Proof of the DOZZ Formula

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DOZZ formula

Dorn, Otto (1994) and Zamolodchikov, Zamolodchikov (1996):

$$C_\gamma(\alpha_1,\alpha_2,\alpha_3) = \left(\pi\,\mu\,\frac{\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\left(\frac{\gamma}{2}\right)^{\frac{4-\gamma^2}{2}}\right)^{\frac{2Q-\bar\alpha}{\gamma}} \times \frac{\Upsilon'(0)\Upsilon(\alpha_1)\Upsilon(\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\frac{\bar\alpha-2Q}{2})\Upsilon(\frac{\bar\alpha-\alpha_1}{2})\Upsilon(\frac{\bar\alpha-\alpha_2}{2})\Upsilon(\frac{\bar\alpha-\alpha_3}{2})}$$

- $\alpha_i \in \mathbb{C}$, $\gamma \in \mathbb{R}$, $\mu > 0$
- $\bar\alpha = \alpha_1+\alpha_2+\alpha_3$, $Q = \frac{\gamma}{2}+\frac{2}{\gamma}$

Υ is an entire function on $\mathbb{C}$ with simple zeros defined by

$$\log\Upsilon(\alpha) = \int_0^\infty \left(\left(\frac{Q}{2}-\alpha\right)^2 e^{-t} - \frac{\sinh^2((\frac{Q}{2}-\alpha)\frac{t}{2})}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})}\right)\frac{dt}{t}$$

Structure Constant

$C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ is the structure constant of **Liouville Conformal Field Theory**

Physics:

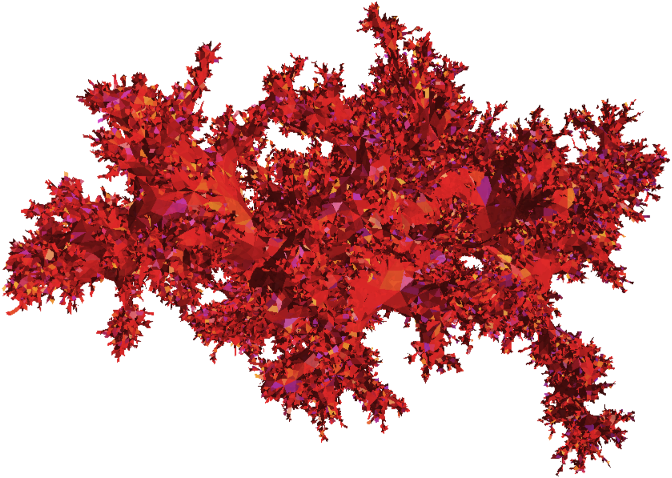
- String theory, 2d quantum gravity,
- 4d Yang-Mills; AGT correspondence

Mathematics:

- Fractal random surfaces
- Quantum cohomology

$\gamma = \sqrt{2}$, Quantum Sphere

Random fractal surfaces



F. David

Quantum Gravity

Polyakov '81: string theory in terms of **gravity on world sheet**

(Euclidean) gravity: Riemannian metric $g = g_{\mu\nu} dx^\mu dx^\nu$.

(Euclidean) **quantum** gravity: g is **random**:

$$\langle F(g, \phi) \rangle = \int F(g, \phi) e^{-S_{gravity}(g) - S_{matter}(g, \Psi)} Dg D\Psi$$

Gravity action

$$S_{gravity}(g) = \frac{1}{2\kappa} \int R_g \sqrt{\det g} dx + \mu \int \sqrt{\det g} dx$$

Ψ: matter fields, say for free scalar field

$$S_{matter}(g, \Psi) = \int g^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi \sqrt{\det g} dx$$

2d Gravity

In **two dimensions** can go to conformal coordinates

$$g = e^{\sigma(z)}((dx)^2 + (dy)^2), \quad z = x + iy.$$

Einstein-Hilbert Action is topological in 2d:

$$\int R_g \sqrt{\det g} dx = 8\pi(1 - \text{genus})$$

so only cosmological constant term $\mu \int e^{\sigma} dz$.

