# Entanglement fluctuations

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ref. J. de Boer, J. Järvelä, E. K-V., arXiv:170?.????

(+ see the refs. in the preprint to appear )

## Thoughts:

- How to characterize quantum fluctuations in quantum states $\rho$ ?
- Observables -> probability density function, moments and cumulants
- von Neumann entropy $S$ as a random variable
- How to compute $\langle S^n \rangle$ , $\langle S^n \rangle_c$ ?
- Specifically: consider a subregion $A$ and entanglement entropy
- Are there special relations among the moments?
- If a gravity dual exists, is there a telltale special relation?

## Results:

- Rényi entropy as a generating function for moments
- Focus on variance of entropy, $\Delta S^2$
- Equals to a heat capacity
- Upper bound on $\Delta S^2$
- Subregion $A$ and entanglement entropy:
- For (2d) CFTs: $\Delta S^2_{EE} = S_{EE}$ (leading terms)
- Holds even throughout quenches, non-equilibrium to thermality
- Violated when deforming away from criticality

Density matrix written with the modular Hamiltonian $K$ (canonical normalization)

$$\rho = e^{-K}$$

von Neumann entropy = $\langle K \rangle$:

$$S_V = -\mathrm{Tr}\rho \log \rho = -\langle \log \rho \rangle = \langle K \rangle$$

Moments of S = moments of $K$:

$$S_V^n = \mathrm{Tr}[\rho(-\log \rho)^n] = \langle K^n \rangle$$

Rényi entropies:

$$S_\alpha = \frac{1}{1-\alpha} \log \mathrm{Tr}\rho^\alpha$$

Well known how to obtain $S_V$:

$$S_V = \lim_{\alpha \to 1} S_\alpha$$

But haven't seen this (?): Rényi entropy as a generating function for the moments:

$$k(\alpha) = \mathrm{Tr}\left(\rho^{\alpha}\right) = (1-\alpha)S_{\alpha}$$

$$S_V^n = \langle K^n \rangle = (-1)^n \frac{d^n k(\alpha)}{d\alpha^n}|_{\alpha=1}$$

The logarithm is then a generating function for the cumulants:

$$\tilde{k}(\alpha) = \log k(\alpha)$$

$$\langle K^n \rangle_c = (-1)^n \frac{d^n \tilde{k}(\alpha)}{d\alpha^n}|_{\alpha=1} \ .$$

In particular, the variance

$$\Delta S_V^2 = \langle K^2 \rangle_c = (-1)^2 \tilde{k}^{(2)}(\alpha = 1) = \frac{k''(1)k(1) - (k'(1))^2}{(k(1))^2}$$

$$= \frac{\langle K^2 \rangle \cdot 1 - \langle K \rangle^2}{1} = S_V^2 - (S_V)^2$$

Condensed matter theory: interest in the full eigenvalue spectrum, the eigenvalues $\{\lambda_m = e^{-\varepsilon_n}\}$ of the density matrix $\rho$ (or the eigenvalues $\varepsilon_n$ of the modular Hamiltonian) and their multiplicities $g_m$. Partition function

$$k(\alpha) = \sum_m g_m \lambda_m^\alpha = \sum_m g_m e^{-\alpha\varepsilon_m} = Z(\beta_\alpha)$$

with 'temperature'

$$T_\alpha = 1/\beta_\alpha = 1/\alpha \ .$$

Thus,

eigenvalue spectrum $\Leftrightarrow$ moments/cumulants of entropy

Different ways to package the same information, but may highlight different features of the system.

A variant of Rényi entropy: *modular entropy* (Xi Dong):

$$\tilde{S}_\alpha \equiv \alpha^2 \partial_\alpha \left( \frac{\alpha - 1}{\alpha} S_\alpha \right)$$

