Quantum integrable systems and hyperplane arrangements

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1. Gaudin models
   - Gaudin hamiltonians
   - Kohno–Drinfeld relations
   - Jucys–Murphy elements
2. Gaudin subalgebras and moduli spaces of stable rational curves
   - Gaudin subalgebras
   - A Grassmannian embedding of $\bar{\mathcal{M}}_{0,n+1}$
   - Tautological bundle and spectrum
3. The case of a general root system
   - Holonomy Lie algebra
   - Generalized Gaudin hamiltonians
   - Principal Gaudin subalgebras
4. General arrangements of hyperplanes

Gaudin models

Gaudin (1976) quantum integrable models of magnetic one-dimensional systems with parameters $z_1, \ldots, z_n \in \mathbb{C}$, $z_i \neq z_j$ for $i \neq j$.

$$H_i(z) = \sum_{j \neq i} \frac{\vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)}}{z_i - z_j}, \quad i = 1, \ldots, n,$$

acting on the spin chain state space $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ ($\vec{\sigma}^{(i)}$ is $\vec{\sigma} = \left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)$ acting on the ith factor).

Theorem (Gaudin)

$H_i(z)H_j(z) = H_j(z)H_i(z)$

Infinitesimal braid relations

The commutation of the Gaudin hamiltonians $H_i(z)$ follows from a more general statement:

Theorem (Kohno, Drinfeld)

Suppose $t_{ij} = t_{ji}$, $1 \leq i \neq j \leq n$ *obey the* infinitesimal braid relations

$$[t_{ij}, t_{kl}] = 0, \qquad [t_{ij}, t_{ik} + t_{jk}] = 0,$$

for all distinct (i, j, k, l), (i, j, k). *Then, for every* $z_1, \ldots, z_n \in \mathbb{C}$ *with* $z_i \neq z_j$ *for* $i \neq j$,

$$H_i(z) = \sum_{j \neq i} \frac{t_{ij}}{z_i - z_j}$$

commute: $[H_i(z), H_j(z)] = 0$, $i, j = 1, \ldots, n$.

Examples of solutions of the Kohno-Drinfeld relations

In each of these cases we have a family of commuting hamiltonians $H_j(z)$:

- Let $\mathfrak{g}$ be a semisimple Lie algebra with invariant symmetric tensor $\Omega \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$. Then $t_{ij} = \Omega^{(ij)} \in (U\mathfrak{g})^{\otimes n}$ obey the infinitesimal braid relations (Gaudin, Knizhnik–Zamolodchikov).

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Questions

What happens when $z_i \to z_j$? What are "all" Gaudin systems in the span of t_{ij}? What happens for general hyperplane arrangements replacing the A_{n-1}-arrangement $\{z_i = z_j\}$?

Jucys–Murphy elements

Consider the hamiltonians in a degenerate limit as $z_1, \ldots, z_{n-1} \to 0$:

$$J_i = \lim_{\epsilon \to 0} \epsilon^{n-j} H_i(\epsilon^{n-1}, \ldots, \epsilon, 1).$$

We get commuting elements, called (for $\mathbb{C}S_n$) *Jucys–Murphy elements*.

$$\begin{aligned} J_1 &= 0, \\ J_2 &= t_{12} \\ J_3 &= t_{13} + t_{23}, \\ J_4 &= t_{14} + t_{24} + t_{34}, \ldots \end{aligned}$$

Jucys–Murphy elements and representation theory of S_n

Theorem (Jucys 1974)

For every irreducible representation V of S_n the simultaneous eigenvectors of $(J_1, \ldots, J_n)$ acting on V are in one-to-one correspondence with Young-tableaux with a fixed shape: the box (i, j) has label k iff the eigenvalue of J_k is $j - i$.

Example

$n = 7$.

