cdot \frac{\partial}{\partial \boldsymbol{r}}\right)\boldsymbol{u}(\boldsymbol{r}, t) + \mu_H \Delta \boldsymbol{H}(\boldsymbol{r}, t), \tag{2}$$

where μ_ρ is the dynamical diffusion coefficient for the density, μ_H is the dynamical diffusion coefficient for the magnetic field, which is related to the medium conductivity, and $\boldsymbol{u}(\boldsymbol{r}, t)$ is the field of turbulent velocities.

Dynamic system density (1) and magnetic field (2) are conservative, and both the total scalar mass $M = \int d\boldsymbol{r}\rho(\boldsymbol{r},t)$ and the magnetic flux $\int d\boldsymbol{r}\,\boldsymbol{H}(\boldsymbol{r},t)$ remain constant during the evolution.

For homogeneous initial conditions $\rho(\boldsymbol{r},0) = \rho_0, \boldsymbol{H}(\boldsymbol{r},0) = \boldsymbol{H}_0$, that we consider here, the following equalities are a corollary of the conservatism of dynamic systems (1) and (2) $\langle\rho(\boldsymbol{r},t)\rangle = \rho_0, \langle\boldsymbol{H}(\boldsymbol{r},t)\rangle = \boldsymbol{H}_0$.

A specific feature of Eqs (1) and (2) is the parametric excitation with time *in each realization* of both the density field $\rho(\boldsymbol{r},t)$ (for a compressible fluid flow) and the magnetic field energy $E(\boldsymbol{r},t) = \boldsymbol{H}^2(\boldsymbol{r},t)$ (for a turbulent fluid flow), which is called the *stochastic dynamo*.

These equations are fairly complicated and depend on a large body of parameters. For a homogeneous initial condition, one can explicitly distinguish two temporal intervals in which problem solutions differ fundamentally. At the first (initial) stage, fields are generated in every particular realization. Effects related to dynamic diffusion are clearly inessential at this stage, and one can omit the corresponding terms in Eqs. (1) and (2).

Thus, at the first stage we arrive at the equations

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{u}(\boldsymbol{r},t)\right)\rho(\boldsymbol{r},t) = 0, \quad \rho(\boldsymbol{r},0) = \rho_0, \tag{3}$$

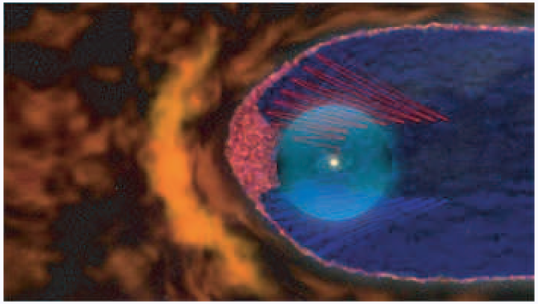
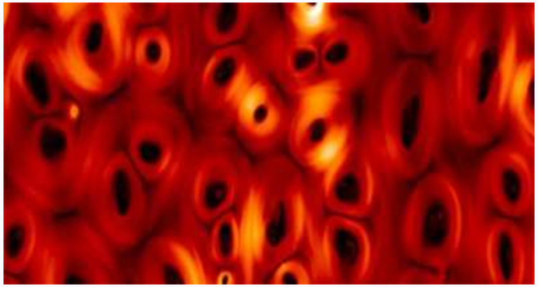
$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\boldsymbol{u}(\boldsymbol{r},t)\right)\boldsymbol{H}(\boldsymbol{r},t) = \left(\boldsymbol{H}(\boldsymbol{r},t)\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)\boldsymbol{u}(\boldsymbol{r},t), \quad \boldsymbol{H}(\boldsymbol{r},0) = \boldsymbol{H}_0. \tag{4}$$

However, it is namely the interval during which *spatial structure formations* can originate in separate realizations!

I illustrate structure formation in magnetic field by the extract from an internet-page:

"*What does puzzle astrophysicists so strongly?*

Contrary to hypotheses formed for fifty years, at the boundary of planetary system observers encountered not a linear and gradually decreasing magnetic field (or magnetic *laminar*), but a boiling foam of locally magnetized areas each of hundreds of millions kilometers in extent, which form a non-stationary cellular structure in which magnetic field lines are permanently breaking and recombining to form new areas—*magnetic "bubbles"* [4].

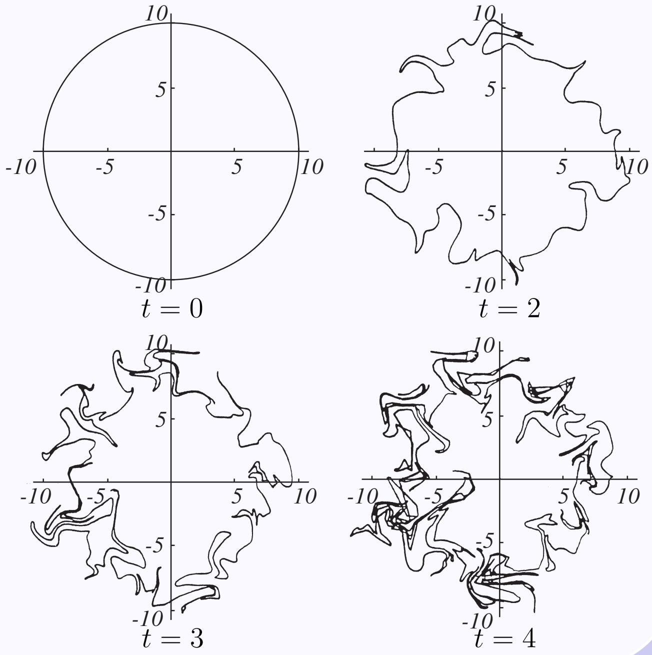
Magnetic situation at the boundary of heliosphere as it most likely looks in reality. Conditional interpretation (left) and the system of magnetic bubbles (right). Diameter of each bubble measures about 100 million kilometers. A computer model.