Reduces to von Neumann entropy as $\alpha = 1$. Satisfies thermodynamic relation:

$$\tilde{S}_\alpha = -\frac{\partial F_\alpha}{\partial T_\alpha} \ ,$$

Define then

$$E_\alpha = \frac{\partial}{\partial \beta_\alpha} \left( \beta_\alpha F_\alpha \right) \ .$$

and a $\alpha$-heat capacity

$$C_\alpha \equiv \frac{\partial E_\alpha}{\partial T_\alpha}$$

Of particular interest is $\alpha = 1$, where modular entropy = von Neumann entropy. Define the heat capacity (related work: Nakaguchi, Nishioka)

$$C = \lim_{\alpha = 1} C_\alpha$$

We show the heat capacity satisfies the usual thermodynamic relation

$$C = \Delta S_V^2 \ .$$

Specifically when $\rho$ is a reduced density matrix, starting from a CFT vacuum, with half-line subregion, can map to a thermal CFT on hyperbolic cylinder (Hung, Myers, Smolkin, Yale). Then:

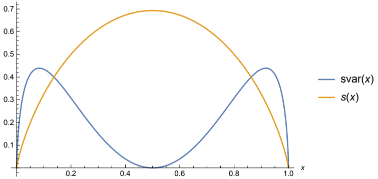
- $T_{\alpha=1} \rightarrow T_{\text{thermal}}$
- $S_V = S_{EE} \rightarrow S_{\text{thermal}}$
- $C \rightarrow C_{\text{thermal}}$
- $C = \Delta^2 S_{EE} \rightarrow C_{\text{thermal}} = \Delta S^2_{\text{thermal}}$

So map to the natural thermodynamic counterparts.

Example 1: Two-qubit system,

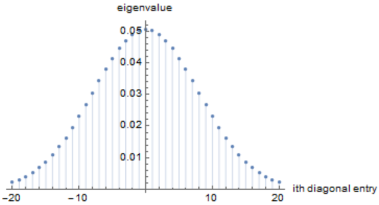
$$|\theta, \phi\rangle = \cos(\theta/2)|10\rangle + e^{i\phi}\sin(\theta/2)|01\rangle$$

trace over one spin:

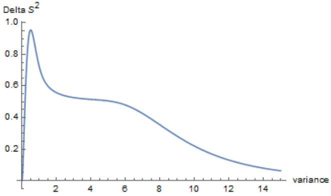


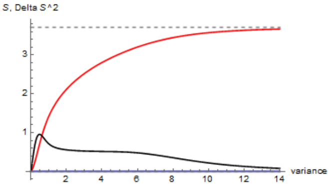

(cf. Feld'man, Yurishchev 2009)

Example 2: Eigenvalue spectrum with $N$ Gaussian distributed eigenvalues $\lambda_i$ of $\rho$:




Entropy and variance: ($\sigma = 0$: pure state, $\sigma = \infty$: max mixed)



**Upper bound on variance?** (Warning: not yet triple-checked!) Consider $N$ eigenvalues $\lambda_i$. Constraint $\sum_i \lambda_i = 1$. Maximize $\Delta S^2$.

Find: can reduce to $n_1, n_2$ eigenvalues $\lambda_1, \lambda_2$, rest =0.

Then,

$$\Delta S^2 \leq \Delta S_{max} = \frac{1}{4}(\log N)^2 = \frac{1}{4} S^2_{\max}$$

where $S_{max} = \log N$ for a max mixed state. Different state gives $\Delta S^2_{max}$!

Anything special about it?

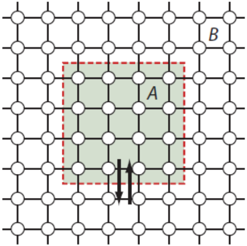
(For max mixed state: moments $S^n = (\ln N)^n$; cumulants vanish for $n > 1$. )

How to measure entanglement? (Klich, Refael, Silva, Levitov,...)

Quantum many-body system in its ground state $|\psi\rangle$

Fluctuating number of (quasi)particles (or $U(1)$ charge) in region $A$, $\langle N_A \rangle = \mathrm{Tr}(\rho_A \hat{N})$.

Cumulants $C_n$ of the number distribution. [H.F.Song *et. al.* PRB.85.035409]




Entanglement associated w/ (entangled) particles in/out of A.

The Rényi entropy of subregion A related to cumulants of particle number,

$$S_\alpha(A) = \sum_{k=1}^{\infty} s_k^{(\alpha)} C_{2k}$$

Simplest case systems $\sim$ free fermions (Song et al, Calabrese et al.).