Eigenvalues $(0, -1, 1, 0, 2, 3, 2) \leftrightarrow$ tableau

1	3	5	6
2	4		
7			

Gaudin subalgebras

Definition

The *Kohno-Drinfeld Lie algebra* $\mathfrak{t}_n$ is the Lie algebra on generators $t_{ij} = t_{ji}$, $1 \leq i \neq j \leq n$, with defining relations

$$[t_{ij}, t_{kl}] = 0, \quad \text{for } i, j, k, l \text{ distinct}$$
$$[t_{ij}, t_{ik} + t_{jk}] = 0, \quad \text{for } i, j, k \text{ distinct}$$

The Lie algebra $\mathfrak{t}_n = \oplus_{j=1}^{\infty} \mathfrak{t}_n^j$ is graded with $\deg t_{ij} = 1$.

$$\mathfrak{t}_n^1 = \operatorname{span} t_{ij}, \qquad \mathfrak{t}_n^2 = \operatorname{span} [t_{ij}, t_{jk}], \ldots$$

Gaudin subalgebras

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Example

The span of Gaudin hamiltonians

$$G_n(z) = \mathrm{span}(H_1, \ldots, H_n) = \{\sum_{i<j} \frac{w_i - w_j}{z_i - z_j} t_{ij},\ w \in \mathbb{C}^n\}$$

is a Gaudin subalgebra for all $z \in \mathbb{C} \setminus D_0$, $D_0 = \cup_{i<j}\{z_i = z_j\}$

Remarks

- $n-1$ is the maximal dimension of an abelian subalgebra in $\mathfrak{t}_n^1$.
- $G_n(z) = G_n(az_1 + b, \ldots, az_n + b)$. Thus $G_n(z)$ is parametrized by

$$z \in (\mathbb{C}^n \smallsetminus D_0)/\mathrm{Aff} \cong (\mathbb{P}^1)^{n+1}/\mathrm{PSL}_2 \cong \mathcal{M}_{0,n+1},$$

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Theorem (L. Aguirre, G. F., A. Veselov 2011)

- *All Gaudin subalgebras of* $\mathfrak{t}_n$ *are obtained as limits of* $G_n(z)$ *in the Grassmannian* $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$.
- *They form a smooth closed subvariety of* $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ *isomorphic to the Deligne–Mumford compactification* $\bar{\mathcal{M}}_{0,n+1}$ *of* $\mathcal{M}_{0,n+1}$

Moduli space of stable rational curves

- The Deligne–Mumford compactification $\bar{\mathcal{M}}_{0,n+1}$ is a smooth variety parametrizing stable rational nodal curves with $n+1$ marked points.

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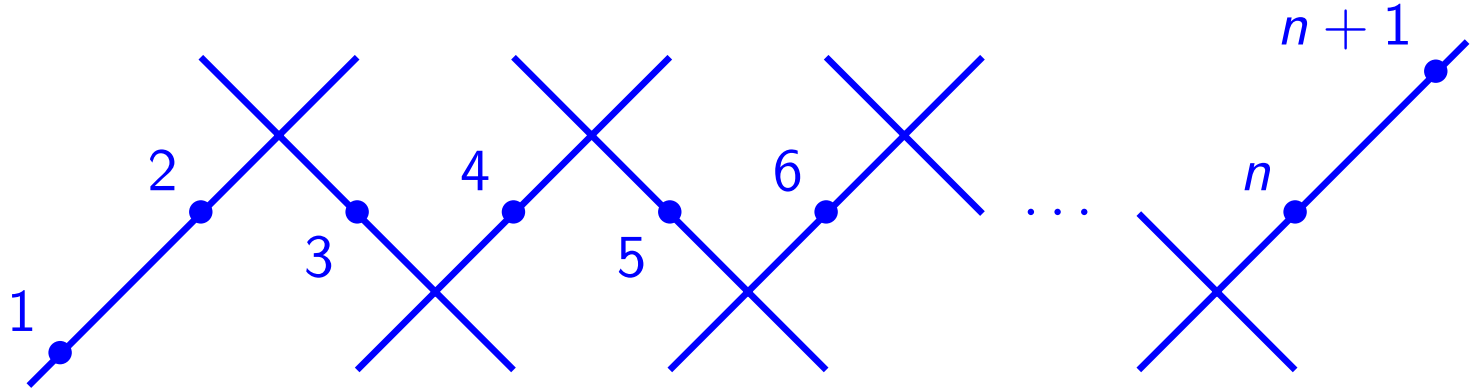
- The Jucys–Murphy elements span the Gaudin subalgebra corresponding to the rational curve



A description of $\bar{\mathcal{M}}_{0,n+1}$

There is a closed embedding $\bar{\mathcal{M}}_{0,n+1} \hookrightarrow \prod_{(i,j,k,l)} \mathbb{P}^1$ sending a point of $\mathcal{M}_{0,n+1}$ to the cross ratios of all distinct quadruples of points.

