From Temperley-Lieb algebras to representations of Virasoro algebras: modules, fusion, and operator algebras

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Helsinki University, October 10, 2012.

Part I: modules and fusion

The quantum group $U_\mathfrak{q} s\ell(2)$

The quantum group $U_\mathfrak{q} s\ell(2)$ with generic $\mathfrak{q}$ has generators E, F, and $K^{\pm 1}$ satisfying the defining relations ($K = \mathfrak{q}^H$ where H is the classical Kartan element)

$$KEK^{-1} = \mathfrak{q}^2 E, \quad KFK^{-1} = \mathfrak{q}^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$

and the Hopf-algebra structure is given by

$$\Delta(E) = \mathbf{1} \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes \mathbf{1}, \quad \Delta(K) = K \otimes K,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1},$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1.$$

The limit of $U_{\mathfrak{q}}s\ell(2)$ to a root of unity

To obtain a Hopf-algebra structure on $U_{\mathfrak{q}}s\ell(2)$ at a root of unity, we use an algebraic approach following Lusztig:

- There is an evident limit $\mathfrak{q} \rightarrow \mathrm{e}^{\frac{\imath\pi}{p}}$ in which E^p, F^p are non-zero and, together with K^p, become central.
- In this limit, the quantum group has the same generators and the same relations.
- The representation theory in this case is very different from generic case: cylcic and semi-cyclic irreducible representations of dimension up to p. Continuum of inequivalent irreps.

The limit of $U_\mathfrak{q} s\ell(2)$ to a root of unity

To obtain a Hopf-algebra structure on $U_\mathfrak{q} s\ell(2)$ at a root of unity, we use an algebraic approach following Lusztig:

- but we consider another limit (so-called Lusztig limit) in which the relations

$$E^p = F^p = 0, \quad K^{2p} = 1$$

are imposed but the generators

$$e = \frac{E^p}{[p]!} \quad \textit{and} \quad f = \frac{F^p}{[p]!}, \qquad [n] = \frac{\mathfrak{q}^n - \mathfrak{q}^{-n}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$

are kept in the limit.

In the limit $\mathfrak{q} \to e^{\frac{\imath\pi}{p}}$, we have $[p]! = 0$ and the ambiguity $\frac{0}{0}$ is solved in such a way that the e and f become generators of the ordinary $s\ell(2)$.

The limit of $U_{\mathfrak{q}} s\ell(2)$ to a root of unity

So, we have two subalgebras in $U_{\mathfrak{q}} s\ell(2)$:

- one is a f.-d. Hopf subalgebra generated by usual letters E, F, and K with relations

$$E^p = F^p = 0, \quad K^{2p} = 1,$$

  it is called the *small quantum group*.

- and an associative (not Hopf) subalgebra $U s\ell(2)$ generated by $e = \frac{E^p}{[p]!}$, $f = \frac{F^p}{[p]!}$ and $h = [e, f]$.

We obtain a Hopf algebra $U_{\mathfrak{q}} s\ell(2)$ that contains
the small quantum group $\overline{U}_{\mathfrak{q}} s\ell(2)$ as a Hopf ideal
and the quotient by it is the $U(s\ell(2))$, the universal enveloping of the $s\ell(2)$

The limit of $U_{\mathfrak{q}} s\ell(2)$ to a root of unity

The Hopf-algebra structure on $U_{\mathfrak{q}} s\ell(2)$ is the following. The defining relations between the E, F, and K generators are the same as in generic case plus $E^p = F^p = 0$ and the usual $s\ell(2)$ relations between the e, f, and h:

$$[h, e] = e, \qquad [h, f] = -f, \qquad [e, f] = 2h,$$

and the "mixed" relations

$$[h, K] = 0, \qquad [E, e] = 0, \qquad [K, e] = 0, \qquad [F, f] = 0, \qquad [K, f] = 0,$$

$$[F, e] \sim (\mathfrak{q}K - \mathfrak{q}^{-1}K^{-1})\, E^{p-1},$$

$$[E, f] \sim (\mathfrak{q}K - \mathfrak{q}^{-1}K^{-1})\, F^{p-1},$$

$$[h, E] = \frac{1}{2}EA, \quad [h, F] = -\frac{1}{2}AF,$$

where A is a projector.

The limit of $U_{\mathfrak{q}} s\ell(2)$ to a root of unity

The comultiplication in $U_{\mathfrak{q}} s\ell(2)$ is

$$\Delta(e) = e \otimes 1 + K^p \otimes e + \tfrac{1}{[p-1]!} \sum_{r=1}^{p-1} \frac{\mathfrak{q}^{r(p-r)}}{[r]} K^p E^{p-r} \otimes E^r K^{-r},$$

$$\Delta(f) = f \otimes 1 + K^p \otimes f + \tfrac{(-1)^p}{[p-1]!} \sum_{s=1}^{p-1} \frac{\mathfrak{q}^{-s(p-s)}}{[s]} K^{p+s} F^s \otimes F^{p-s},$$

an explicit form of $\Delta(h) = \frac{1}{2}[\Delta(e), \Delta(f)]$ is very bulky and we do not give it here.

The antipode S and the counity ϵ are

$$S(e) = -K^p e, \qquad S(f) = -K^p f, \qquad S(h) = -h,$$

$$\epsilon(e) = \epsilon(f) = \epsilon(h) = 0.$$

The Casimir operator and Temperley–Lieb algebra

As a module over $U_{\mathfrak{q}}s\ell(2)$, the XXZ spin chain $\mathcal{H}_N$ is a tensor product of N copies of the "fundamental" two-dimensional simple module $\mathbb{C}^2 = \{\uparrow, \downarrow\}$ such that the genereators are represented on $\mathbb{C}^2$ as

$$E = \sigma^+, \quad F = \sigma^- \quad \text{and} \quad K = \tfrac{\mathfrak{q}+\mathfrak{q}^{-1}}{2}\mathbf{1} + \tfrac{\mathfrak{q}-\mathfrak{q}^{-1}}{2}\sigma^z.$$

Using $(N-1)$-folded comultiplications we obtain the XXZ representation of $U_{\mathfrak{q}}s\ell(2)$

$$\rho_{\mathfrak{q},N} : U_{\mathfrak{q}}s\ell(2) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{H}_N), \qquad \mathcal{H}_N = \bigotimes_{j=1}^{N} \mathbb{C}^2$$

The Casimir operator and Temperley–Lieb algebra

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Using $(N-1)$-folded comultiplications we obtain the XXZ representation of $U_{\mathfrak{q}} s\ell(2)$ with expressions for the generators

$$\rho_{\mathfrak{q},N}(E) = \sum_{j=1}^{N} \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{j-1} \otimes \sigma^+ \otimes \mathfrak{q}^{\sigma^z} \otimes \ldots \otimes \mathfrak{q}^{\sigma^z},$$

$$\rho_{\mathfrak{q},N}(F) = \sum_{j=1}^{N} \underbrace{\mathfrak{q}^{-\sigma^z} \otimes \ldots \otimes \mathfrak{q}^{-\sigma^z}}_{j-1} \otimes \sigma^- \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}.$$

The Casimir operator and Temperley–Lieb algebra

We recall that the Casimir element

$$\boldsymbol{C} = FE + \frac{\mathfrak{q}K + \mathfrak{q}^{-1}K^{-1}}{(\mathfrak{q} - \mathfrak{q}^{-1})^2}$$

commutes with the full (Lusztig) quantum group $U_{\mathfrak{q}}s\ell(2)$ and it is represented on $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$\Delta(\boldsymbol{C}) = \tfrac{1}{2}\Big(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \tfrac{\mathfrak{q} + \mathfrak{q}^{-1}}{2}\sigma^z \otimes \sigma^z\Big) \\ + \tfrac{\mathfrak{q} - \mathfrak{q}^{-1}}{4}\Big(\mathbf{1} \otimes \sigma^z - \sigma^z \otimes \mathbf{1}\Big) + \tfrac{3\mathfrak{q}^3 + \mathfrak{q} + \mathfrak{q}^{-1} + 3\mathfrak{q}^{-3}}{4(\mathfrak{q} - \mathfrak{q}^{-1})^2}\mathbf{1} \otimes \mathbf{1}.$$

This gives the usual XXZ coupling and the hamiltonian is defined as a sum over the densities $\Delta(\boldsymbol{C})_{i,i+1}$ acting on ith and $(i+1)$th tensorands:

$$H(\mathfrak{q}) = \sum_{i=1}^{N-1} \Delta(\boldsymbol{C})_{i,i+1} - (N-1)\tfrac{3\mathfrak{q}^3 + \mathfrak{q} + \mathfrak{q}^{-1} + 3\mathfrak{q}^{-3}}{4(\mathfrak{q} - \mathfrak{q}^{-1})^2}.$$

The Casimir operator and Temperley–Lieb algebra

Introducing more convenient notation

$$e_i = -\Delta(\boldsymbol{C})_{i,i+1} + \frac{\mathfrak{q}^3 + \mathfrak{q}^{-3}}{(\mathfrak{q} - \mathfrak{q}^{-1})^2}\mathbf{1}, \qquad 1 \leqslant i \leqslant N-1,$$

the hamiltonian takes the usual form

$$H(\mathfrak{q}) = -\sum_{i=1}^{N-1} e_i + (N-1)\frac{\mathfrak{q} + \mathfrak{q}^{-1}}{4}$$

It is straightforward to check that the operators e_i, for $1 \leqslant i \leqslant N-1$, satisfy the defining relations of the Temperley–Lieb algebra $\mathcal{TL}_{\mathfrak{q},N}$

$$e_i^2 = (\mathfrak{q} + \mathfrak{q}^{-1})e_i,$$

$$e_i e_{i\pm1} e_i = e_i,$$

$$e_i e_j = e_j e_i, \quad |i-j| > 1.$$

The Casimir operator and Temperley–Lieb algebra

- The relation of the Temperley–Lieb generators e_i with the quantum-group Casimir operator and the coassociativity of the quantum-group comultiplication give the well-known result (Pasquier–Saleur, 1990)

$$[U_{\mathfrak{q}}s\ell(2), \mathcal{TL}_{\mathfrak{q},N}] = 0$$

including the divided powers.