Find:

$$S_{EE}(A) = \sum_{k=1}^{\infty} \frac{(2\pi)^{2k}|B_{2k}|}{(2k)!} C_{2k} = \frac{\pi^2}{3} C_2 + \frac{\pi^4}{45} C_4 + \cdots$$

$$\Delta S_{EE}^2(A) = \frac{\pi^2}{3} C_2 + \sum_{k=2}^{\infty} \frac{2(2\pi)^{2k}|B_{2k}|}{(2k-1)!} C_{2k} = \frac{\pi^2}{3} C_2 + \frac{8\pi^4}{45} C_4 + \cdots ,$$

In the limit of large particle number (density), $C_2$ term dominates, so

$$\Delta S^2_{EE} = S_{EE} \ ! \quad (N \to \infty) \qquad (*)$$

Speculation: assume the system has a gravity dual. Typically invoke large $\mathcal{N}$ limit of particle species, thus average particle number

$$N_A = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} N_A^{(i)} \ ,$$

in the large $\mathcal{N}$ limit central limit theorem -> Gaussian $N_A$ -> $(*)$!

Caveats:

- Strong coupling
- Non-local observables (*e.g.* Wilson loops in gauge sector)

1+1 CFTs: General analytical results for the Rényi entropies exist,

$$\mathrm{Tr}(\rho_A^\alpha) = c_\alpha e^{-\frac{c}{12}\left(\alpha - \frac{1}{\alpha}\right) W_A}$$

implying (leading terms)

$$\Delta S^2_{EE} = S_{EE} = \frac{c}{6} W_A + \cdots$$

Examples:

$$W_A = \begin{cases} 2\log\left(\frac{l}{\epsilon}\right) & \text{infinite system}, \quad T = 0 \\ 2\log\left(\frac{L}{\pi\epsilon}\sin\frac{l\pi}{L}\right) & \text{finite system, size } L, \quad T = 0 \\ 2\log\left(\frac{\beta}{\pi\epsilon}\sinh\frac{l\pi}{\beta}\right) & \text{infinite system}, \quad T > 0\ ; \\ \log\left(\frac{\beta}{\pi\epsilon}\cosh(2\pi t/\beta)\right) & \text{global quench at } t = 0 \\ \log\left(\frac{t^2+\lambda^2}{\epsilon\lambda/2}\right) & \text{local quench at } t = 0\ , \end{cases}$$

In higher dimensions, can derive

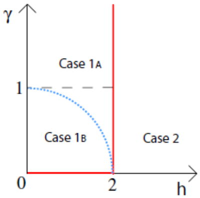
$$\Delta S_{EE}^2 \sim S_{EE}$$

but more subtleties; associated with sensitivity to different regularization schemes

What happens away from criticality / conformal symmetry?

Consider the Heisenberg quantum XY spin chain

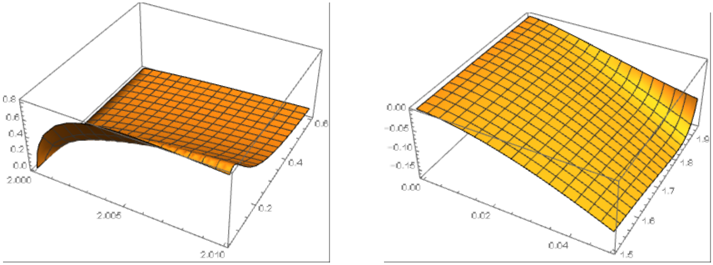
$$H = -\sum_{j=-\infty}^{\infty} \left[(1+\gamma)\sigma_j^x \sigma_{j+1}^x + (1-\gamma)\sigma_j^y \sigma_{j+1}^y + h\sigma_j^z\right],$$




Franchini, Its, Korepin; arXiv:0707.2534

Franchini, Its, Korepin computed the Rényi entropies. Derive from that $S_{EE} - \Delta S_{EE}^2$.

Entropy, variance deviate deforming away from critical lines:




**(A)** **(B)**

Kuva: Samples of the difference $S_{EE} - \Delta S_{EE}^2$ near the critical line $h_c = 2$ (**A**) and near the critical line $\gamma = 0$ (**B**) .