Gerritzen, Herrlich and van der Put (1988) give a set of quadratic and cubic equations defining the image.

Remark

The same holds if one restricts the product to distinct quadruples (i, j, k, l) such that $l = n + 1$: we still get an embedding cut out by a subset of the GHvdP equations.

Ingredients of the proof

For (i, j, k) distinct let $p_{ijk} \colon \mathfrak{t}_n^1 \to \mathrm{span}(t_{ij}, t_{ik}, t_{jk})$ be the projection.

Lemma

If G is a maximal abelian subalgebra in $\mathfrak{t}_n^1$ then $p_{ijk}(G)$ is at most two-dimensional and contains $t_{ij} + t_{ik} + t_{jk}$.

Thus for each maximal abelian subalgebra in $\mathfrak{t}_n^1$ there exist linear forms $\ell_{ijk} \in \mathbb{P}^2$ vanishing on $p_{ijk}(G)$.

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Lemma

If $\ell_{ijk} \in \mathbb{P}^2$ are S_3-equivariant forms vanishing on $(1, 1, 1)$ then $\cap_{(i,j,k)} \mathrm{Ker}(\ell_{ijk} \circ p_{ijk})$ is an abelian subalgebra.

It turns out that the conditions on $\ell_{ijk} \in \mathbb{P}^1 = (1, 1, 1)^\perp \subset \mathbb{P}^2$ to yield a subalgebra of maximal dimension are equivalent to the Gerritzen–Herrlich–van der Put relations for cross ratios $(i, j, k, n + 1)$.

Tautological bundle

The variety $\bar{\mathcal{M}}_{0,n+1}$ carries a tautological vector bundle $\mathcal{G}_n$, whose fiber at a point is the corresponding Gaudin subalgebra.

Theorem (AFV 2011)

The tautological vector bundle $\mathcal{G}_n$ on $X = \bar{\mathcal{M}}_{0,n+1}$ is an extension

$$0 \to \mathcal{O}_X \to \mathcal{G}_n \to T_X\langle -D\rangle \to 0$$

of the logarithmic tangent bundle $T_X\langle -D\rangle$ of the compactification divisor $D = \bar{\mathcal{M}}_{0,n+1} \smallsetminus \mathcal{M}_{0,n+1}$. The embedding of the trivial bundle $\mathcal{O}_X$ sends 1 to the central element $c_n = \sum_{i<j} t_{ij}$.

Spectrum

Let V be a finite dimensional representation of $\mathfrak{t}_k$ such that the central element $c_n = \sum_{i<j} t_{ij}$ is mapped to α.

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Theorem (AFV 2011, cf. Reshetikhin–Varchenko)

The common spectrum of the Gaudin subalgebras on V is a coisotropic subscheme of $T^*\langle D\rangle_\alpha \bar{\mathcal{M}}_{0,n+1}$. Its intersection with the symplectic subvariety $T^*\langle D\rangle_\alpha \mathcal{M}_{0,n+1} \cong T^*_\alpha \mathcal{M}_{0,n+1}$ is Lagrangian.

Holonomy Lie algebra

Sofar we considered the complement of the hyperplanes $z_i = z_j$, which are the hyperplanes of the A_n root system. What of this extends to a general root or Coxeter system?

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Definition (Kohno)

Let $\Delta \subset \mathfrak{h}^*$ be a set of positive roots of a root system. The *holonomy Lie algebra* $\mathfrak{t}_\Delta$ is the Lie algebra with generators $(t_\alpha)_{\alpha \in \Delta}$ and a relation

$$[t_\alpha, \sum_{\beta \in V_2 \cap \Delta} t_\beta] = 0,$$

for each pair (α, V_2) with $\alpha \in \Delta$, and $\alpha \in V_2 \subset \mathfrak{h}^*$ a two-dimensional subspace.