- It was shown by P.P. Martin (1992) that $\mathcal{TL}_{\mathfrak{q},N}$ **is the centralizer** (i.e. the algebra of intertwiners or all commuting operators) of the representation of $U_{\mathfrak{q}}s\ell(2)$ on $\mathcal{H}_N$ and vice versa, i.e. they are mutual centralizers,

$$\mathcal{TL}_{\mathfrak{q},N} \cong \mathrm{End}_{U_{\mathfrak{q}}s\ell(2)}(\mathcal{H}_N), \qquad \rho_{\mathfrak{q},N}\big(U_{\mathfrak{q}}s\ell(2)\big) \cong \mathrm{End}_{\mathcal{TL}_{\mathfrak{q},N}}(\mathcal{H}_N),$$

for any $\mathfrak{q}$ at the root of unity cases.

The Casimir operator and Temperley–Lieb algebra

- The $\mathcal{TL}_{\mathfrak{q},N}$ **is the centralizer** of the representation of $U_{\mathfrak{q}}s\ell(2)$ on $\mathcal{H}_N$ and vice versa

$$\mathcal{TL}_{\mathfrak{q},N} \cong \mathrm{End}_{U_{\mathfrak{q}}s\ell(2)}(\mathcal{H}_N), \qquad \rho_{\mathfrak{q},N}\big(U_{\mathfrak{q}}s\ell(2)\big) \cong \mathrm{End}_{\mathcal{TL}_{\mathfrak{q},N}}(\mathcal{H}_N),$$

for any $\mathfrak{q}$ at the root of unity cases.

This statement is a quantum analogue of the classical Schur–Weyl duality between the symmetric group S_N and the special linear group $SL(2)$.

The quantum Schur–Weyl duality (generic case)

When $\mathfrak{q}$ is generic, the Hilbert space of the Hamiltonian densities decomposes onto the so-called standard modules of $\mathcal{TL}_{\mathfrak{q},N}$ (generically irreducible)

$$\mathcal{H}_N|_{\mathcal{TL}_{\mathfrak{q},N}} \cong \bigoplus_{j=(N \bmod 2)/2}^{N/2} (2j+1)\mathsf{S}_j[N],$$

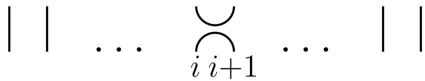
where the degeneracies $2j+1$ are dimensions of the *Weyl modules* $\mathcal{W}_j$ (which are also generically irreducible) over the centralizer of $\mathcal{TL}_{\mathfrak{q},N}$, which is a f.-d. homomorphic image of the quantum group $U_{\mathfrak{q}}s\ell(2)$ called the $\mathfrak{q}$-Schur algebra.

The space $\mathcal{H}_N$ as a semi-simple bi-module over the pair $\mathcal{TL}_{\mathfrak{q},N} \boxtimes U_{\mathfrak{q}}s\ell(2)$

$$\mathcal{H}_N|_{\mathcal{TL}_{\mathfrak{q},N} \boxtimes U_{\mathfrak{q}}s\ell(2)} \cong \bigoplus_{j=(N \bmod 2)/2}^{N/2} \mathcal{W}_j \boxtimes \mathsf{S}_j[N].$$

The quantum Schur–Weyl duality (standard modules)

The TL algebra is best understood diagrammatically. Introducing the notation

$e_i =$ 

the defining TL relations can now be interpreted geometrically, the composition law corresponding to stacking the diagrams of the e_i's where it is assumed that every closed loop carries a weight $\mathfrak{q} + \mathfrak{q}^{-1}$.

Within this geometrical setup, the algebra $\mathcal{TL}_{\mathfrak{q},N}$ itself can be thought of as an algebra of diagrams.

The quantum Schur–Weyl duality (standard modules)

For a (half-)integer $0 \leqslant j \leqslant N/2$, we define a standard module $\mathsf{S}_j[N]$, which is irreducible for $\mathfrak{q}$ generic, as the span of link diagrams with $2j$ through-lines (also called “strings”) which are not allowed to be contracted by the TL generators. The action of the generators on these modules is again interpreted as stacking the various diagrams.

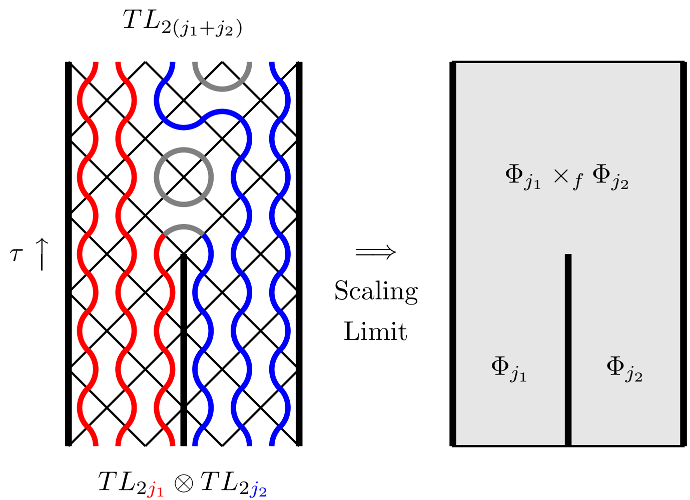
The dimension of the standard modules does not depend on $\mathfrak{q}$

$$\dim(\mathsf{S}_j[N]) = \binom{N}{N/2+j} - \binom{N}{N/2+j+1}$$

For $N = 4$ for instance, there are four standard modules $\mathsf{S}_2[4] = \{\, ||||\, \}$, $\mathsf{S}_1[4] = \{\, |\cup| \,,\, \cup|| \,,\, ||\cup \,\}$, and $\mathsf{S}_0[4] = \{\, \smile\!\!\!\!\!\cup \,,\, \cup\cup \,\}$ with the TL action

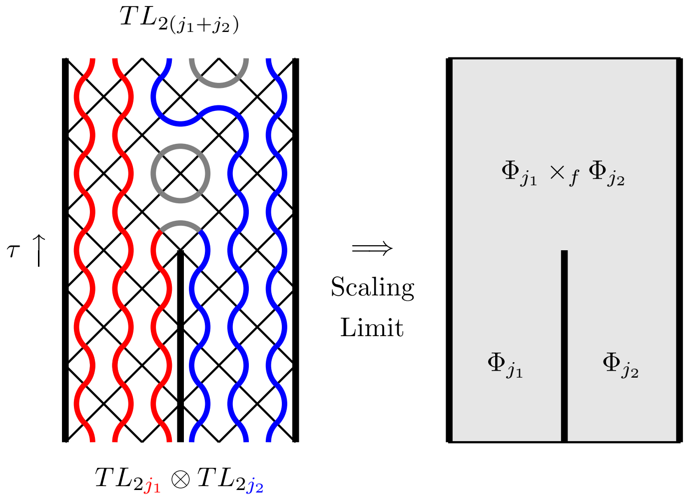
$$e_2 \; |\cup| = (\mathfrak{q} + \mathfrak{q}^{-1}) \; |\cup| \,, \quad e_2 \; \cup|| = \; |\cup| \,, \quad \text{and} \quad e_3 \; \cup|| = 0.$$

Temperley–Lieb fusion (informal definition)



The fusion of two standard TL modules $\mathsf{S}_{j_1}[N_1]$ and $\mathsf{S}_{j_2}[N_2]$ (in the picture, $N_1 = 2j_1$ and $N_2 = 2j_2$ so that both standard modules are one-dimensional). The induction procedure can be seen as an event in imaginary time τ, consisting in “joining” the two standard modules by acting with an additional TL generator.

Temperley–Lieb fusion (informal definition)



The idea is that in the continuum limit, we expect this construction to coincide with the usual fusion procedure or OPE of boundary fields, here Φ_{j_1} and Φ_{j_2}, living in the corresponding Virasoro modules.

Temperley–Lieb fusion (formal definition)

Our approach relies on the so-called *induction functor* which associates with any pair of modules over the algebras $\mathcal{TL}_{\mathfrak{q},N_1}$ and $\mathcal{TL}_{\mathfrak{q},N_2}$ a module over the bigger algebra $\mathcal{TL}_{\mathfrak{q},N_1+N_2}$.

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Let M_1 and M_2 be two modules over $\mathcal{TL}_{\mathfrak{q},N_1}$ and $\mathcal{TL}_{\mathfrak{q},N_2}$. Then, the tensor product $M_1 \otimes M_2$ is a module over the product $\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}$ of the two algebras. (This product of algebras is naturally a subalgebra in $\mathcal{TL}_{\mathfrak{q},N_1+N_2}$.)