What is the probability law of $\sigma(z)$? I.e. find $S_{effective}(\sigma)$:

$$\langle F(g) \rangle = \int F(e^{\sigma}) e^{-S_{effective}(\sigma)} D\sigma.$$

For **conformal** matter the answer is known (conjectured!).

Liouville Gravity

Knizhnik, Polyakov, and Zamolodchikov '88:

Let the matter be **conformal field theory** with central charge $c \leq 1$. Then

$$\sigma = \gamma\phi$$

and the distribution of ϕ is given by

$$\langle f(\phi)\rangle = \int f(\phi)e^{-S(\phi)}D\phi$$

with $S(\phi)$ the **Liouville action functional**:

$$S(\phi) = \int (|\partial_z\phi(z)|^2 + \mu e^{\gamma\phi(z)})dz$$

- $c = 25 - 6Q^2$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$
- $c \leq 1 \Leftrightarrow Q^2 \geq 4 \Leftrightarrow \gamma \in \mathbb{R}$

Random Surfaces

Random triangulation T of S^2:

$$\mathbb{P}(T) = e^{-\mu|T|} Z(T)$$

$|T|$ number of triangles, $Z(T)$ **partition function** of a spin system

- Pure gravity (no matter): $Z(T) = 1$
- Ising model: $\sigma : \{\textit{triangles of } T\} \to \{-1, 1\}$

$$Z(T) = \sum_{\sigma} e^{\beta \sum_{t \sim t'} \sigma_t \sigma_{t'}}$$

Scaling limit: let $(\beta, \mu) \to (\beta_c, \mu_c)$ and rescale

Conjecture Obtain Liouville theory in the limit with $\gamma = \sqrt{8/3}$ (pure gravity), $\gamma = \sqrt{3}$ (Ising)

There are numerical checks

Gravitational Dressing

Example: Ising model $\gamma = \sqrt{3}$. Let

- σ be scaling limit of Ising spin
- $\tilde{\sigma}$ be scaling limit of Ising spin on a random triangulation

Then

$$\tilde{\sigma}(z) = e^{\alpha\phi(z)}\sigma(z)$$

with ϕ the $\gamma = \sqrt{3}$ Liouville field and $\alpha = \frac{5}{2\sqrt{3}}$.

Similar formuli for all $c \leq 1$ models (Potts, tricritical Ising, etc.)

Hence we need to understand correlation functions of **vertex operators** $e^{\alpha\phi(z)}$in Liouville theory:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = \int \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} e^{-S(\phi)} D\phi$$

Conformal Field Theory

Conformal Field Theory is a massless 2d quantum field theory

Belavin, Polyakov, Zamolodchikov '84: Conformal Field Theory is determined by

- **Spectrum**: the set of primary fields Ψ_i, $i \in I$
  - Transform like tensors under conformal transformations
  - E.g. in Ising model spin and energy are primary fields
- **Three point functions** $\langle \Psi_i(z_1)\Psi_j(z_2)\Psi_k(z_3)\rangle$

By Möbius invariance suffices to find **structure constants**

$$C(i,j,k) = \langle \Psi_i(0)\Psi_j(1)\Psi_k(\infty)\rangle$$

BPZ: spectrum and structure constants determine all correlation functions by **Conformal Bootstrap**

Conformal Bootstrap

Basic postulate of BPZ: in correlation functions **operator product expansion** (OPE)holds:

$$\Psi_i(z)\Psi_j(w) = \sum_k C_{ij}^k(z,w)\Psi_k(w)$$

where $C_{ij}^k(z,w)$ are given in terms of structure constants $C(i,j,k)$. Iterating this n-point function is given in terms of structure constants.

BPZ found $C(i,j,k)$ for **minimal models** (e.g. Ising).

Liouville model should be a CFT so can one solve it?