Note that the partial differential equations (*Eulerian description*) (3), (4) are equivalent to the system of characteristic equations for particles (*Lagrangian description*) which are the simplest purely *kinematic equations*

$$\frac{d}{dt}\boldsymbol{r}(t) = \boldsymbol{u}\left(\boldsymbol{r}(t), t\right), \quad \boldsymbol{r}(0) = \boldsymbol{r}_0.$$

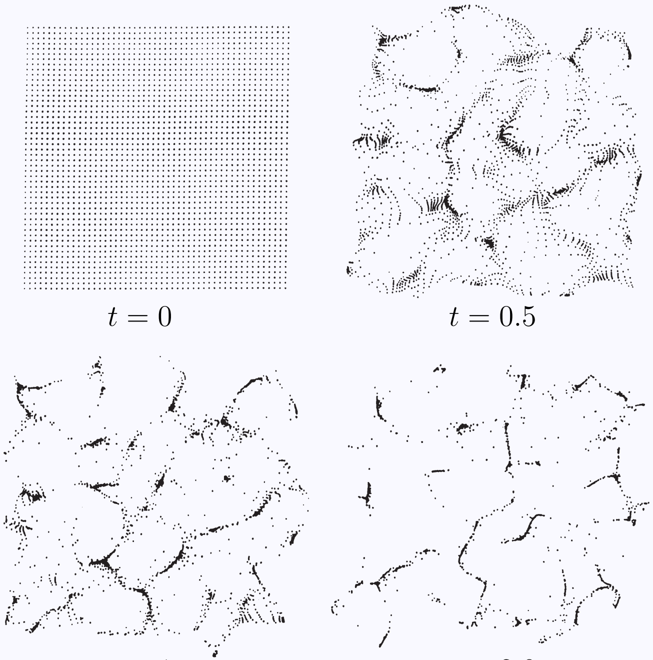
Numerical simulations show that the behavior of a system of particles essentially depends on whether the random field of velocities is *nondivergent* or *divergent*.

By way of example, this figure shows a schematic of the evolution of the two-dimensional system of particles uniformly distributed within the circle for a particular realization of the nondivergent steady field $\boldsymbol{u}(\boldsymbol{r})$. Here, we use the dimensionless time related to statistical parameters of field $\boldsymbol{u}(\boldsymbol{r})$. In this case, the area of surface patch within the contour remains intact and particles relatively uniformly fill the region within the deformed contour. The only feature consists in the *fractal-type* irregularity of the deformed contour. This phenomenon—called *chaotic advection*—is under active study now.



On the contrary, in the case of the potential velocity field $\boldsymbol{u}(\boldsymbol{r})$, particles uniformly distributed in the square at the initial instant will *form clusters* during the temporal evolution. Results simulated for this case are shown in this figure. We emphasize that the formation of clusters in this case is purely a *kinematic effect.*

This feature of particle dynamics disappears on averaging over an ensemble of realizations of random velocity field.

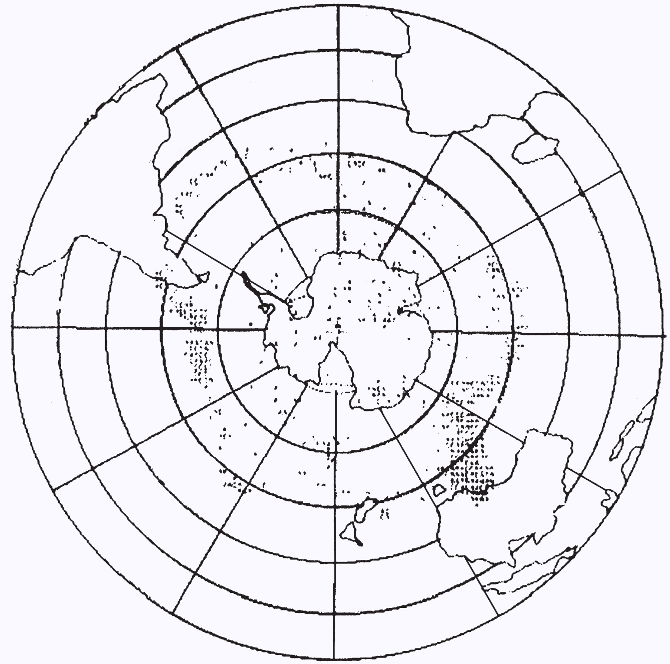


$t = 0$ $t = 0.5$ $t = 1$ $t = 2.0$

Note that such type clustering in a system of particles was found, to all appearance for the first time, as a result of numerical simulating the so-called *Eole experiment* with the use of the simplest equations of atmospheric dynamics.

In this global experiment, 500 constant-density balloons were launched in Argentina in 1970-1971; these balloons traveled at a height of about 12 km and spread along the whole of the southern hemisphere.

This figure shows the balloon distribution over the southern hemisphere for day 105 from the beginning of this process simulation; this distribution clearly shows that balloons are concentrated in groups, which just corresponds to *clustering*.



Another example of problems pertaining to parametric excitation of dynamic systems is the problem on propagation of a monochromatic plane wave in random multidimensional media in terms of the complex *scalar Leontovich parabolic equation*

$$\frac{\partial}{\partial x}u(x,\boldsymbol{R}) = \frac{i}{2k}\Delta_{\boldsymbol{R}}u(x,\boldsymbol{R}) + \frac{ik}{2}\varepsilon(x,\boldsymbol{R})u(x,\boldsymbol{R}), \quad u(x,\boldsymbol{R}) = u_0, \tag{5}$$

where function $\varepsilon(x, \boldsymbol{R})$ is the fluctuating portion (deviation from unity) of dielectric permittivity, x–axis is directed along the initial direction of wave propagation, and vector $\boldsymbol{R}$ denotes the coordinates in the transverse plane.

Note that this equation is the *Schrödinger equation* with a random potential $\varepsilon(x, \boldsymbol{R})$, where coordinate x plays the role of time t.

If we introduce the amplitude and phase of the wave field by the formula $u(x, \boldsymbol{R}) = A(x, \boldsymbol{R}) \exp\{iS(x, \boldsymbol{R})\}$, then we can derive the equation for wave field intensity $I(x, \boldsymbol{R}) = |u(x, \boldsymbol{R})|^2$

$$\frac{\partial}{\partial x} I(x, \boldsymbol{R}) + \frac{1}{k} \boldsymbol{\nabla}_{\boldsymbol{R}} \{\boldsymbol{\nabla}_{\boldsymbol{R}} S(x, \boldsymbol{R}) I(x, \boldsymbol{R})\} = 0, \quad I(0, \boldsymbol{R}) = I_0, \tag{6}$$

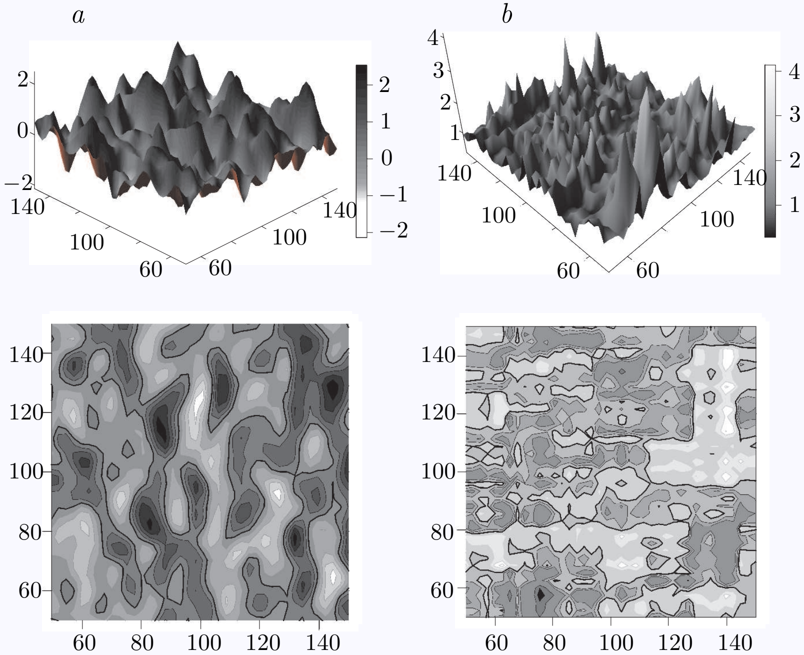
which coincides in form with the equation for the tracer density field in a random potential flow, and, hence, the wave field intensity undergoes clustering which is manifested as appearance of *field caustic structure*. In this case all one-point statistical characteristics are independent of variable $\boldsymbol{R}$.