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The fusion $\times_f$ of two modules M_1 and M_2 is defined as the module induced from this subalgebra, $i.e.$

$$M_1 \times_f M_2 = \mathcal{TL}_{\mathfrak{q},N_1+N_2} \otimes_{\left(\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}\right)} M_1 \otimes M_2$$

Temperley–Lieb fusion (formal definition)

The fusion $\times_f$ of two modules M_1 and M_2 is defined as the module induced from the subalgebra $\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}$

$$M_1 \times_f M_2 = \mathcal{TL}_{\mathfrak{q},N_1+N_2} \otimes_{\left(\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}\right)} M_1 \otimes M_2$$

Recall: the balanced product $\otimes_A$ (of right and left modules) over an algebra A is defined as a quotient of the usual tensor product by the relations

$$v_1 \triangleleft a \otimes v_2 = v_1 \otimes a \triangleright v_2, \qquad \text{for all} \quad a \in A,$$

where left and right actions of A are denoted by $\triangleright$ and $\triangleleft$, — any element from A can *pass through* the tensor-product symbol from right to left.

Temperley–Lieb fusion (formal definition)

The fusion $\times_f$ of two modules M_1 and M_2 is defined
as the module induced from the subalgebra $\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}$

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Recall: the balanced product $\otimes_A$ over an algebra A is

$$v_1 \triangleleft a \otimes v_2 = v_1 \otimes a \triangleright v_2, \qquad \text{for all} \quad a \in A.$$

In our context, A is $\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}$ and we consider $\mathcal{TL}_{\mathfrak{q},N_1+N_2}$ as a left-right module over itself, with the left and right actions given by the multiplication, and in particular it is a right module over the subalgebra A.

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The fusion $\times_f$ of two modules M_1 and M_2 is defined
as the module induced from the subalgebra $\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}$

$$M_1 \times_f M_2 = \mathcal{TL}_{\mathfrak{q},N_1+N_2} \otimes_{\left(\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}\right)} M_1 \otimes M_2$$

We call the decomposition of the induced module (into indecomposable direct summands) as *the fusion rules* for the pair of left modules M_1 and M_2 over $\mathcal{TL}_{\mathfrak{q},N_1}$ and $\mathcal{TL}_{\mathfrak{q},N_2}$ for any choice of N_1 and N_2 such that $N_1 + N_2 = N$.

Temperley–Lieb fusion (Example I)

The fusion $\mathsf{S}_0[2] \times_f \mathsf{S}_1[2]$ results in a 3-dim $\mathcal{TL}_{\mathfrak{q},4}$-module with the basis

$$\mathsf{S}_0[2] \times_f \mathsf{S}_1[2] = \langle \cup \otimes ||, e_2 \cup \otimes ||, e_3 e_2 \cup \otimes || \rangle$$

Indeed, these states are the only ones allowed, because of the relation

$$e_1 e_2 \cup \otimes || = \delta^{-1} e_1 e_2 e_1 \cup \otimes || = \delta^{-1} e_1 \cup \otimes || = \cup \otimes ||, \qquad \delta = \mathfrak{q} + \mathfrak{q}^{-1},$$

which follows from the relation on 2 sites $\cup = (\mathfrak{q} + \mathfrak{q}^{-1})^{-1} e_1 \cup$. So for example, we have $e_1 e_3 e_2 \cup \otimes || = 0$.

The fusion module is isomorphic to $\mathsf{S}_1[4]$, and we thus obtain

$$\mathsf{S}_0[2] \times_f \mathsf{S}_1[2] = \mathsf{S}_1[4]$$

Temperley–Lieb fusion (Example II)

Consider a more interesting example where the fusion yields indecomposable modules that consist of a gluing of two standard TL modules.

Consider the fusion $\mathsf{S}_1[2] \times_f \mathsf{S}_1[2]$, where $\mathsf{S}_1[2]$ has the basis $\{\,||\,\}$ with $e_1\,|| = 0$. The induction results in a six-dimensional $\mathcal{TL}_{\mathfrak{q},4}$-module with the basis

$$\mathsf{S}_1[2] \times_f \mathsf{S}_1[2] = \langle\, l,\ e_2 l,\ e_1 e_2 l,\ e_3 e_2 l,\ e_1 e_3 e_2 l,\ e_2 e_1 e_3 e_2 l \,\rangle,$$

with $l = \,||\, \otimes \,||$. This module is decomposed for $\mathfrak{q}$ generic as

$$\mathsf{S}_1[2] \times_f \mathsf{S}_1[2] = \mathsf{S}_0[4] \oplus \mathsf{S}_1[4] \oplus \mathsf{S}_2[4],$$

where the two-dimensional invariant subspace $\mathsf{S}_0[4]$ is spanned by $e_1 e_3 e_2 l$ and $e_2 e_1 e_3 e_2 l$ which may be identified with the link states $\cup\,\cup$ and $\smile$, respectively.

Temperley–Lieb fusion (Example II)

$$\mathsf{S}_1[2] \times_f \mathsf{S}_1[2] = \langle\, l,\ e_2 l,\ e_1 e_2 l,\ e_3 e_2 l,\ e_1 e_3 e_2 l,\ e_2 e_1 e_3 e_2 l \,\rangle$$
$$= \mathsf{S}_0[4] \oplus \mathsf{S}_1[4] \oplus \mathsf{S}_2[4]$$

with $l = {}_{|\,|} \otimes {}_{|\,|}$.

- The invariant one-dimensional subspace $\mathsf{S}_2[4]$ is spanned, after solving a simple system of linear equations, by

$$\mathrm{inv}(\delta) = l + \tfrac{1}{\delta^2 - 2}\Big(e_1 e_2 l + e_3 e_2 l - \delta e_2 l + \tfrac{1}{\delta^2 - 1}(e_2 e_1 e_3 e_2 l - \delta e_1 e_3 e_2 l)\Big),$$

with

$$e_j \mathrm{inv}(\delta) = 0, \qquad \text{for} \quad j = 1, 2, 3.$$

- Three remaining linearly independent states contribute to the 3-dim irreducible direct summand isomorphic to $\mathsf{S}_1[4]$.

Temperley–Lieb fusion (Example II)

Note that the fusion states can be identified with link states in the following way: $l = \textcolor{red}{|\,|}\,\textcolor{blue}{|\,|}$, $e_2 l = \textcolor{red}{|} \cup \textcolor{blue}{|}$, $e_1 e_2 l = \cup\, \textcolor{red}{|}\,\textcolor{blue}{|}$, $e_3 e_2 l = \textcolor{red}{|}\,\textcolor{blue}{|} \cup$, $e_1 e_3 e_2 l = \cup \cup$, and $e_2 e_1 e_3 e_2 l =$ . When a Temperley-Lieb generator acts on two through lines with two different colors, it comes with a weight 1 instead of 0. In other words, one can fuse a red trough-line with a blue one with weight 1. For example, one has

$$e_2\, \textcolor{red}{|\,|}\,\textcolor{blue}{|\,|} = \textcolor{red}{|} \cup \textcolor{blue}{|}$$

while

$$e_1\, \textcolor{red}{|\,|}\,\textcolor{blue}{|\,|} = 0$$

With these rules in hand, the calculations become easy and can be done geometrically in terms of diagrams.

Temperley–Lieb fusion (Example II)

We see that the submodules $\mathsf{S}_0[4]$ and $\mathsf{S}_1[4]$ (or their basis elements) have a well-defined limit $\delta \to 1$ ($p = 3$, percolation) while the invariant $\mathrm{inv}(\delta)$ spanning $\mathsf{S}_2[4]$ is not defined at the limit – it has a diverging term at $\delta \to 1$ or $\delta \to \sqrt{2}$ – or in other words the system of linear equations

$$e_j\big(\,||||+a\,|\cup|+b\,||\cup+c\,\cup||+d\,\cup\cup+e\,\bigcup\!\!\!\!\!\!\cup\,\,\big)=0,\qquad \text{with } j=1,2,3,$$

has no solutions at $p = 3, 4$, with the normalization chosen such that the coefficient in front of $||||$ equals 1.

The divergence observed above indicates that we should prepare our basis before taking the limit in order to eliminate the "$\delta \to 1$ catastrophe".

Temperley–Lieb fusion (Example II)

The divergence in $\delta \to 1$ can be resolved by taking an appropriate combination of $\mathrm{inv}(\delta)$ with a state that has the same eigenvalue (which is 0 here) with respect to the Hamiltonian $H = -(e_1 + e_2 + e_3)$ in the limit $\delta = 1$:

$$t(\delta) = \mathrm{inv}(\delta) - \frac{1}{(\delta^2-2)(\delta^2-1)}\big(e_2e_1e_3e_2l + a_- e_1e_3e_2l\big),$$

where $a_- = \delta/2 - \sqrt{2+(\delta/2)^2}$. Now, the state $t(\delta)$ is well-defined and the divergence is resolved in the state

$$t(\delta) = |\,|\,|\,| + \frac{1}{\delta^2-2}\left(|\,|\,\cup + \cup\,|\,| - \delta\,|\,\cup\,| - \frac{4}{3\delta+\sqrt{8+\delta^2}}\cup\cup\right)$$

with the limit

$$t \equiv t(1) = |\,|\,|\,| - \cup\,|\,| - |\,|\,\cup + |\,\cup\,| - \frac{2}{3}\cup\cup$$

Temperley–Lieb fusion (Example II)

The state t is the "logarithmic partner" of the state $T = \cup\,\cup$ - ◡. Indeed, we find a Jordan cell for the Hamiltonian acting between these two states

$$Ht = \frac{2}{3}T,$$

We will also say that T is the "descendant" of the vacuum state $|\mathsf{vac}\rangle =$ ◡ $+2\,\cup\,\cup$ as the standard module S_0 has the following indecomposable structure at $\delta = 1$:

$$\mathsf{S}_0 = |\mathsf{vac}\rangle \rightarrow T$$

(where the arrow corresponds to the action of the TL algebra.)