BPZ **failed** to find structure constants for Liouville

Conformal Field Theory is an "unsuccesful attempt to solve the Liouville model" (Polyakov)

DOZZ Conjecture

The **spectrum** of Liouville was conjectured to be (Braaten, Curtright, Thorn, Gervais, Neveu, 1982):

$$\Psi_\alpha = e^{\alpha\phi}, \quad \alpha = Q + iP, \quad P > 0 \quad (Q = \tfrac{\gamma}{2} + \tfrac{2}{\gamma})$$

'94-96 DOZZ gave an explicit formula for Liouville structure constants

$$C(\alpha_1, \alpha_2, \alpha_3) = \langle e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle$$

Its original derivation was somewhat mysterious:

"It should be stressed that the arguments of this section have nothing to do with a derivation. These are rather some motivations and we consider the expression proposed as a guess which we try to support in the subsequent sections"

Attempts to derive of DOZZ formula

1. Perturbation theory in cosmological constant μ (DOZZ)
   - Order by order ∞, interpret terms as residues of poles in α, "analytically continue" from integers; cf. cite above
   - Problem: μ is not a perturbative parameter (KPZ scaling)
2. Assume the full machinery of CFT (Teschner '95)
   - Fusion rules of degenerate fields
   - Bootstrap of 4-point functions to 3-point functions
   - A mysterious **reflection relation** $\Psi_\alpha = R(\alpha)\Psi_{2Q-\alpha}$
3. Attempts for quantum integrability (Teschner '01)
4. Functional integral (Harlow, Maltz, Witten 2011)

What does it mean to prove the DOZZ formula?

1. Bootstrap. **Define** CFT by assuming spectrum, OPE and conformal Ward identites. Then structure constants have to satisfy consistency conditions:

$$\langle \Psi_i \Psi_j \Psi_k \Psi_l \rangle = \sum_m C_{ij}^m C_{mkl} = \sum_m C_{ik}^m C_{mjl}$$

DOZZ formula satisfies this quadratic equation.

2. Our proof of DOZZ:

- **Rigorous construction** of Liouville functional integral DKRV2014
- **Proof** of the CFT machinery (Ward identities, BPZ equations) KRV2016
- Probabilistic derivation of **reflection relation** KRV2017

Probabilistic Liouville Theory

What is the meaning of

$$e^{-\int (|\partial_z \phi(z)|^2 + \mu e^{\gamma\phi(z)})dz} D\phi \quad ?$$

- $e^{-\int (|\partial_z \phi(z)|^2} D\phi$ can be defined in terms of **Gaussian Free Field (GFF)**
- GFF has a Gaussian distribution with covariance (2-point function) $-\Delta^{-1}$
- Work on sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$.
- On $\mathbb{S}^2$ $Ker(\Delta) = \{constants\}$
- $\phi = c + \psi$, c constant and $\psi \perp Ker(\Delta)$

Then define the Liouville functional integral as

$$\langle F(\phi) \rangle := \int_{\mathbb{R}} \mathbb{E}\left[F(c+\psi) e^{-\mu \int e^{\gamma(c+\psi(z))} dz} \right] dc$$

Gaussian Multiplicative Chaos (GMC)

Problem: ψ is **not** a function:

$$\mathbb{E}\psi(z)\psi(z') \sim_{z\to z'} \log|z-z'|^{-1}$$

so that $\mathbb{E}\psi(z)^2 = \infty$. What does $e^{\gamma\psi(z)}$ mean?

Regularize: $\psi \to \psi_\epsilon$ with UV cutoff ϵ and **renormalize**

$$\lim_{\epsilon\to 0} e^{\gamma\psi_\epsilon(z)-\frac{\gamma^2}{2}\mathbb{E}\psi_\epsilon(z)^2} dz = dM(z) \quad \textit{almost surely}$$

M is a **random measure** on $\mathbb{S}^2$ and $M \neq 0 \Leftrightarrow \gamma < 2$

It has nontrivial **multifractal spectrum**.