A similar situation should also be observed in the case of the monochromatic nonlinear *problem on wave self-interaction* in random inhomogeneous media described by the *nonlinear parabolic equation* (*nonlinear Schrödinger equation*)

$$\frac{\partial}{\partial x} u(x, \boldsymbol{R}) = \frac{i}{2k} \Delta_{\boldsymbol{R}} u(x, \boldsymbol{R}) + \frac{ik}{2} \varepsilon(x, \boldsymbol{R}; I(x, \boldsymbol{R})) u(x, \boldsymbol{R}), \quad u(0, \boldsymbol{R}) = u_0(\boldsymbol{R}),$$

because equation (6) is independent of the shape of the function $\varepsilon(x, \boldsymbol{R}; I(x, \boldsymbol{R}))$.

Elements of the statistical topography of random fields



Randomness of medium parameters in dynamic systems gives rise to a stochastic behavior of physical fields. Indeed, individual samples of scalar two-dimensional fields $f(\boldsymbol{R}, t)$, where $\boldsymbol{R} = (x, y)$, resemble a rough mountainous terrain with randomly scattered peaks, troughs, ridges and saddles. The figure shows examples of realizations of (a) Gaussian and (b) lognormal random fields whose level curves are characterized by different statistical structures.

Phenomenon of clustering random fields can be detected and described only on the basis of the ideas of statistical topography. Similarly to common topography of mountain ranges, the statistical topography studies the systems of contours (level lines in the 2D case and surfaces of constant values in the 3D case) specified by the equality $f(\boldsymbol{r}, t) = f = \text{const}$.

For analyzing a system of contours (for simplicity, we will deal with the two-dimensional case and assume $\boldsymbol{r} = \boldsymbol{R}$), we introduce the singular *indicator function* $\varphi(\boldsymbol{R}, t; f) = \delta(f(\boldsymbol{R}, t) - f)$ concentrated on these contours. The convenience of this function consists, in particular, in the fact that it allows simple expressions for quantities such as the total area of regions where $f(\boldsymbol{R}, t) > f$ (i.e., within level lines $f(\boldsymbol{R}, t) = f$)

$$S(t; f) = \int \theta(f(\boldsymbol{R}, t) - f) d\boldsymbol{R} = \int_{f}^{\infty} df' \int d\boldsymbol{R}\, \varphi(\boldsymbol{R}, t; f'),$$

and the total 'mass' of the field within these regions

$$M(t; f) = \int f(\boldsymbol{R}, t)\theta(f(\boldsymbol{R}, t) - f) d\boldsymbol{R} = \int_{f}^{\infty} f' df' \int d\boldsymbol{R}\, \varphi(\boldsymbol{R}, t; f'),$$

where $\theta(f(\boldsymbol{R}, t) - f)$ is the Heaviside theta function.

The mean value of indicator function over an ensemble of realizations of random field $f(\boldsymbol{R}, t)$ determines the one-time (in time) and one-point (in space) probability density $P(\boldsymbol{R}, t; f) = \langle \delta\left(f(\boldsymbol{R}, t) - f\right) \rangle$.

Consequently, this probability density immediately determines ensemble-averaged values of the above expressions $S(t;f)$ and $M(t;f)$:

$$\langle S(t;f)\rangle = \int\limits_f^\infty df' \int d\boldsymbol{R}\, P(\boldsymbol{R},t;f'), \quad \langle M(t;f)\rangle = \int\limits_f^\infty f'df' \int d\boldsymbol{R}\, P(\boldsymbol{R},t;f').$$

If we include into consideration the spatial gradient $\boldsymbol{p}(\boldsymbol{R},t) = \boldsymbol{\nabla} f(\boldsymbol{R},t)$, we can obtain additional information on details of the structure of field $f(\boldsymbol{R},t)$. For example, quantity $l(t;f) = \int d\boldsymbol{R}\,|\boldsymbol{p}(\boldsymbol{R},t)|\,\delta(f(\boldsymbol{R},t)-f)$ is the total length of contours. Inclusion of second-order spatial derivatives into consideration allows estimating the total number of contours $f(\boldsymbol{R},t) = f = \text{ const}$ by the approximate formula

$$N(t;f) = \frac{1}{2\pi}\int d\boldsymbol{R}\kappa(t,\boldsymbol{R};f)\,|\boldsymbol{p}(\boldsymbol{R},t)|\,\delta\left(f(\boldsymbol{R},t)-f\right),$$

where $\kappa(\boldsymbol{R},t;f)$ is the curvature of the level line.

In the case of the spatially homogeneous field $f(\boldsymbol{R},t)$, the corresponding probability density $P(\boldsymbol{R},t;f)$ is independent of $\boldsymbol{R}$. In this case, statistical averages of the above expressions (without integration over $\boldsymbol{R}$) will characterize the corresponding specific (per unit area) values of these quantities. In this case, random field $f(\boldsymbol{R},t)$ is statistically equivalent to the random process whose statistical characteristics coincide with the spatial one-point characteristics of field $f(\boldsymbol{R},t)$.

Consider now the conditions of occurrence of stochastic structure formation. It is clear that, for a *positive field* $f(\boldsymbol{R},t)$, the condition of clustering with a probability of one, i.e., almost in all realizations, is formulated *in the general case* as simultaneous tendency of fulfillment of the following asymptotic equalities for $t \to \infty$

$$\langle S(t;f)\rangle \to 0, \quad \langle M(t;f)\rangle \to \int d\boldsymbol{R}\,\langle f(\boldsymbol{R},t)\rangle\,.$$

On the contrary, simultaneous tendency of fulfillment of the asymptotic equalities for $t \to \infty$

$$\langle S(t;f)\rangle \to \infty, \quad \langle M(t;f)\rangle \to \int d\boldsymbol{R}\,\langle f(\boldsymbol{R},t)\rangle$$

corresponds to the absence of structure formation.

In the case of a spatially homogeneous field $f(\boldsymbol{R},t)$, the corresponding probability density $P(\boldsymbol{R},t;f)$ is independent of $\boldsymbol{R}$. In this case, statistical averages of the above expressions (without integration over $\boldsymbol{R}$) will characterize the corresponding specific (per unit area) values of these quantities.