Temperley–Lieb fusion (Example II)

We see that the standard modules $\mathsf{S}_0[4]$ and $\mathsf{S}_2[4]$ arising in the generic fusion rules are "glued" together at $\delta = 1$ into a bigger indecomposable module with the TL action given by the diagram

$$t \longrightarrow |\mathsf{vac}\rangle \longrightarrow T$$

. The subquotient structure of this module reads

$$\mathsf{X}_2 \longrightarrow \mathsf{X}_0 \longrightarrow \mathsf{X}_2$$

where each subquotient is one-dimensional and X_j denotes the irreducible top of $\mathsf{S}_j[N]$. This module is called *tilting module* over the TL algebra. We shall denote it $\mathsf{P}_2[4]$. Finally, the fusion rules at $\delta = 1$ reads

$$\mathsf{S}_1[2] \times_f \mathsf{S}_1[2] = \mathsf{S}_1[4] \oplus \mathsf{P}_2[4], \qquad \text{for } p = 3.$$

The quantum Schur–Weyl duality (bimodules)

For any $\mathfrak{q}$, the q-Schur–Weyl duality is expressed by "Hom-functor" – a map from the set (actually, category) $\mathcal{C}$ of left finite-dimensional $U_{\mathfrak{q}}s\ell(2)$-modules to the set $\mathcal{D}$ of left $\mathcal{TL}_{\mathfrak{q},N}$-modules. Let M be a $U_{\mathfrak{q}}s\ell(2)$-module then this Hom-mapping is defined as

$$\mathcal{H} : \mathcal{C} \to \mathcal{D}, \qquad M \mapsto \mathrm{Hom}_{U_{\mathfrak{q}}s\ell(2)}(\mathcal{H}_N, M),$$

where we consider the spin-chain vector space $\mathcal{H}_N$ as a left module over $U_{\mathfrak{q}}s\ell(2)$ and as a right module over $\mathcal{TL}_{\mathfrak{q},N}$. The left action of $\mathcal{TL}_{\mathfrak{q},N}$ on $\mathcal{H}(M)$ is defined as $a \triangleright \phi(v) = \phi(v \triangleleft a)$ for any $v \in \mathcal{H}_N$ and any ϕ from the Hom space. We denote by $\triangleright$ and $\triangleleft$ left and right actions of $\mathcal{TL}_{\mathfrak{q},N}$.

The quantum Schur–Weyl duality (bimodules)

The equivalence means that there exists a functor from $\mathcal{H}(\mathcal{C})$ to $\mathcal{C}$ which composed with $\mathcal{H}$ is isomorphic to the identity. This functor is the adjoint of the functor $\mathcal{H}$. We denote this functor ("a map") as $\mathcal{T}$ and it is given by

$$\mathcal{T} : \mathcal{D} \supset \mathcal{H}(\mathcal{C}) \to \mathcal{C}, \qquad M \mapsto \mathcal{H}_N \otimes_{\mathcal{TL}_{\mathfrak{q},N}} M.$$

The image of $\mathcal{T}$ is obviously in the category of $U_{\mathfrak{q}}s\ell(2)$-modules because the quantum group acts on the left side of the bimodule $\mathcal{H}_N$, and we take the balanced product over $\mathcal{TL}_{\mathfrak{q},N}$ of the right module $\mathcal{H}_N$ and the left module M.

The quantum Schur–Weyl duality (fusion)

Let $M_1 \otimes M_2$ be a module over the product $\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}$ of the two TL algebras. We can then demonstrate the connection between TL fusion and its quantum-group counterpart, which is the restriction operation, by the diagram

$$\begin{array}{ccc} M_1 \otimes M_2 & \xrightarrow{\mathrm{Ind}} & M_1 \times_f M_2 \\ {\scriptstyle \mathcal{H}\otimes\mathcal{H}}\uparrow \; \downarrow{\scriptstyle \mathcal{T}\otimes\mathcal{T}} & & {\scriptstyle \mathcal{T}}\downarrow \; \uparrow{\scriptstyle \mathcal{H}} \\ \mathcal{T}(M_1) \otimes \mathcal{T}(M_2) & \xrightarrow{\mathrm{Res}} & \mathcal{T}(M_1 \times_f M_2) \end{array}$$

where the top horizontal arrow corresponds to the TL fusion map Ind which sends $M_1 \otimes M_2$ to its fusion $\mathcal{TL}_{\mathfrak{q},N_1+N_2} \otimes_{\left(\mathcal{TL}_{\mathfrak{q},N_1} \otimes \mathcal{TL}_{\mathfrak{q},N_2}\right)} M_1 \otimes M_2$ while the bottom horizontal arrow corresponds to the tensor-product decomposition of $U_{\mathfrak{q}} s\ell(2)$-modules which is the restriction for corresponding f.-d. quotients of $U_{\mathfrak{q}} s\ell(2)$.

The quantum Schur–Weyl duality (fusion)

Then, the composition $\mathcal{T} \circ \mathrm{Ind}$ gives a $U_{\mathfrak{q}} s\ell(2)$-module

$$\mathcal{T} \circ \mathrm{Ind}(M_1 \otimes M_2) = \mathcal{H}_N \otimes_{TL_N} \left(TL_N \otimes_{TL_{N_1} \otimes TL_{N_2}} M_1 \otimes M_2 \right)$$

$$\cong \left(\mathcal{H}_N \otimes_{TL_N} TL_N \right) \otimes_{TL_{N_1} \otimes TL_{N_2}} M_1 \otimes M_2 \cong \left(\mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2} \right) \otimes_{TL_{N_1} \otimes TL_{N_2}} M_1 \otimes M_2$$

The first equality holds just by definition while we use associativity of the tensor product for the isomorphism in the second line, the first '$\cong$'. For the second '$\cong$', we use the usual isomorphism $\mathcal{H}_N \otimes_{TL_N} TL_N \cong \mathcal{H}_N : v \otimes a \mapsto v \triangleleft a = v'$ (any element of TL can be passed through the balanced tensor product). It is then easy to see that the final result is **isomorphic** to the image of another composition $\mathrm{Res} \circ (\mathcal{T} \otimes \mathcal{T})$ which uses comultiplication on the quantum-group side.

This shows that the diagram composed of solid arrows is commutative.

The quantum Schur–Weyl duality (fusion)

$$\begin{array}{ccc}
M_1 \otimes M_2 & \xrightarrow{\text{Ind}} & M_1 \times_f M_2 \\
\mathcal{H}\otimes\mathcal{H} \uparrow \downarrow \mathcal{T}\otimes\mathcal{T} & & \mathcal{T} \downarrow \uparrow \mathcal{H} \\
\mathcal{T}(M_1) \otimes \mathcal{T}(M_2) & \xrightarrow{\text{Res}} & \mathcal{T}(M_1 \times_f M_2)
\end{array}$$

Having this “fusion-correspondence” diagram at hand, we can easily recover the generic TL fusion

$$\mathsf{S}_{j_1}[N_1] \times_f \mathsf{S}_{j_2}[N_2] = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathsf{S}_j[N_1+N_2]$$

The full space of states is a bimodule over $TL_N(\mathfrak{q}) \otimes U_{\mathfrak{q}}s\ell(2)$

- the states are organized into indecomposables for $U_{\mathfrak{q}}s\ell(2)$ (or $TL_N(\mathfrak{q})$)

The decomposition of $\mathcal{H}_N$ over the quantum group can be written as

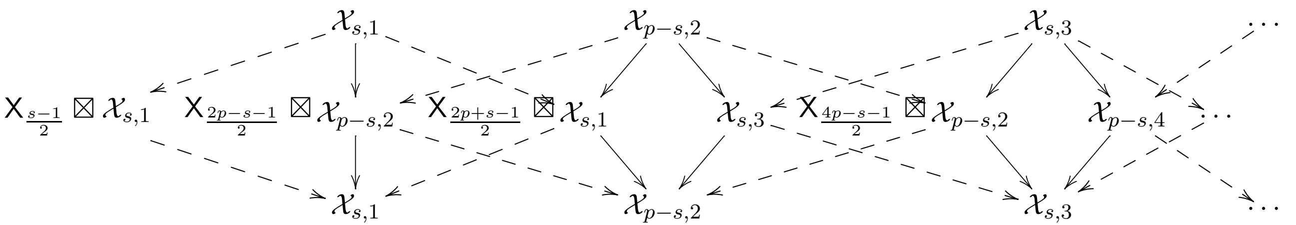
$$\mathcal{H}_N\big|_{U_{\mathfrak{q}}s\ell(2)} \cong \bigoplus_{\substack{s=N \bmod 2+1,\\ s+N=1 \bmod 2}}^{p-1} \dim\left(\mathsf{X}_{\frac{s-1}{2}}\right)\mathcal{X}_{s,1} \oplus \bigoplus_{r=1}^{r_m-1} \bigoplus_{\substack{s=0,\\ rp+s+N=1 \bmod 2}}^{p-1} \dim\left(\mathsf{X}_{\frac{rp+s-1}{2}}\right)\mathcal{P}_{p-s,r}$$

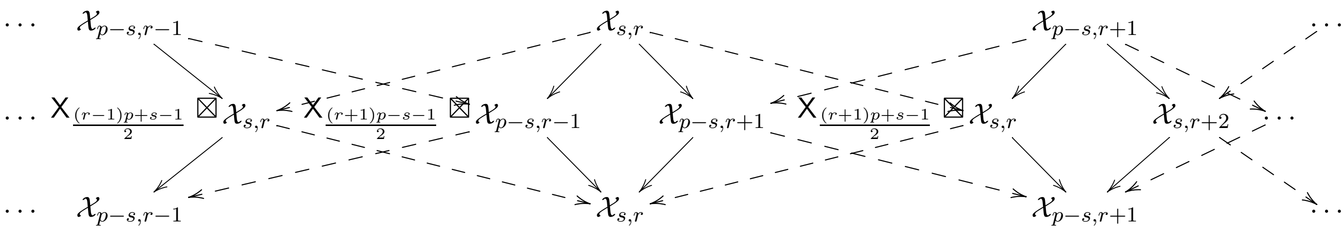
$$\oplus \bigoplus_{\substack{s=0,\\ s+s_m=1 \bmod 2}}^{s_m+1} \dim\left(\mathsf{X}_{\frac{r_mp+s-1}{2}}\right)\mathcal{P}_{p-s,r_m}\,,$$

where we defined $N = r_m p + s_m$, for $r_m \in \mathbb{N}$ and $-1 \leqslant s_m \leqslant p-2$.