Existence of Liouville correlations

Need to renormalize vertex operators too: we define

$$\langle\prod_{i=1}^{n} e^{\alpha_i\phi(z_i)}\rangle := \lim_{\epsilon\to 0}\int dc\,\mathbb{E}(\prod_{i=1}^{n} e^{\alpha_i c} e^{\alpha_i\psi_\epsilon(z_i)-\frac{\alpha_i^2}{2}\mathbb{E}\psi_\epsilon(z_i)^2} e^{-\mu e^{\gamma c}\int_{\mathbb{S}^2} dM(z)})$$

Theorem (DKRV 2014) *The limit (1) exists and is nontrivial if and only if:*

$$(A)\ \ \forall i : \alpha_i < Q \quad \text{and} \quad (B)\ \ \sum_i \alpha_i > 2Q$$

Remarks

- Recall that $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$
- (A), (B) are called **Seiberg bounds**
- (A), (B) $\implies$ $n \geq 3$: **1- and 2-point functions are ∞**.

Proof is based on representation in terms of GMC

0-mode

Integrate first over the constant c:

$$\langle\prod_{i=1}^{n} e^{\alpha_i\phi(z_i)}\rangle = \mathbb{E}\left[\prod_i e^{\alpha_i\psi(z_i)}\int_{\mathbb{R}} e^{\sum_i \alpha_i c} e^{-2Qc} e^{-\mu e^{\gamma c}\int e^{\gamma\psi}dz} dc\right]$$

e^{-2Qc} for topological reasons.

The c-integral converges **if** $\sum_i \alpha_i > 2Q$:

$$\langle\prod_{i=1}^{n} e^{\alpha_i\phi(z_i)}\rangle = \frac{\Gamma(s)}{\mu^s\gamma}\mathbb{E}\left[\prod_i e^{\alpha_i\psi(z_i)}(\int e^{\gamma\psi}dz)^{-s}\right]$$

where $s := (\sum_i \alpha_i - 2Q)/\gamma$.

This explains Seiberg bound (B)

Modulus of Chaos

Shift (Cameron-Martin theorem)

$$\psi(z) \to \psi(z) + \sum_i \alpha_i \underbrace{\mathbb{E}\psi(z)\psi(z_i)}_{=-\log|z-z_i|}$$

Result: Liouville correlations are given by

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = \frac{\Gamma(s)}{\mu^s \prod_{i<j} |z_i - z_j|^{\alpha_i \alpha_j}} \mathbb{E} \left(\int \prod_i \frac{1}{|z - z_i|^{\gamma \alpha_i}} dM(z) \right)^{-s}$$

Seiberg bound (A): The measure $dM = e^{\gamma\psi} dz$ has **scaling dimension** $\gamma Q = 2 + \frac{\gamma^2}{2} \implies$ **almost surely** we have

$$\int \frac{1}{|z - z_i|^{\gamma \alpha_i}} dM(z) < \infty \Leftrightarrow \alpha_i < Q.$$

So surprisingly

$$" e^{\alpha\phi} \equiv 0" \quad \alpha \geq Q$$

Liouville Conformal Field Theory

Liouville model defines indeed a Conformal Field Theory:

Theorem (DKRV 2014, KRV 2016)

$$\langle \prod_i e^{\alpha_i \phi(f(z_i))} \rangle = \prod_i |f'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_i e^{\alpha_i \phi(z_i)} \rangle$$

where $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius map and $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

They satisfy **conformal Ward identities** and the **Weyl anomaly** formula with **central charge**

$$c = 1 + 6Q^2$$

Structure constants

We obtain a probabilistic expression for the structure constants

$$C(\alpha_1, \alpha_2, \alpha_3) \propto \mathbb{E}\left(\int \frac{(\max(|z|, 1))^{\gamma\bar{\alpha}}}{|z|^{\gamma\alpha_1}|1-z|^{\gamma\alpha_2}} M(dz)\right)^{\frac{2Q-\bar{\alpha}}{\gamma}}$$

$\bar{\alpha} := \alpha_1 + \alpha_2 + \alpha_3$.

The DOZZ formula is an explicit conjecture for this expectation.

Note!