So, the specific mean area $\langle S_{\mathbf{hom}}(t;f)\rangle$ over which the random field $f(\boldsymbol{R},t)$ exceeds a given level f, coincides with the probability of the event $f(\boldsymbol{R},t) > f$ at any spatial point, i.e., $\langle S_{\mathbf{hom}}(t;f)\rangle = \langle\theta(f(\boldsymbol{R},t) - f)\rangle = \mathsf{P}\{f(\boldsymbol{R},t) > f\}$ and therefore the mean specific area offers a geometric interpretation of the probability of the event $f(\boldsymbol{R},t) > f$, which is apparently independent of the point $\boldsymbol{R}$. Consequently, in the case of a *homogeneous* field, conditions of clustering are reduced to the tendency of asymptotic equalities for $t \to \infty$

$$\langle S_{\mathbf{hom}}(t;f)\rangle = \mathsf{P}\{f(\boldsymbol{r},t) > f\} \to 0, \quad \langle M_{\mathbf{hom}}(t;f)\rangle \to \langle f(t)\rangle\,.$$

Absence of clustering corresponds to the tendency of asymptotic equalities for $t \to \infty$

$$\langle S_{\mathbf{hom}}(t;f)\rangle = \mathsf{P}\{f(\boldsymbol{r},t) > f\} \to 1, \quad \langle M_{\mathbf{hom}}(t;f)\rangle \to \langle f(t)\rangle\,.$$

Thus, in spatially homogeneous problems, clustering is the physical phenomenon (realized with probability one, i.e., occurred in almost all realizations of a positive random field) generated by a rare event whose probability tends to zero.

Namely availability of these rare events is the trigger that starts the process of structure formation.

In the conditions of developed clustering, the field is simply absent in the most part of space!

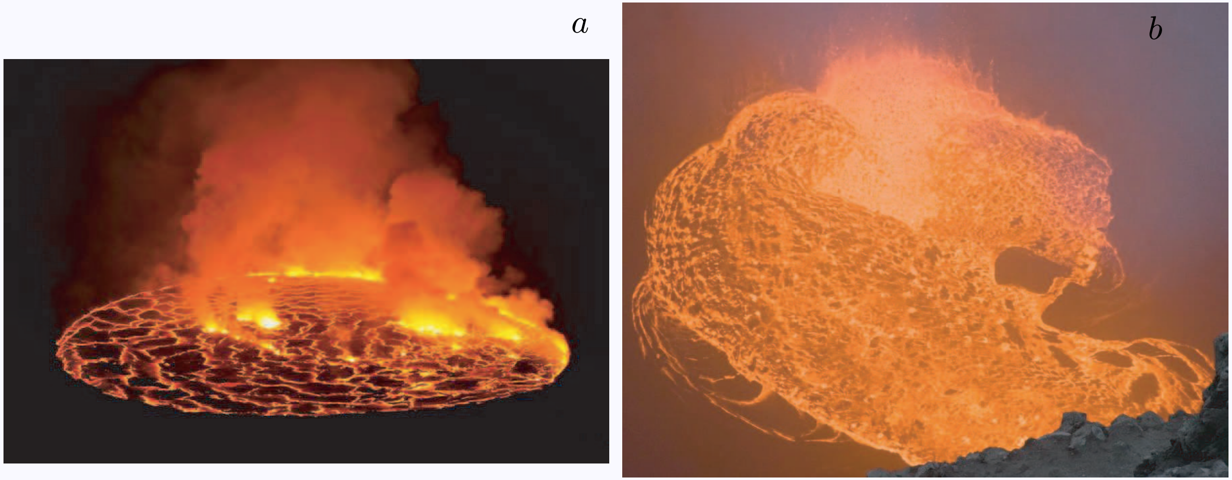
As for setup time of such spatial structure formation, it depends on limiting behavior of the right-hand expressions in all above asymptotic equalities.

It is clear that the above conditions of presence and absence of clustering field $f(\boldsymbol{R},t)$ bear no relation to parametric growth in time of the field statistical characteristics such as moment and correlation functions of arbitrary order.

The above criterion of '*ideal*' clustering (analogously to *ideal hydrodynamic*) describes dynamics of cluster formation in the dynamic systems described in general by the first-order partial differential equations. This *ideal* structure originates in the form of very *thin belts* (in the two-dimensional case) or very *thin tubes* (in the three-dimensional case).

As for *actual* physical systems, various additional factors come to play with time; they are related to generation of random field spatial derivatives like spatial diffusion or diffraction, which *deform* the pattern of clustering, but not *dispose* it.

In particular, a possible situation can occur when the probability density rapidly approaches its steady-state regime $P(\boldsymbol{R};f)$ for $t\to\infty$. In this case, functionals like $\langle S(f)\rangle=\int\limits_f^\infty df'\int d\boldsymbol{R}\,P(\boldsymbol{R};f')$ and $\langle M(f)\rangle=\int\limits_f^\infty f'df'\int d\boldsymbol{R}\,P(\boldsymbol{R};f')$ cease to describe further deformation of the clustering pattern, and we must study temporal evolution of functionals related to the spatial derivatives of field $f(\boldsymbol{R},t)$, like the total length of contours and the number of contours.



As an illustration of '*ideal*' and '*deformed*' clustering we mention the lava lake boiling in the depths of Nyiragongo Crater (Great Lakes region of Africa) (a) [5] and lava lakes in the depth of Kilauea Crater (Hawaiian National Volcano Park) (b) [6].

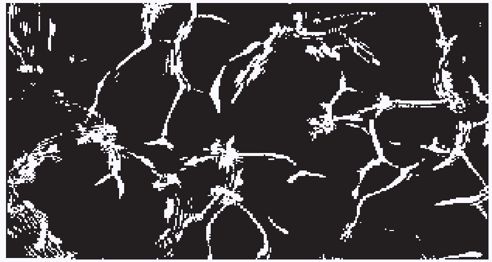
Another illustrations we have from the problem of the *waves propagation in random medium* (5). With increasing the distance statistical characteristics of wave intensity approach the saturated regime. In this region we have

$$\langle I^n(x)\rangle = n!, \quad P(x, I) = e^{-I}.$$

In this case, the mean specific contour length and mean specific number of wave intensity contours continue to grow with distance; consequently, contour subdivision occurs, which was observed

in laboratory experiments

a



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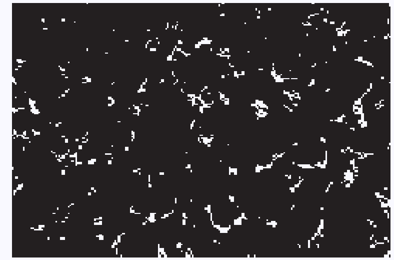
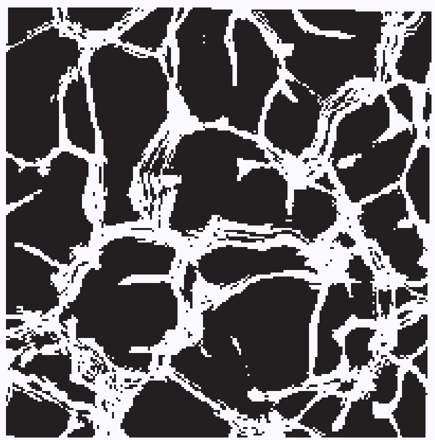


and in numerical simulations.