The full space of states is a bimodule over $TL_N(\mathfrak{q}) \otimes U_{\mathfrak{q}} s\ell(2)$

- the states are organized into <u>indecomposables</u> for $U_{\mathfrak{q}} s\ell(2)$ (or $TL_N(\mathfrak{q})$)





The quantum Schur–Weyl duality (images of $\mathcal{H}$)

Proposition *The functor $\mathcal{H}$ from the category $\mathcal{C}$ of left modules over (a finite-dimentional image of) $U_{\mathfrak{q}}s\ell(2)$ to the category $\mathcal{D}$ of left $\mathcal{TL}_{\mathfrak{q},N}$-modules has the following images*

$$\mathcal{H}(\mathcal{X}_{s,1}) = \mathsf{S}_{\frac{s-1}{2}}, \qquad \mathcal{H}(\mathcal{X}_{s,r}) = \mathsf{X}_{\frac{(r+1)p-s-1}{2}},$$

$$\mathcal{H}(\mathcal{W}^*_{s,r}) = \mathsf{S}_{\frac{(r+1)p-s-1}{2}},$$

$$\mathcal{H}(\mathcal{W}_{s,1}) = \mathsf{K}_{\frac{s-1}{2};1}, \qquad \mathcal{H}(\mathcal{W}_{s,r}) = \mathsf{S}^*_{\frac{(r+1)p-s-1}{2}}$$

$$\mathcal{H}(\mathcal{P}_{s,r}) = \mathsf{P}_{\frac{(r+1)p-s-1}{2}},$$

where we imply that values of the s-index satisfy $(r+1)p+s+N = 1 \bmod 2$; here, the module S^*_j is the contragredient of the standard module S_j (also called costandard).

TL fusion at roots of unity

Now, we can easily compute TL fusion for any modules and for any root of unity using the correspondence with the quantum group modules. As example, we have

Proposition *The fusion of two standard modules over $\mathcal{TL}_{\mathfrak{q},N}$ is*

$$\mathsf{S}_{\frac{r_1p+s_1-1}{2}} \times_f \mathsf{S}_{\frac{r_2p+s_2-1}{2}} = \bigoplus_{\substack{s=|s_1-s_2|+1\\ \text{step}=2}}^{p-|p-s_1-s_2|-1} \mathsf{S}_{\frac{(r_1+r_2)p+s-1}{2}} \oplus \bigoplus_{\substack{r=|r_1-r_2|+1\\ \text{step}=2}}^{r_1+r_2-1} \bigoplus_{\substack{s=\gamma_2\\ \text{step}=2}}^{p-|s_1-s_2|-1} \mathsf{P}_{\frac{rp+s-1}{2}}$$

$$\oplus \bigoplus_{\substack{s=\gamma_2\\ \text{step}=2}}^{s_1+s_2-p-1} \mathsf{P}_{\frac{(r_1+r_2+1)p+s-1}{2}} \oplus \bigoplus_{\substack{r=|r_1-r_2+\text{sg}(s_1-s_2)|+1\\ \text{step}=2}}^{r_1+r_2} \bigoplus_{\substack{s=\gamma_1\\ \text{step}=2}}^{|s_1-s_2|-1} \mathsf{P}_{\frac{rp+s-1}{2}},$$

It turns out that the fusion rules are stable when N is growing, and thus persist in the "continuum limit" $N \to \infty$.

TL fusion at roots of unity: motivation

The fusion rules for a pair of standard TL modules is of most physical interest, as it turns out to have a deep connection at the continuum limit $N \to \infty$ with the operator product expansions of primary fields in the corresponding (logarithmic) CFT at critical central charges $c = c_{p-1,p} < 1$.

Part II: From TL to Virasoro, operator algebras

Rough conjecture

about relations between Temperley–Lieb and Virasoro algebras.

There exist embeddings or, more formally, **inductive systems** *for TL algebras*

$$TL_2 \xrightarrow{\phi_2} TL_4 \xrightarrow{\phi_4} \ldots \xrightarrow{\phi_{N-2}} TL_N \xrightarrow{\phi_N} TL_{N+2} \xrightarrow{\phi_{N+2}} \ldots \xrightarrow{N\to\infty} \mathsf{Vir}$$

that give in the (inductive) limit – when number of sites goes to ∞ – an infinite-dimensional operator algebra realizing a Virasoro algebra representation.

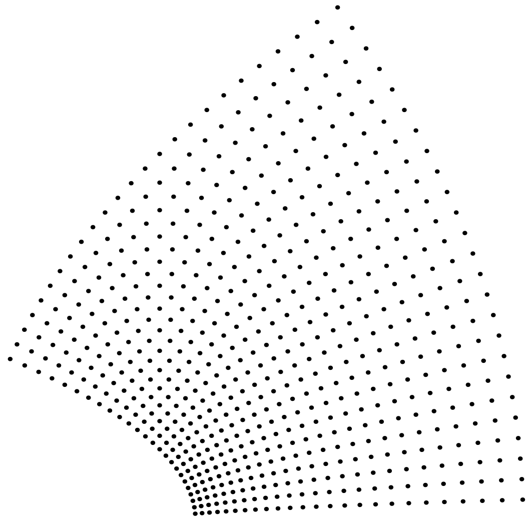
Some physical motivation

Some physical motivation

Why we have the conjecture about TL and Virasoro algebras?

Some physical motivation

The origin is in critical phenomena in statistical physics problems.



To describe universality classes of 2D statistical systems

at a **critical point** ($\xi \to \infty$)

Some physical motivation

Morally,

- the lattice models are discretizations of CFTs and
- TL algebra gives a regularization of the energy–momentum tensor $T(z)$:

its modes L_n are obtained in a scaling limit

from the hamiltonian densities e_j (generators of TL algebra) keeping

"higher" hamiltonians $H(n) = \sum_j \exp(i\pi nj/N)\boldsymbol{e_j}$ —

— due the Koo–Saleur results, 1994

The model: $\mathfrak{gl}(1|1)$ spin-chains

We consider $\mathfrak{gl}(1|1)$ SUSY spin-chains

with open and closed boundary conditions (b.c.)

The space of states is the tensor product space $\mathcal{H} = (V \otimes V^*)^{\otimes L}$ of $N = 2L$ tensorands labelled $j = 1, \ldots, 2L$ with the fundamental representation $V = \mathbb{C}^{1|1}$ for $\mathfrak{gl}(1|1)$ on even sites and the dual V^* on odd sites.

Nearest-neighbour interaction is given by e_j's – projectors on the $\mathfrak{gl}(1|1)$-invariant in the product $V \otimes V^*$ of two neighbour tensorands.

Open $\mathfrak{gl}(1|1)$ spin-chain: free fermions

- The open $\mathfrak{gl}(1|1)$ spin-chain has a free fermion representation based on operators f_j and $f_j^\dagger$ acting non-trivially only on jth tensorand and obeying

$$\{f_j, f_{j'}\} = 0, \quad \{f_j, f^\dagger_{j'}\} = (-1)^j \delta_{jj'},$$

where the '$-$' sign for an odd j is due to the dual representations of $\mathfrak{gl}(1|1)$.

- Nearest-neighbour interaction is then

$$e_j = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \qquad 1 \leqslant j \leqslant 2L-1,$$

- The critical hamiltonian $H = \sum_{j=1}^{2L-1} e_j$ is hermitian but acts on an indefinite inner product space $\mathcal{H} = (V \otimes V^*)^{\otimes L}$ because of the sign factor.

Open $\mathfrak{gl}(1|1)$ spin-chain: TL algebra

- the Hamiltonian densities satisfy TL algebra $TL_N(m)$ relations:

$$e_j^2 = m e_j, \qquad e_j e_{j\pm 1} e_j = e_j,$$

$$e_j e_k = e_k e_j, \qquad (j \neq k,\ k \pm 1),$$

- with $m = 0$ for $\mathfrak{gl}(1|1)$ spin-shains –

  – the algebra is **non semi-simple**

- These open chains provide a **faithful** representation of $TL_N(m)$.