Generalized Gaudin hamiltonians

Proposition

Let $z \in \mathfrak{h}_{\mathrm{reg}} := \mathfrak{h} \setminus \cup_{\alpha \in \Delta} \mathrm{Ker}\, \alpha$ The subspace

$$G_\Delta(z) = \left\{ \sum_{\alpha \in \Delta} \frac{\alpha(w)}{\alpha(z)} t_\alpha, w \in \mathfrak{h} \right\} \subset \mathfrak{h}^*,$$

is a maximal abelian subalgebra of $\mathfrak{t}_\Delta$

Examples

- Let e_α be root vectors in a semisimple Lie algebra $\mathfrak{g}$. The elements $t_\alpha = e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha \in U\mathfrak{g}$, $\alpha \in \Delta_+$. obey the relations of the holonomy Lie algebra.

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- The reflections s_α of the Weyl group of Δ obey the same relations.

Principal Gaudin subalgebras

Definition

The space of *principal Gaudin subalgebra* is the closure of the set of abelian subalgebras $G_\Delta(z)$, $z \in \mathbb{P}(\mathfrak{h}_{\mathrm{reg}})$ in the Grassmannian $\mathrm{Gr}(r, \mathfrak{t}^1_\Delta)$, $r = \dim \mathfrak{h}$.

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De Concini and Procesi defined and studied a smooth compactification $\bar{\mathcal{M}}_\Delta$ of $\mathcal{M}_\Delta := \mathbb{P}(\mathfrak{h}_{\mathrm{reg}})$ generalizing $\bar{\mathcal{M}}_{0,n+1}$.

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Theorem (AFV 2014)

The principal Gaudin subalgebras form a smooth subvariety of $\mathrm{Gr}(r, \mathfrak{t}^1_\Delta)$ *isomorphic to the De Concini–Procesi compactification* $\bar{\mathcal{M}}_\Delta$.

Remarks

- In general principal Gaudin subalgebras are just one component of the variety of r-dimensional abelian subalgebras in $\mathrm{Gr}(r, \mathfrak{t}^1_\Delta)$. Also, in some cases r is not the maximal dimension.
- The statements about tautological bundle and spectrum extend to principal Gaudin subalgebras for general root systems.

Holonomy Lie algebras and Gaudin subalgebras

Let Δ be a finite set of pairwise non-collinear vectors in $\mathfrak{h}^*$, thought of as the set of normal vectors to an arrangement of hyperplanes in $\mathfrak{h}$. With the same definitions as for root systems, we have the holonomy Lie algebra t_Δ, the parameter space $\mathcal{M}_\Delta = \mathbb{P}(\mathfrak{h}_{\mathrm{reg}})$, the De Concini–Procesi compactification $\bar{\mathcal{M}}_\Delta$ and the set of principal Gaudin subalgebras, closure of the abelian subalgebras $G_\Delta(z), z \in M_\Delta$.

Aguirre's criterion

A *flat* $A \subset \Delta$ is a subset so that $A = \Delta \cap \mathrm{span}_{\mathbb{C}} A$. A flat is *reducible* if it is the union of two flats whose spans are in direct sum, otherwise *irreducible*. The *rank* of A is the dimension of its span.

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Condition A

For all irreducible flats A of Δ and corank 1 subflats $F \subset A$, $\mathrm{span}_{\mathbb{C}}(F) \cap \mathrm{span}_{\mathbb{C}}(A \smallsetminus F)$ is not the span of a proper subflat of A.

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Theorem (Aguirre 2014)

There is a morphism $\bar{\mathcal{M}}_\Delta \to \mathrm{Gr}(r, \mathfrak{t}_\Delta)$ restricting to an embedding of $\mathcal{M}_\Delta$ with image $\cup_{z \in \mathfrak{h}_{\mathrm{reg}}} G_\Delta(z)$. It is a closed embedding if and only if Δ obeys Condition A.

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Condition A is satisfied both for $n \geq r$ vectors in an r-dimensional space in general position and for (normal vectors of reflection hyperplanes of) Coxeter systems. There is a counterexample with 5 vectors in 3-space.

Paljon onnea päivänäsi!

Ihanaa syntymäpäivää, Antti!