- DOZZ formula is defined for α in **spectrum** i.e. $\alpha = Q + iP$
- Probabilistic formula is defined for **real** α_i satisfying Seiberg bounds
- Real α_i are relevant for random surfaces, complex α_i for Liouville CFT

Dilemma

The DOZZ proposal $C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$ is a **meromorphic** function of $\alpha_i \in \mathbb{C}$. In particular for **real** α's

$$C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3) \neq 0 \quad \text{if} \quad \alpha_i > Q$$

The probabilistic $C(\alpha_1, \alpha_2, \alpha_3)$ is **identically zero** in this region:

$$C(\alpha_1, \alpha_2, \alpha_3) \equiv 0 \quad \alpha_i \geq Q$$

What is going on? DOZZ is too beautiful to be wrong!

Remark. *One can renormalize $e^{\alpha\phi}$ for $\alpha \geq Q$ so that $C(\alpha_1, \alpha_2, \alpha_3) \neq 0$. However the result does not satisfy DOZZ.*

Analyticity

Theorem (KRV 2017)

The structure constants $C(\alpha_1, \alpha_2, \alpha_3)$ are analytic in α_i in a neighborhood of $\alpha_i \in (0, Q)$ and they have a unique analytic continuation to meromorphic function ***satisfying the DOZZ conjecture.***

Note! Due to renormalization $|e^{i\beta\phi(x_i)}| = \infty$ so that analyticity even near real α_i is subtle!

Belavin-Polyakov-Zamolodchicov equation

Consider a **4-point function**

$$F(u) := \langle e^{-\chi\phi(u)} e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle$$

Theorem (KRV2016) *For $\chi = \frac{\gamma}{2}$ **or** $\chi = \frac{2}{\gamma}$, F satisfies a hypergeometric equation*

$$\partial_u^2 F + \frac{a}{u(1-u)} \partial_u F - \frac{b}{u(1-u)} F = 0$$

Proof: Gaussian integration by parts & regularity estimates.

Remark: In CFT jargon $e^{-\chi\phi(u)}$ are **level 2 degenerate fields**. In bootstrap approach this is postulated, we **prove** it.

Solution

We have the multiplicative chaos representation:

$$F(u) \propto \mathbb{E}\left(\int \frac{(\max(|z|,1))^{\gamma\bar{\alpha}}}{|z-u|^{-\gamma\chi}|z|^{\gamma\alpha_1}|1-z|^{\gamma\alpha_2}} M(dz) \right)^{\frac{2Q-\bar{\alpha}}{\gamma}}$$

Asymptotics as $u \to 0$: **if** $\alpha_1 + \chi < Q$ then

$$F(u) = C(\alpha_1-\chi,\alpha_2,\alpha_3) + A(\alpha_1)C(\alpha_1+\chi,\alpha_2,\alpha_3)|u|^{\frac{\gamma}{2}(Q-\alpha_1)} + \dots$$

with A explicit ratio of 12 Γ-functions!

This can be stated as a **fusion rule**:

$$e^{-\chi\phi} \times e^{\alpha\phi} = e^{(\alpha-\chi)\phi} + A(\alpha)e^{(\alpha+\chi)\phi}$$

which is postulated in the bootstrap approach.

Periodicity

The BPZ equation implies a relation between the coefficients

$$C(\alpha_1+\chi,\alpha_2,\alpha_3)=D(\chi,\alpha_1,\alpha_2,\alpha_3)C(\alpha_1-\chi,\alpha_2,\alpha_3)$$

with $\chi=\frac{\gamma}{2}$ **or** $\frac{2}{\gamma}$. and $D(\chi,\alpha_1,\alpha_2,\alpha_3)$ explicit.

$$D(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=-\frac{1}{\pi\mu}\frac{\Gamma(-\chi^2)\Gamma(-\alpha_0\alpha_1)\Gamma(\chi\alpha_1-\chi^2)\Gamma(\frac{\chi}{2}(\bar{\alpha}-2\alpha_1))}{\Gamma(\frac{\chi}{2}(\bar{\alpha}-2Q))\Gamma(\frac{\chi}{2}(\bar{\alpha}-2\alpha_3))\Gamma(\frac{\chi}{2}(\bar{\alpha}-2\alpha_2))}$$
$$\times\frac{\Gamma(1+\frac{\chi}{2}(2Q-\bar{\alpha}))\Gamma(1+\frac{\chi}{2}(2\alpha_3-\bar{\alpha}))\Gamma(1+\frac{\chi}{2}(2\alpha_2-\bar{\alpha}))}{\Gamma(1+\chi^2)\Gamma(1+-\chi\alpha_1)\Gamma(1-\chi\alpha_1+\chi^2)\Gamma(1+\frac{\chi}{2}(2\alpha_1-\bar{\alpha}))}$$