Open $\mathfrak{gl}(1|1)$ spin-chain: the centralizer Read–Saleur, 2007

Important concept — **the full symmetry algebra** Z_{TL} —
the centralizer of the "hamiltonian densities" algebra TL (the centralizer is a largest algebra that commutes with TL, i.e. technically is defined as $\mathrm{End}_{TL}(\mathcal{H})$)

In the open $\mathfrak{gl}(1|1)$ spin-chain, Z_{TL} is generated by the identity and

$$F_{(1)} = \sum_{1\leqslant j\leqslant N} f_j, \qquad F^{\dagger}_{(1)} = \sum_{1\leqslant j\leqslant N} f^{\dagger}_j,$$

$$F_{(2)} = \sum_{1\leqslant j<j'\leqslant N} f_j f_{j'}, \qquad F^{\dagger}_{(2)} = \sum_{1\leqslant j<j'\leqslant N} f^{\dagger}_{j'} f^{\dagger}_j,$$

$$S^z = \sum_{1\leqslant j\leqslant N} (-1)^j f^{\dagger}_j f_j - L.$$

Open $\mathfrak{gl}(1|1)$ spin-chain: the centralizer

- In the open $\mathfrak{gl}(1|1)$ spin-chain, Z_{TL} is generated by the identity and

$$F_{(1)} = \sum_{1 \leqslant j \leqslant N} f_j, \qquad F^\dagger_{(1)} = \sum_{1 \leqslant j \leqslant N} f^\dagger_j,$$

$$F_{(2)} = \sum_{1 \leqslant j < j' \leqslant N} f_j f_{j'}, \qquad F^\dagger_{(2)} = \sum_{1 \leqslant j < j' \leqslant N} f^\dagger_{j'} f^\dagger_j,$$

$$S^z = \sum_{1 \leqslant j \leqslant N} (-1)^j f^\dagger_j f_j - L.$$

Why this algebra generates the full symmetry algebra?

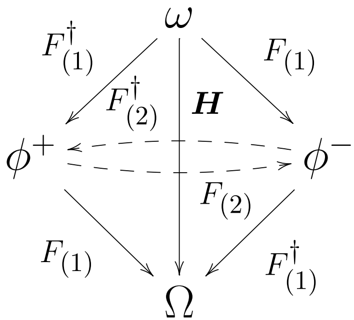
- Note that the formulas give a representation of the quantum group $U_{\mathfrak{q}} s\ell(2)$ with $\mathfrak{q} = i$. The fermionic generators $F_{(1)}$ and $F^\dagger_{(1)}$ are from the nilpotent part and the bosonic ones form the $s\ell(2)$ subalgebra in $U_{\mathfrak{q}} s\ell(2)$.

Open $\mathfrak{gl}(1|1)$ spin-chain: relation to XX model

- Jordan–Wigner transformation gives an isomorphism

between the open $\mathfrak{gl}(1|1)$ and XX spin-chains and between Z_{TL} and $U_i s\ell(2)$

$$\mathfrak{q}^{S^z} S^{\pm} \mathfrak{q}^{-S^z} = \mathfrak{q} S^{\pm},$$
$$[S^+, S^-] = \frac{\mathfrak{q}^{S^z} - \mathfrak{q}^{-S^z}}{\mathfrak{q} - \mathfrak{q}^{-1}},$$
$$F^{\dagger}_{(1)} = S^+, \quad F_{(1)} = S^-,$$
$$F^{\dagger}_{(2)} = \frac{(S^+)^2}{[2]!}, \quad F_{(2)} = \frac{(S^-)^2}{[2]!}.$$



the vacuum Ω and the state ω form a 2-dim Jordan cell of the lowest eigenvalue for $\boldsymbol{H}$

Open $\mathfrak{gl}(1|1)$ spin-chain: generalized Hamiltonians

We start by defining the family of operators

$$H_n^0 = \sum_{j=1}^{N-1} \cos \pi \frac{nj}{N} e_j, \qquad n \in \mathbb{Z}.$$

The Hamiltonian in this notation is H_0^0.

The adjoint action by the Hamiltonian H_0^0 on H_n^0 generates the family H_n^r with $r \in \mathbb{N}_0$ and $n \in \mathbb{Z}$:

$$[H_0^0, H_m^r] = -4 \sin \pi \frac{m}{2N} H_m^{r+1}, \qquad r = 0, 1, 2, \ldots.$$

Open $\mathfrak{gl}(1|1)$ spin-chain: scaling limit

Scaling limit $=$ keep terms up to the order $1/N$ in the $N \to \infty$ decomposition:

$$\frac{N}{\pi} H_n^0 \longrightarrow L_n + L_{-n} + o\big(\tfrac{1}{N}\big),$$

We can write formal series decomposition of H_n^r in $\frac{1}{N}$:

$$H_n^r = \sum_{k=1}^{\infty} H_n^r[k] \left(\frac{\pi}{N}\right)^{2k-1}.$$

Open $\mathfrak{gl}(1|1)$ spin-chain: scaling limit

$$H_n^r = \sum_{k=1}^{\infty} H_n^r[k] \left(\frac{\pi}{N}\right)^{2k-1}$$

with first several terms in the form

$$H_n^r[1] = (-1)^r L_n + L_{-n}$$

Open $\mathfrak{gl}(1|1)$ spin-chain: scaling limit

$$H_n^r = \sum_{k=1}^{\infty} H_n^r[k] \left(\frac{\pi}{N}\right)^{2k-1}$$

with first several terms in the form

$$\begin{aligned} H_n^r[2] = \frac{1}{24}\Big(& 4(3r+1)\big((-1)^r (L^2)_n + (L^2)_{-n}\big) \\ & - 3r\big((-1)^r (\partial^2 L)_n + (\partial^2 L)_{-n}\big) \\ & + 3(r+2)\big((-1)^r (\partial L)_n + (\partial L)_{-n}\big) \\ & + 4\big((-1)^r L_n + L_{-n}\big)\Big) \end{aligned}$$

Open $\mathfrak{gl}(1|1)$ spin-chain: scaling limit

$$H_n^r = \sum_{k=1}^{\infty} H_n^r[k] \left(\frac{\pi}{N}\right)^{2k-1}$$

with first several terms in the form

$$\begin{aligned}
H_n^r[3] = \frac{1}{5760}\Big(&-64(15r^2+1)\big((-1)^r (L^3)_n + (L^3)_{-n}\big)+\\
&+8(2+15r(r-1))\big((-1)^r (\partial^2 L^2)_n + (\partial^2 L^2)_{-n}\big)+\\
&+(-4+15r(r+2))\big((-1)^r (\partial^4 L)_n + (\partial^4 L)_{-n}\big)-\\
&-1080r(r+1)\big((-1)^r (\partial^2 L^2)_n + (\partial^2 L^2)_{-n}\big)+\\
&+10(-4+3r(9r+8))\big((-1)^r (\partial^3 L)_n + (\partial^3 L)_{-n}\big)-\\
&-80(9r^2+24r-1)\big((-1)^r (L^2)_n + (L^2)_{-n}\big)+\\
&+45(r^2+6r-4)\big((-1)^r (\partial^2 L)_n + (\partial^2 L)_{-n}\big)-\\
&-45(r^2+6r+4)\big((-1)^r (\partial L)_n + (\partial L)_{-n}\big)-\\
&-48\big((-1)^r L_n + L_{-n}\big)\Big)
\end{aligned}$$

Open $\mathfrak{gl}(1|1)$ spin-chain: commutation relations for H_n^r

The family of "hamiltonians" H_n^r generates a Lie algebra with the commutation relations for any finite N:

$$[H_n^0, H_m^0] = 2\sin\pi\frac{n-m}{2N}H^1_{n+m} + 2\sin\pi\frac{n+m}{2N}H^1_{n-m},$$

$$[H_n^0, H_m^1] = 2\sin\pi\frac{2n-m}{2N}H^2_{n+m} - 2\sin\pi\frac{2n+m}{2N}H^2_{n-m}-$$
$$-2\sin\pi\frac{n}{2N}\Big(\cos\pi\frac{n-m}{2N}H^0_{n+m} - \cos\pi\frac{n+m}{2N}H^0_{n-m}\Big),$$

$$[H_n^1, H_m^1] = 2\sin\pi\frac{n-m}{N}H^3_{n+m} - 2\sin\pi\frac{n+m}{N}H^3_{n-m}-$$
$$-2\sin\pi\frac{n-m}{2N}(\cos\pi\frac{n+m}{2N} + \sin\pi\frac{n}{2N}\sin\pi\frac{m}{2N})H^1_{n+m}+$$
$$+2\sin\pi\frac{n+m}{2N}(\cos\pi\frac{n-m}{2N} - \sin\pi\frac{n}{2N}\sin\pi\frac{m}{2N})H^1_{n-m}.$$

Open $\mathfrak{gl}(1|1)$ spin-chain: commutation relations at $N \to \infty$

In the limit $N \to \infty$ the commutators of H_n^r give the commutation relations of the Virasoro.

The commutation relations of L_n as the limit

$$[L_n, L_m] = \lim_{N\to\infty} \frac{N^2}{4\pi^2}\left([H_n^0, H_m^0] - [H_n^0, H_m^1] + [H_m^0, H_n^1] + [H_n^1, H_m^1]\right)$$

What is our problem?

- Consider a quotient of Hecke algebras – the (affine) Temperley–Lieb algebras
- Consider its tensor-product representations – open (closed) XXZ spin-chains
- Construct inductive systems (system of embeddings) of these spin-chains and the (affine) Temperley–Lieb algebras

$$\dots \xrightarrow{\phi_{N-2}} \Big(\mathcal{E}_{m,n}\Big)_{N\times N} \xrightarrow{\phi_N} \begin{pmatrix} 0 & 0 \dots 0 & 0 \\ 0 & \mathcal{E}_{m,n} & 0 \\ 0 & 0 \dots 0 & 0 \end{pmatrix}_{(N+2)\times(N+2)} \xrightarrow{\phi_{N+2}} \dots$$

- Find field-theoretic description of the inductive limits.

What is our motivation?

Virasoro representation theory and understanding Logarithmic CFT.

What is our motivation?