a



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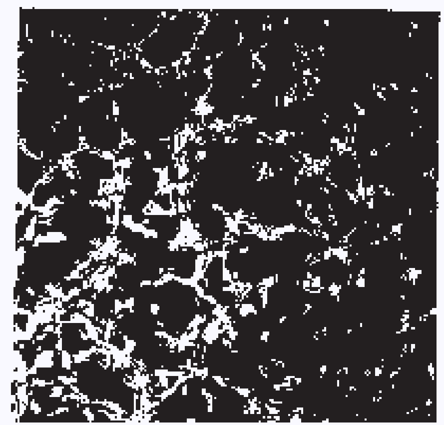


Lognormal positive random fields

The pattern of *ideal clustering* is realized for positive random lognormal fields $E(\boldsymbol{r},t)$ closely related to lognormal processes whose one-point probability density $P(\boldsymbol{r},t;E)$ satisfies in general case the equation $(E(\boldsymbol{r},0)=E_0(\boldsymbol{r}))$

$$\left(\frac{\partial}{\partial t}-D_0\frac{\partial^2}{\partial \boldsymbol{r}^2}\right)P(\boldsymbol{r},t;E)=\left\{\alpha\frac{\partial}{\partial E}E+D\frac{\partial}{\partial E}E\frac{\partial}{\partial E}E\right\}P(\boldsymbol{r},t;E),\quad P(\boldsymbol{r},0;E)=\delta(E-E_0(\boldsymbol{r})),$$

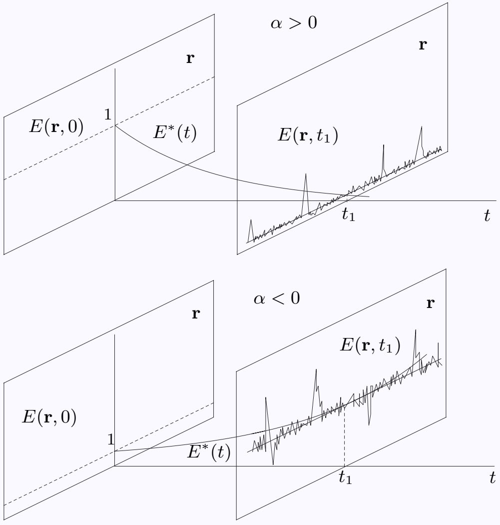
where coefficient D_0 characterizes diffusion in $\boldsymbol{r}$-space and coefficients α and D characterize diffusion in E-space. Here, parameter α can be both positive and negative.

The probability density $P(\boldsymbol{r},t;E)$ for homogeneous case $(E_0(\boldsymbol{r})=E_0)$is independent of $\boldsymbol{r}$ and satisfies the equation

$$\frac{\partial}{\partial t}P(t;E)=\left\{\alpha\frac{\partial}{\partial E}E+D\frac{\partial}{\partial E}E\frac{\partial}{\partial E}E\right\}P(t;E), \qquad (7)$$

For definiteness, we will term field $E(\boldsymbol{r},t)$ 'energy'. Figure, schematically shows random realizations of energy for different signs of parameter α at arbitrary spatial point. The solution of this equation is given by

$$P(t;E)=\frac{1}{2E\sqrt{\pi Dt}}\exp\left\{-\frac{\ln^2\left[Ee^{\alpha t}/E_0\right]}{4Dt}\right\}. \qquad (8)$$



The corresponding asymptotic expressions ($t \to \infty$) for the specific values of the volume of large fluctuations and their total specific values of the energy become ($2D - \alpha > 0$)

$$\langle V_{\mathrm{hom}}(t,E)\rangle = \mathsf{P}\{E(\boldsymbol{r},t;\alpha) > E\} \approx \begin{cases} \dfrac{1}{\alpha}\sqrt{\dfrac{D}{\pi t}\left(\dfrac{E_0}{E}\right)^{\alpha/D}} e^{-\alpha^2 t/(4D)} \; (\alpha > 0), \\ 1 - \dfrac{1}{|\alpha|}\sqrt{\dfrac{D}{\pi t}\left(\dfrac{E}{E_0}\right)^{|\alpha|/D}} e^{-\alpha^2 t/(4D)} \; (\alpha < 0), \end{cases}$$

$$\langle E_{\mathrm{hom}}(t,E)\rangle \approx E_0 e^{(D-\alpha)t}\left[1 - \frac{1}{(2D-\alpha)}\sqrt{\frac{D}{\pi t}\left(\frac{E}{E_0}\right)^{(2D-\alpha)/D}} e^{-(2D-\alpha)^2 t/(4D)}\right].$$

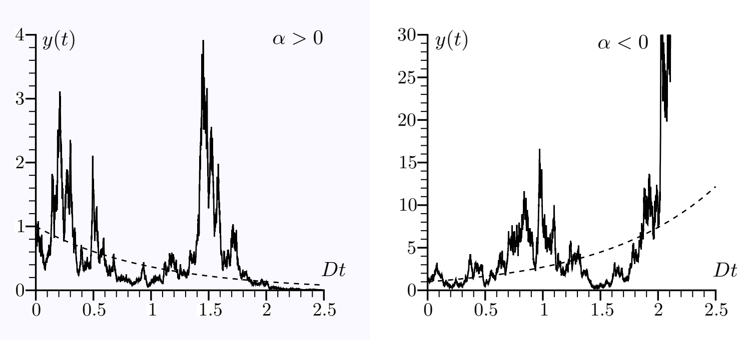
So clustering of random field of energy $E(\boldsymbol{r},t;\alpha)$ will happen with a probability one (i.e., almost in each realization) under the condition $\alpha > 0$.

Note that the simplest Markovian lognormal process

$$y(t;\alpha) = \exp\left\{-\alpha t + \int\limits_0^t d\tau z(\tau)\right\}, \tag{9}$$

where $z(t)$ is the Gaussian 'white noise' process with the parameters $\langle z(t)\rangle = 0$, $\langle z(t)z(t')\rangle = 2D\delta(t-t')$ is statistically equivalent to Eq. (7).