Teschner '95: DOZZ is the **unique analytic** solution of these equations **if** we could extend $C(\alpha_1,\alpha_2,\alpha_3)$ beyond the region $\alpha_i<Q$.

Reflection

What happens **if** $\alpha_1 + \chi > Q$? $F(u)$ has same asymptotics with the replacement:

$$C(\alpha_1 + \chi, \alpha_2, \alpha_3) \to R(\alpha + \chi)C(2Q - (\alpha_1 + \chi)), \alpha_2, \alpha_3)$$

This can be stated informally as the **reflection relation**

$$e^{\alpha\phi} = R(\alpha)e^{(2Q-\alpha)\phi}, \;\; \alpha > Q.$$

This is postulated in bootstrap since DOZZ satisfies it.

We prove it with a **probabilistic** expression for R.

Reflection coefficient

$R(\alpha)$ is given in terms of **tail behavior** of multiplicative chaos:

Let $\alpha < Q$, D any neighborhood of origin and

$$Z_D := \int_D \frac{1}{|z|^{\gamma\alpha}} M(dz)$$

Then

$$\mathbb{P}(Z > x) = R(\alpha)|x|^{-\frac{2(Q-\alpha)}{\gamma}}(1 + o(z))$$

$R(\alpha)$ enters since asymptotics of the four point function

$$F(u) = \langle e^{-\chi\phi(u)} e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle$$

as $u \to 0$ is controlled by the singularity at α_1.

Integrability

We use R to obtain **analytic continuation** of $C(\alpha_1, \alpha_2, \alpha_3)$ to $\alpha_1 > Q$:

$$C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1) C(2Q - \alpha_1, \alpha_2, \alpha_3)$$

DOZZ formula satifies this with

$$R_{DOZZ}(\alpha) = -((\tfrac{\gamma}{2})^{\frac{\gamma^2}{2}-2} \tilde{\mu})^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(\frac{\gamma}{2}(\alpha - Q))\Gamma(\frac{2}{\gamma}(\alpha - Q))}{\Gamma(\frac{\gamma}{2}(Q - \alpha))\Gamma(\frac{2}{\gamma}(Q - \alpha))}.$$

To conclude the proof of DOZZ formula we need to show the probabilistic R equals R_{DOZZ}.

This is done by proving analyticity of R in the probabilistic region $\alpha < Q$ and deriving shift identities for R that determine it.

Outlook

We have shown DOZZ follows from the functional integral. This settles also two puzzles:

- **Duality**: DOZZ invariant under $\gamma \to \frac{4}{\gamma}$ with:

$$\mu \to \tilde{\mu} = (\mu\pi\ell(\tfrac{\gamma^2}{4}))^{\frac{4}{\gamma^2}}(\pi\ell(\tfrac{4}{\gamma^2}))^{-1} \quad (\ell(x) = \tfrac{\Gamma(x)}{\Gamma(1-x)}).$$

  but Liouville action is not!
- **Reflection** $e^{\alpha\phi} = R(\alpha)e^{(2Q-\alpha)\phi}, \ \ \alpha > Q$.

Numerical evidence: DOZZ structure constants are essentially the **unique solutions** to $c > 1$ bootstrap equations and Liouville CFT is the unique unitary $c > 1$ CFT (Collier et al)

For $c < 1$ the minimal model structure constants can be be recovered from DOZZ with γ **imaginary** (Ribault, Santa-Chiara)

It would be nice to prove also bootstrap, and integrability of Liouville theory as well as analytic continuation in γ!