In a log CFT model, we encounter (at least) two **problems**:

(1) The space of states (indefinite inner-product space) is decomposed onto complicated **indecomposables** over Virasoro but their structure is not known apriori – a problem in constructing even a consistent chiral theory.

(2) a problem in combining chiral and antichiral parts to construct the full space of states of a local theory (**non-chiral** theory) in order to describe, say, 2D percolation on a torus.

Open $\mathfrak{gl}(1|1)$ spin-chain: irreducible TL representations

- **generic $\mathfrak{q}$:** The space $\mathcal{H}_N$ as a semi-simple module over $TL_{\mathfrak{q},N}$:

$$\mathcal{H}_N|_{TL_{\mathfrak{q},N}} \cong \bigoplus_{j=(N \bmod 2)/2}^{N/2} (2j+1)\mathsf{S}_j,$$

The dimension of the standard (generically irreducible) modules S_j:

$$\dim(\mathsf{S}_j) \equiv d_j = \binom{N}{N/2+j} - \binom{N}{N/2+j+1}.$$

- **non-generic $\mathfrak{q} = i$:** TL-modules S_j are reducible with structure $\mathfrak{L}_j \longrightarrow \mathfrak{L}_{j+1}$

Dimensions of the irreducible modules:

$$\dim \mathfrak{L}_j = \sum_{j' \geqslant j} (-1)^{j'-j} d_{j'} = \binom{N-2}{N/2-j} - \binom{N-2}{N/2-j-2}.$$

Open $\mathfrak{gl}(1|1)$ spin-chain: Lie algebra structure

- **non-generic $\mathfrak{q} = i$:** TL-modules S_j are reducible with structure $\mathcal{L}_j \to \mathcal{L}_{j+1}$

Dimensions of the irreducible TL-modules

$$\dim \mathcal{L}_j = \binom{N-2}{N/2-j} - \binom{N-2}{N/2-j-2}.$$

are the dimensions of the one-column

(fundamental) representations of $\mathfrak{sp}_{N-2}$!

Theorem. *The simple modules $\mathcal{L}_j$, with $1 \leqslant j \leqslant L$, over TL_{2L} are simple modules over the enveloping algebra for $\mathfrak{sp}_{2L-2}$ and they correspond to all highest-weight representations of $\mathfrak{sp}_{2L-2}$ of the weights $(1,1,\dots,1,0,\dots,0)$, which are sequences of 0's and 1's of length $L-1$, with $j-1$ of 0's.*

Open $\mathfrak{gl}(1|1)$ spin-chain: Lie algebra structure

Recall that the e_j generators of the TL_N at $\mathfrak{q} = i$ are linear combinations of bilinears in the fermions f_j and $f_j^\dagger$

$$e_j^{\mathfrak{gl}} = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \qquad 1 \leqslant j \leqslant N-1,$$

- The commutators of these combinations of bilinears can be expressed again in fermionic bilinears, and of course belong to the TL algebra. Obviously, they should generate a finite-dimensional Lie algebra.

- The spin-chain images of the TL algebra contain many operators which are not bilinears. These non-bilinear operators are generated by the Lie algebra elements because the e_j generators belong to the Lie algebra. We can thus expect that the TL algebra can be described as an enveloping algebra of a Lie algebra.

Open $\mathfrak{gl}(1|1)$ spin-chain: Clifford algebra

Consider the Clifford algebra generated by f_j and $f_j^\dagger$ and consider new fermions ψ_n^α which are linear combinations of f_j and $f_j^\dagger$ (Fourier transforms) and satisfy

$$\{\psi_n^\alpha, \psi_m^\beta\} = J^{\alpha\beta}\omega_{n,m}\,, \qquad \alpha, \beta \in \{1, 2\}, \qquad -\tfrac{N}{2} + 1 \leqslant n, m \leqslant \tfrac{N}{2} - 1,$$

with symplectic forms

$$J^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \omega_{n,m} = n\delta_{n,-m}.$$

The fermions ψ_n^α diagonalize the Hamiltonian.

The zero modes ψ_0^1 and ψ_0^2 anti-commute: $\{\psi_0^1, \psi_0^2\} = 0$

and they are the two generators $F_{(1)}$ and $F_{(1)}^\dagger$

of the $\mathfrak{gl}(1|1)$-symmetry.

Open $\mathfrak{gl}(1|1)$ spin-chain: Howe duality

- Consider an action of two Lie algebras $\mathfrak{sp}_2$ and $\mathfrak{sp}_{N-2}$:

$$\mathfrak{sp}_2 : \ \omega^{n,m}\psi_n^\alpha\psi_m^\beta, \qquad \alpha,\beta \in \{1,2\}$$

$$\mathfrak{sp}_{N-2} : \ J_{\alpha,\beta}\psi_n^\alpha\psi_m^\beta, \qquad -\frac{N}{2}+1 \leqslant n,m \leqslant \frac{N}{2}-1, \quad n,m \neq 0$$

- The two Lie algebras commute

$$\left[\mathfrak{sp}_2, \mathfrak{sp}_{N-2}\right] = 0.$$

The $\mathfrak{sp}_2$ generates the $Us\ell(2)$ part of the centralizer $U_is\ell(2)$,
i.e. the $\mathfrak{sp}_2$ commutes with the TL algebra.

Open $\mathfrak{gl}(1|1)$ spin-chain: Howe duality

Consider a subspace in $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$ — the image of ψ_0^1 and ψ_0^2 $\longrightarrow$

$\longrightarrow$ the $\mathfrak{sp}_2$ and $\mathfrak{sp}_{N-2}$ centralize each other on this subspace!

$$\psi_0^1\psi_0^2\mathcal{H}_N = \bigoplus_j [j\text{-dim irrep of } \mathfrak{sp}_2] \otimes [j\text{-th fundamental irrep of } \mathfrak{sp}_{N-2}]$$

This is just the Howe duality
in the symplectic case.

- But the $\mathfrak{sp}_2$ also centralizes the TL action $\longrightarrow$

$\longrightarrow$ The semi-simple part of the TL algebra
is an envloping algebra of $\mathfrak{sp}_{N-2}$.

Open $\mathfrak{gl}(1|1)$ spin-chain: a special basis in TL algebra

Introduce the standard basis in $\mathfrak{sp}_{N-2}$

$$\mathcal{A}_{m,n} = J_{\alpha,\beta}\psi_n^\alpha\psi_m^\beta \;\; (n>0, m<0), \quad \mathcal{B}_{m,n} = J_{\alpha,\beta}\psi_n^\alpha\psi_m^\beta \;\; (n,m>0),$$

$$\mathcal{C}_{m,n} = J_{\alpha,\beta}\psi_n^\alpha\psi_m^\beta \;\; (n,m<0).$$

- The generators $\mathcal{A}_{n,n}$ span a basis in the Cartan subalgebra of the $\mathfrak{sp}_{N-2}$.

The diagonal part of the Hamiltonian H has a very simple expression in terms of these generators:

$$H = 2\sum_{m=1}^{N/2-1} \sin\tfrac{m\pi}{N}\,\mathcal{A}_{m,m} + 2\psi_0^1\psi_0^2.$$

- The representation of $U\mathfrak{sp}_{N-2}$ is a subalgebra in TL_N.

Open $\mathfrak{gl}(1|1)$ spin-chain: a special basis in TL algebra

- The generators $\mathcal{A}_{n,n}$ span a basis in the Cartan subalgebra of the $\mathfrak{sp}_{N-2}$.

The diagonal part of the Hamiltonian H has a very simple expression in terms of these generators:

$$H = 2\sum_{m=1}^{N/2-1} \sin\tfrac{m\pi}{N}\,\mathcal{A}_{m,m} + 2\psi_0^1\psi_0^2.$$

- TL_N simple modules as highest-weight representations of $\mathfrak{sp}_{N-2}$:

highest-weight vectors are the charged vacua
for the Hamiltonian

The vacuum state $\mathbf{\Omega}$ in the space of ground states coincides with the highest-weight vector of the unique $\mathfrak{sp}_{N-2}$ module of the weight $(1,1,\ldots,1,1)$, the next ones at $S^z = \pm 1$ correspond to the weight $(1,1,\ldots,1,0)$, *etc.*

Open $\mathfrak{gl}(1|1)$ spin-chain: a special basis in TL algebra

- To find a Lie algebra for full TL we should include zero modes.

Definition. *Lie algebra $\mathfrak{S}_N$ is generated by $\mathcal{A}_{m,n}$, $\mathcal{B}_{m,n}$, $\mathcal{C}_{m,n}$ and the unit matrices $\mathcal{E}_{0,n}$, $\mathcal{E}_{0,\frac{N}{2}}$, $\mathcal{E}_{0,\frac{N}{2}+n}$, $\mathcal{E}_{m,\frac{N}{2}}$, and $\mathcal{E}_{\frac{N}{2}+m,\frac{N}{2}}$. This Lie algebra can be schematically depicted by matrices of the form (in the standard basis of $\mathfrak{gl}_N$)*

$$\mathfrak{S}_N : \quad \begin{pmatrix} 0 & \times & \dots & \times & \times & \times & \dots & \times \\ 0 & & & & \times & & & \\ 0 & & \mathcal{A}_{m,n} & & \times & & \mathcal{B}_{m,n} & \\ 0 & & & & \times & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & & & & \times & & & \\ 0 & & \mathcal{C}_{m,n} & & \times & & & \\ 0 & & & & \times & & & \end{pmatrix},$$

where the crosses $\times$ stand for the corresponding elements $\mathcal{E}_{0,n}$, $\mathcal{E}_{0,\frac{N}{2}}$, $\mathcal{E}_{0,\frac{N}{2}+n}$, $\mathcal{E}_{m,\frac{N}{2}}$, and $\mathcal{E}_{\frac{N}{2}+m,\frac{N}{2}}$.