This figure displays realizations of process (9) for positive and negative parameter α and $D = |\alpha|$ (the dashed curves show the functions $\exp\{-Dt\}$ and $\exp\{Dt\}$). The figure shows the presence of rare but strong fluctuations relative to the dashed curves towards both large values and zero. Such a property of random processes is called *intermittency*. This property is common to all random processes. The curve with respect to which the fluctuations are observed is referred to as the *typical realization curve*. The concept of *typical realization curve* of arbitrary random process $z(t)$ concerns the fundamental features of the behavior of a separate process realization as a whole for temporal intervals of arbitrary duration.



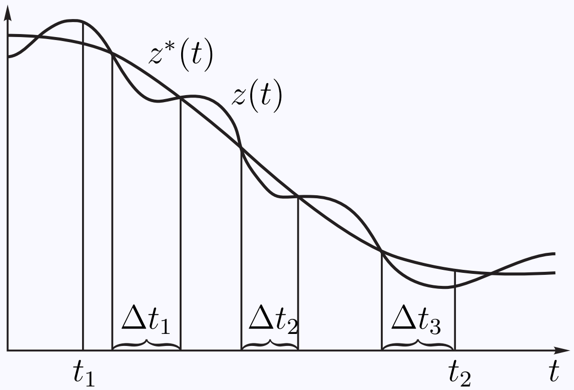
Typical realization curve

Statistical characteristics of a random process $z(t)$ at a fixed instant t are described by the one-time probability density $P(z,t)$ and the probability distribution function $F(t,Z) = P(z(t) < Z) = \int\limits_{-\infty}^{Z} dz P(z,t)$. The deterministic typical realization curve $z^*(t)$ is determined as a solution of the algebraic equation $F(t, z^*(t)) = 1/2$.

This means, on the one hand, that for any t

$$\mathsf{P}\{z(t) > z^*(t)\} = \mathsf{P}\{z(t) < z^*(t)\} = 1/2.$$

On the other hand, this curve has a specific property that, for any time interval (t_1, t_2), the random process $z(t)$ 'winds' around the curve $z^*(t)$ such that the mean times are



$$\left\langle T_{z(t)>z^*(t)} \right\rangle = \left\langle T_{z(t)<z^*(t)} \right\rangle = \frac{1}{2}\left(t_2 - t_1\right).$$

Curve $z^*(t)$ can significantly differ from any particular realization of process $z(t)$ and cannot describe possible magnitudes of spikes. Nevertheless, the definitional domain of typical realization curve $z^*(t)$ derived from the one-point probability density of random process $z(t)$ is the whole temporal axis $t \in (0, \infty)$.

The typical realization curve for a Gaussian random process $z(t)$ coincides with the mean of the process $z(t)$, i.e., $z^*(t) = \langle z(t) \rangle$, while the typical realization curve for the *lognormal* random process $y(t) = e^{z(t)}$ is defined by the equality $y^*(t) = e^{\langle z(t) \rangle} = e^{\langle \ln y(t) \rangle}$.

The typical realization curve is therefore a deterministic curve with respect to which the intermittency is unfolding. But it carries no information about the amplitudes of excursions of the random process relative it.

The one-time probability density of the lognormal process $y(t;\alpha)$ (9) is

$$P(y,t;\alpha)=\frac{1}{2y\sqrt{\pi Dt}}\exp\left\{-\frac{\ln^2\left(ye^{\alpha t}\right)}{4Dt}\right\}.$$

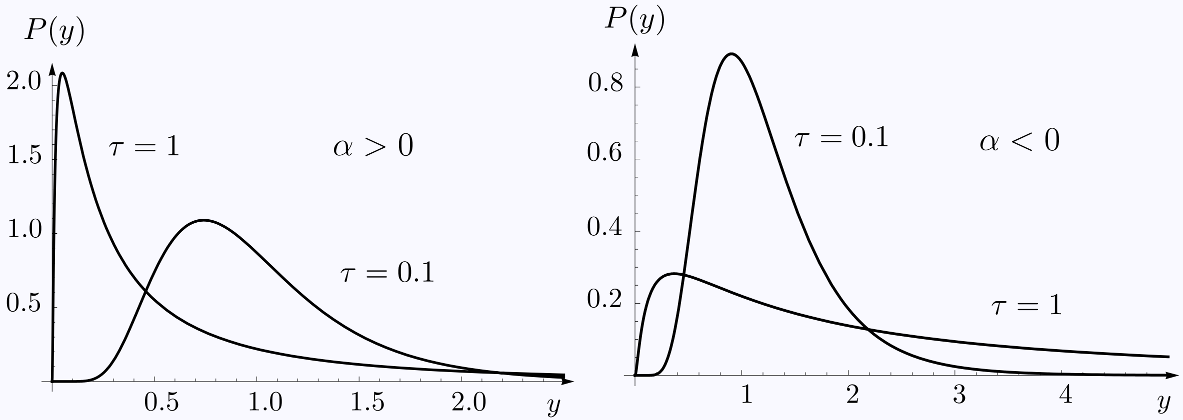
One can easily derive the expression for the moment functions of processes $y(t;\alpha)$ ($n=1,\,2,\cdots$) $\langle y^n(t;\alpha)\rangle=e^{n(n-\alpha/D)Dt}$, from which follows that moments exponentially grow with time for positive and negative parameter α. Also one can easily obtain the equality $\langle\ln y(t)\rangle=-\alpha t$. Consequently, parameter α is the *Lyapunov characteristic index* of the Markovian lognormal random process $y(t;\alpha)$. So, the typical realization curve of lognormal processes $y(t;\alpha)$ is

$$y^*(t)=e^{\langle\ln y(t;\alpha)\rangle}=e^{-\alpha t},$$

which are the exponentially decaying curve if $\alpha>0$ and the exponentially increasing curve in the case of $\alpha<0$.

Consequently, the exponential increase of moments of random process $y(t;\alpha)$ is caused by deviations of this process from the typical realization curves $y^*(t;\alpha)$ towards both large and small values of y.

These figures show the lognormal probability density functions for positive and negative parameters α ($|\alpha|/D = 1$) and dimensionless times $\tau = Dt = 0.1$ and 1. Structurally, these probability distribution functions are absolutely different. The only common feature of these distributions consists in the existence of long flat *tails* that appear in distributions at $\tau = 1$; these tails increase the role of high peaks of process $y(t;\alpha)$ in the formation of the one-time statistics.



In is clear that the following theorem holds:

Under parametric excitation, arbitrary positive conservative field shows the phenomenon of clustering with probability one, i.e. almost in all realizations of the field.

Indeed, in this case $f(\boldsymbol{r},t) = e^{\ln f(\boldsymbol{r},t)}$ and, consequently,

$$\langle f(\boldsymbol{r},t)\rangle = \left\langle e^{\ln f(\boldsymbol{r},t)} \right\rangle = \exp\left\{ \langle \ln f(\boldsymbol{r},t)\rangle + \frac{1}{2}\sigma^2_{\ln f(\boldsymbol{r},t)} \right\},$$

where $\sigma^2_{\ln f(\boldsymbol{r},t)}$ is the variance of random field $\ln f(\boldsymbol{r},t)$. Taking now into account the fact that

$$\left\langle \ln \frac{f(\boldsymbol{r},t)}{f_0} \right\rangle + \frac{1}{2}\sigma^2_{\ln f(\boldsymbol{r},t)} = 0,$$

in view of conservative property, we obtain that the typical realization curve (it coincides with the Lyapunov exponent) has the form

$$f^*(\boldsymbol{r},t) = e^{\langle \ln f(\boldsymbol{r},t)\rangle} = f_0 \exp^{-\alpha t},$$

where

$$\alpha = \lim_{t\to\infty} \frac{1}{2t}\sigma^2_{\ln f(\boldsymbol{r},t)}$$

is the Lyapunov characteristic parameter, and the problem consists in the calculation of this parameter from the corresponding dynamic equation.

Let's go now to statistical description of stochastic transport equations for density field (3) and energy of magnetic field (4) in random velocity field. The problem now is to calculate the statistical Lyapunov characteristic index α and the diffusion coefficient D in equation (7).

Stochastic transport in random velocity field

The random field $\boldsymbol{u}(\boldsymbol{r},t)$ is in general assumed divergent, Gaussian ($\langle \boldsymbol{u}(\boldsymbol{r},t)\rangle = 0$), and statistically homogeneous, being stationary and having the correlation and spectral solenoidal and potential tensors

$$B_{ij}(\boldsymbol{r}-\boldsymbol{r}',t-t') = \langle u_i(\boldsymbol{r},t)u_j(\boldsymbol{r}',t')\rangle = \int d\boldsymbol{k}\, E_{ij}(\boldsymbol{k},t-t')e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')},$$

$$E_{ij}(\boldsymbol{k},t) = \frac{1}{(2\pi)^d}\int d\boldsymbol{r}\, B_{ij}(\boldsymbol{r},t)e^{-i\boldsymbol{k}\boldsymbol{r}}, \quad E_{ij}(\boldsymbol{k},t) = E^{\mathrm{s}}_{ij}(\boldsymbol{k},t) + E^{\mathrm{p}}_{ij}(\boldsymbol{k},t),$$

where d is the space dimension, and the spectral components of the velocity field tensor have the structure

$$E^{\mathrm{s}}_{ij}(\boldsymbol{k},t) = E^{\mathrm{s}}(k,t)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \quad E^{\mathrm{p}}_{ij}(\boldsymbol{k},t) = E^{\mathrm{p}}(k,t)\frac{k_i k_j}{k^2}.$$

In addition, we use the delta-correlated approximation of velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ in time.

Introduce the following parameters

$$D^{\mathrm{s}} = \frac{1}{d-1}\int_0^\infty d\tau\, \langle \boldsymbol{\omega}(\boldsymbol{r},t+\tau)\boldsymbol{\omega}(\boldsymbol{r},t)\rangle, \quad D^{\mathrm{p}} = \int_0^\infty d\tau\, \langle \mathrm{div}\, \boldsymbol{u}(\boldsymbol{r},t+\tau)\mathrm{div}\, \boldsymbol{u}(\boldsymbol{r},t)\rangle \qquad (10)$$

related to the velocity field ($\boldsymbol{\omega}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{u}(\boldsymbol{r},t)$ is the curl of the velocity field), which are just the basic statistical parameters in this approximation.

Probabilistic description of the density field

The one-point probability density of the density field $\rho(\boldsymbol{r}, t)$ in this case satisfies the equation

$$\frac{\partial}{\partial t}P(t;\rho) = D_\rho \frac{\partial^2}{\partial \rho^2}\rho^2 P(t;\rho), \; P(0;\rho) = \delta(\rho - \rho_0), \tag{11}$$

where the diffusion coefficient in ρ-space $D_\rho = D^{\mathrm{p}}$ is given by Eq. (10). This equation coincides with Eq. (7) with parameters $\alpha = D = D_\rho = D^{\mathrm{p}}$, and, consequently, the one-point probability density of the density field is lognormal and assumes the form

$$P(t;\rho) = \frac{1}{2\rho\sqrt{\pi\tau}} \exp\left\{-\frac{\ln^2\left(\rho e^{\tau}/\rho_0\right)}{4\tau}\right\}, \tag{12}$$

where parameter $\tau = D_\rho t$.

All moment functions at $n > 0$ and $n < 0$ grow exponentially with time: $\langle\rho(\boldsymbol{r}, t)\rangle = \rho_0$, $\langle\rho^n(\boldsymbol{r}, t)\rangle = \rho_0^n e^{n(n-1)\tau}$, and the typical realization curve for the density field, which coincides with the Lyapunov exponent, decays exponentially with time at any fixed spatial location:

$$\rho^*(t) = e^{\langle\ln(\rho(\boldsymbol{r},t))\rangle} = \rho_0 e^{-\tau},$$

which tells us that the medium density fluctuations have a cluster character in arbitrary divergent flow with a probability one (i.e., almost in all realizations).

The formation of the density field statistics at any fixed spatial location is maintained by density fluctuations around this curve.

So, for a compressible flow (in a divergent velocity field), the density field always undergoes clustering with probability one. The characteristic time of cluster formation is estimated as $D^{\mathrm{p}} t \sim 1$, where quantity D^{p} is determined by the potential constituent of the spectral component of the velocity field.

We note that even for an incompressible fluid in hydrodynamical flows, the density field experiences clustering for a 'buoyant' tracer, when a finite inertia of the tracer field is considered, or for multiphase flows, i.e., always when a potential component arises in the spectrum of the tracer velocity field, which is different from the velocity of the fluid itself.

In addition, the mean specific area (volume) in which $\rho(r,t) > \rho$ asymptotically decreases at large time $\tau \gg 1$ according to the law

$$\langle s_{\mathrm{hom}}(t,\rho)\rangle = \mathsf{P}\{\rho(\boldsymbol{r},t) > \rho\} \approx \sqrt{\frac{\rho_0}{\pi\rho\tau}} e^{-\tau/4}$$

and this domain confines practically the whole of the mass,

$$\langle m_{\mathrm{hom}}(t,\rho)\rangle / \rho_0 \approx 1 - \sqrt{\frac{\rho}{\pi\rho_0\tau}} e^{-\tau/4},$$

which, according to the above given criterion, just corresponds to the *physical phenomenon of clustering* the density field in random velocity field.

Probabilistic description of magnetic field

Consider now the probabilistic description of magnetic field on the basis of dynamic equation (4). In this case the one-point probability density $P(t;\boldsymbol{H})$ satisfies the equation

$$\frac{\partial}{\partial t}P(t;\boldsymbol{H}) = \left\{D_1\frac{\partial^2}{\partial H_k\partial H_l}H_lH_k + D_2\frac{\partial^2}{\partial H_l\partial H_l}H_k^2\right\}P(t;\boldsymbol{H}), \qquad (13)$$

where $D_1 = \dfrac{\left(d^2-2\right)D^{\mathrm{p}} - 2D^{\mathrm{s}}}{d(d+2)}$ and $D_2 = \dfrac{(d+1)D^{\mathrm{s}} + D^{\mathrm{p}}}{d(d+2)}$ are the diffusion coefficients.