Open $\mathfrak{gl}(1|1)$ spin-chain: a special basis in TL algebra

- $\mathfrak{S}_N$ is a non-semisimple Lie algebra and admits $\mathfrak{sp}_{N-2}$ spanned by $\mathcal{A}_{m,n}$, $\mathcal{B}_{m,n}$, and $\mathcal{C}_{m,n}$ as a Lie subalgebra. The dimension of $\mathfrak{S}_N$ is

$$\dim \mathfrak{S}_N = \frac{(N-1)(N+2)}{2} - 1.$$

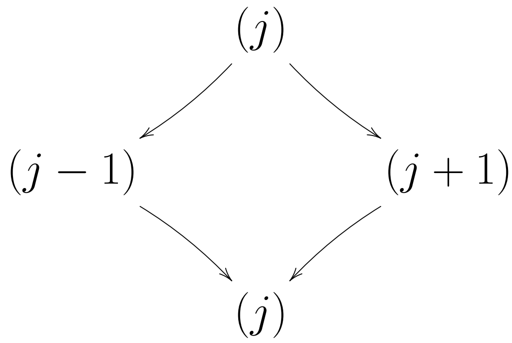
- The Lie radical of $\mathfrak{S}_N$ is generated by $\mathcal{E}_{0,n}$, $\mathcal{E}_{0,L}$, $\mathcal{E}_{0,L+n}$, $\mathcal{E}_{m,L}$, and $\mathcal{E}_{L+m,L}$: these are the generators corresponding to the crosses $\times$.

Theorem. *The image of the representation of the enveloping algebra $U\mathfrak{S}_N$ is isomorphic to the image of the Temperley–Lieb algebra $TL_{i,N}$ in the open $\mathfrak{gl}(1|1)$ spin-chain (or XX spin-chain with the quantum-group symmetry).*

Taking products of the $\mathfrak{S}_N$ basis elements we obtain a special basis in the TL_N which we use to take the scaling limit of the TL algebras.

Open $\mathfrak{gl}(1|1)$ spin-chain: decomposition and tilting TL-modules

- The spin-chain $\mathcal{H}_N$ is decomposed on TL-modules or $\mathfrak{S}_N$-modules



The radical of the $\mathfrak{S}_N$ maps states from j-th fundamental representation (j) of $\mathfrak{sp}_{N-2}$ to $(j \pm 1)$.

The indecomposables are tilting TL modules.

Open $\mathfrak{gl}(1|1)$ spin-chain: scaling limit of TL and spin-chains

- Define first an inductive (direct) system of the matrix algebras $\mathfrak{gl}_N$ in our basis of fermionic bilinears

$$\ldots \xrightarrow{\phi_{N-2}} \left(\mathcal{E}_{m,n}\right)_{N\times N} \xrightarrow{\phi_N} \begin{pmatrix} 0 & 0 \ldots 0 & 0 \\ 0 & \mathcal{E}_{m,n} & 0 \\ 0 & 0 \ldots 0 & 0 \end{pmatrix}_{(N+2)\times(N+2)} \xrightarrow{\phi_{N+2}} \ldots$$

where the elements $\mathcal{E}_{i,j}$ denote usual unit matrices, with $-\frac{N}{2} \leqslant n, m \leqslant \frac{N}{2}$.

The inductive limit $\varinjlim_N \mathfrak{gl}_N$ gives the infinite-dimensional Lie algebra $\mathfrak{gl}_\infty$ – an algebra of infinite matrices with a finite number of non-zero elements.

Open $\mathfrak{gl}(1|1)$ spin-chain: scaling limit of TL and spin-chains

- The inductive system for $\mathfrak{gl}_N$ gives inductive systems of embeddings for the

Lie subalgebras $\mathfrak{sp}_{2L-2}$ and $\mathfrak{S}_N$:

$$\mathfrak{sp}_2 \xrightarrow{\phi_4} \mathfrak{sp}_4 \xrightarrow{\phi_6} \ldots \xrightarrow{\phi_{N-2}} \mathfrak{sp}_{N-2} \xrightarrow{\phi_N} \mathfrak{sp}_N \xrightarrow{\phi_{N+2}} \ldots$$

and

$$\mathfrak{S}_2 \xrightarrow{\phi_2} \mathfrak{S}_4 \xrightarrow{\phi_4} \ldots \xrightarrow{\phi_{N-2}} \mathfrak{S}_N \xrightarrow{\phi_N} \mathfrak{S}_{N+2} \xrightarrow{\phi_{N+2}} \ldots$$

These inductive systems give in the limits $\varinjlim_L \mathfrak{sp}_N$ and $\varinjlim_L \mathfrak{S}_N$ what we call the $\mathfrak{sp}_\infty$ and $\mathfrak{S}_\infty$ Lie algebras.

- For the enveloping algebras, we get the inductive system for TL algebras

$$TL_2 \xrightarrow{\phi_2} TL_4 \xrightarrow{\phi_4} \ldots \xrightarrow{\phi_{N-2}} TL_N \xrightarrow{\phi_N} TL_{N+2} \xrightarrow{\phi_{N+2}} \ldots$$

where ϕ_i here are embeddings of the finite-dimensional associative algebras.

Open $\mathfrak{gl}(1|1)$ spin-chain: scaling limit of TL and spin-chains

- The embeddings for TL algebras

$$TL_2 \xrightarrow{\phi_2} TL_4 \xrightarrow{\phi_4} \ldots \xrightarrow{\phi_{N-2}} TL_N \xrightarrow{\phi_N} TL_{N+2} \xrightarrow{\phi_{N+2}} \ldots$$

are different from the obvious ones (in terms of diagrams).

So, in the special basis the system of embeddings gives
the inductive limit $\varinjlim_L TL_N$ – an enveloping algebra of $\mathfrak{S}_\infty$ –
which we call **the scaling limit of TL** algebras.

- An appropriate completion of this inductive limit $\varinjlim_L TL_N$
is isomorphic to the symplectic-fermion representation of
the Virasoro algebra with $c = -2$!

Open $\mathfrak{gl}(1|1)$ spin-chain: scaling limit of TL and spin-chains

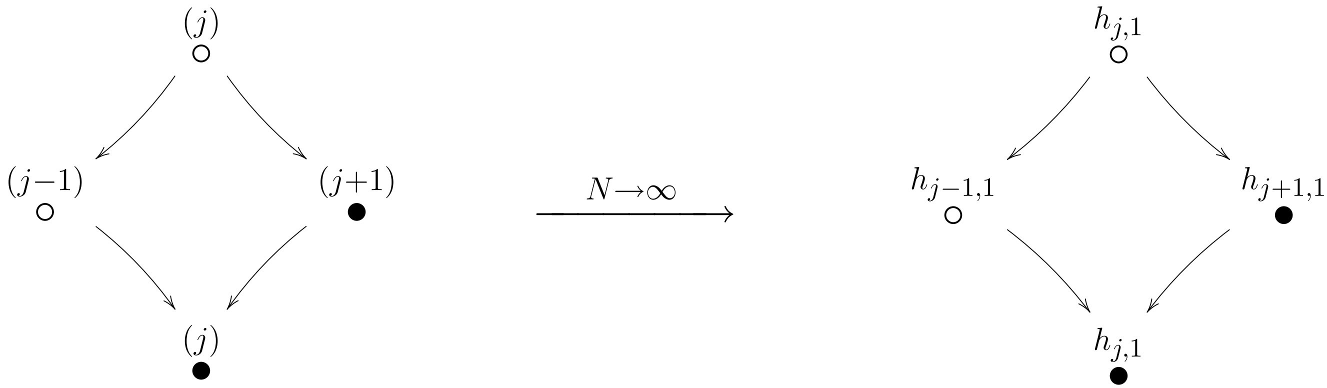
- The second inductive system is for the spin-chains $\mathcal{H}_N$

$$\mathcal{H}_2 \xrightarrow{\phi_2} \mathcal{H}_4 \xrightarrow{\phi_4} \dots \xrightarrow{\phi_{N-2}} \mathcal{H}_N \xrightarrow{\phi_N} \mathcal{H}_{N+2} \xrightarrow{\phi_{N+2}} \dots$$

The inductive limit $\varinjlim_N \mathcal{H}_N$ then coincides with the space $\mathcal{H}$ of scaling states or finite-energy states in the conformal field theory of symplectic fermions described by the action $S = \int d^2z J_{\alpha\beta}\partial_\mu \Phi^\alpha \partial^\mu \Phi^\beta$.

The decomposition of $\mathcal{H} = \varinjlim_N \mathcal{H}_N$ onto $U\mathfrak{S}_\infty$-modules is exactly the same as the decomposition over the Virasoro algebra (which can be obtained independently).

Tilting TL-modules and staggered Vir-modules



Tilting TL-modules go over to

staggered modules for chiral Virasoro

Conclusion of the Part II

(1) We found a special basis in TL algebras at the root of unity $\mathfrak{q} = i$ – open XX chains. In this basis, we constructed an inductive system (or embeddings) of algebras and spin-chains and we found a field-theoretic description of the inductive limits when the number of sites goes to infinity.

(2) The limits turned out to be logarithmic conformal field theory in the chiral case. The TL algebra in the limit – which is an ∞-dim operator algebra – is isomorphic to a representation (to an enveloping) of the Virasoro algebra at central charge $c = -2$, or in terms of fields the inductive limit is generated by the energy-momentum tensor $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

Thank You!