Corollaries of Eq. (13) are the dynamics of mean energy in time and the expression for correlations of magnetic field components $\langle W_{ij}(t)\rangle = \langle H_i(t)H_j(t)\rangle$

$$\langle E(t)\rangle = E_0\exp\left\{2\frac{d-1}{d}\left(D^{\mathrm{s}}+D^{\mathrm{p}}\right)t\right\},$$

$$\frac{\langle W_{ij}(t)\rangle}{\langle E(t)\rangle} = \frac{1}{d}\delta_{ij} + \left(\frac{W_{ij}(0)}{E_0} - \frac{1}{d}\delta_{ij}\right)\exp\left\{-2\frac{(d+1)D^{\mathrm{s}}+D^{\mathrm{p}}}{d+2}t\right\}.$$

So, it follows that the mean energy grows exponentially with time. This growth is accompanied by isotropization of the magnetic field, in accordance with an exponential law. We note that the components of the velocity field enter the respective exponent *in an additive way*. Obviously, this feature is preserved for any other correlations of the *magnetic field and its energy!* To this significant crucial question we'll return later!

Probabilistic description of magnetic field energy

The probability density of magnetic field energy $P(\boldsymbol{r}, t; E)$ is defined as the quantity

$$P(\boldsymbol{r}, t; E) = \langle \delta(E(\boldsymbol{r}, t) - E) \rangle_{\boldsymbol{u}} = \left\langle \delta(\boldsymbol{H}^2(\boldsymbol{r}, t) - E) \right\rangle_{\boldsymbol{H}} .$$

Consequently, in the case of spatially homogeneous problem, the probability density $P(t; E)$ satisfies the equation (7), with the parameters

$$\alpha = 2\frac{d-1}{d+2}\left(D^{\mathrm{p}} - D^{\mathrm{s}}\right), \quad D = 4(d-1)\frac{(d+1)\,D^{\mathrm{p}} + D^{\mathrm{s}}}{d(d+2)}.$$

Parameter α can be both positive and negative.

So, for spatially homogeneous problem the probability density of the magnetic field energy $P(t; E)$ is lognormal, and all moments of magnetic energy are functions exponentially increasing with time (for both positive $n > 0$ and negative $n < 0$ values of n):

$$\langle E^n(t) \rangle = E_0^n \exp\left\{ -2n\frac{d-1}{d+2}\left(D^{\mathrm{p}} - D^{\mathrm{s}}\right) t + 4n^2(d-1)\frac{(d+1)\,D^{\mathrm{p}} + D^{\mathrm{s}}}{d(d+2)} t \right\}.$$

In the particular case $n = 1$, the specific average energy is given by the equality

$$\langle E(t) \rangle = E_0 e^{\gamma t}, \quad \gamma = \frac{2(d-1)}{d}(D^{\mathrm{p}} + D^{\mathrm{s}}). \tag{14}$$

Moreover, quantity

$$\langle \ln\left(E(t)/E_0\right)\rangle = -\alpha t = -2\frac{d-1}{d+2}\left(D^{\mathrm{p}} - D^{\mathrm{s}}\right)t,$$

so that parameter α is the *characteristic Lyapunov exponent*. In addition, the *typical realization curve* of random process $E(t)$, which determines the behavior of magnetic energy at arbitrary spatial point in separate realizations, is the exponential function

$$E^*(t) = E_0 e^{-\alpha t} = E_0 \exp\left\{-2\frac{d-1}{d+2}\left(D^{\mathrm{p}} - D^{\mathrm{s}}\right)t\right\}$$

that can both increase and decrease with time. Indeed, for $\alpha > 0$, the typical realization curve exponentially decreases at every spatial point, which is indicative of cluster structure of energy field. Otherwise, for $\alpha < 0$, the typical realization curve exponentially increases with time, which is evidence of general increase of magnetic energy at every spatial point.

Parameters D^{p} and D^{s} characterize statistics of the random velocity field and *appear as additive terms in all statistical moment and correlation functions of magnetic field energy*. Being a consequence of linearity of Eqs. (2) and (4), this fact means that all principal regularities in such statistical description *make no difference between the effects of the solenoidal and potential components* of the random velocity field.

In other words, all regularities obtained for the above statistical quantities have identical structure for both noncompressible flow ($D^{\mathrm{p}} = 0$) and potential flow ($D^{\mathrm{s}} = 0$). Since *clustering is absent* in the first case and *present* in the second one, it becomes clear that the mentioned statistical characteristics *include no data about stochastic structure formation in separate realizations of magnetic field energy.*

In addition, the initial induction equation (2) holds in the framework of applicability of the kinematic approximation. In the presence of clustering, magnetic field is absent in the most portion of space, and its aftereffect on the velocity field is, naturally, insignificant.

On the contrary, in the absence of clustering, magnetic field is generated everywhere in space; in these conditions, the kinematic approximation can be expected to be valid only on a sufficiently short temporal interval, *and any discussion of the effect of the dynamic diffusion coefficient on the formation of magnetic field energy statistics during such intervals is, in my opinion, unfounded*.

Conclusion

To conclude with, I note that a point commonly accepted in many works suggests that, for an event to happen, it is required that this event was most probable.

For example, in recent work in *Physics – Uspekhi* (2010), Prof. G.R. Ivanitskii calculated certain probabilities and came out with a *hypothesis on origin of life* from the perspective of physics:

'Life can be defined as resulting from a game involving interactions of matter one part of which acquires the ability to remember the success (or failure) probabilities from the previous rounds of the game, thereby increasing its chances for further survival in the next rounds. This part of matter is currently called living matter.'

I cannot agree with the idea that *origin of life* is a game process. I believe that *origin of life* is an event happened with probability one.

THANK YOU VERY MUCH!

References

The last survey paper is:

1. Klyatskin V.I., *Integral characteristics: a key to understanding structure formation in stochastic systems*, Physics – Uspekhi, 54(5), 441, 2011.

The last book in English is:

2. Klyatskin V.I., *Lectures on Dynamics of Stochastic Systems*, Elsevier, 2011.

The last book in Russian is:

3. Кляцкин В.И., *Очерки по динамике стохастических систем*, Серия *Синергетика*, URSS, 2012.

Internet pages are:

4. http://www.gazeta.ru/science/2011/06/17_a_3664677.shtml

5. http://www.boston.com/bigpicture/2011/02 /nyiragongo_crater_journey_to_t.html

6. http://pacificislandparks.com/2010/01/20/more-amazing-lava-lake-photos/