Optimal Transport point of view of the Fluctuation theory

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Phenomenology of Fluctuation: Onsager, Bertini et al.m Maes et al. etc...

- $(X, \Omega, \mathbb{P})$ an abstract fluctuating quantity $X \in \Omega$ the space of possible outcomes (path space by example).
- Θ a symmetry involution, i.e $\Theta^2 = \Theta \circ \Theta^* = Id$.

i.e the value of the variable depends of some hidden degrees of freedom.
The fluctuating variable X is said Θ-stable, scalable or commensurable if the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}(\Theta X)}{\mathrm{d}\mathbb{P}(X)} = e^{-A(X)}$$

with A the well defined action functional.

Problem :

$$A + A \circ \Theta = 0$$

This action contains only the anti-symmetric part of the information (Maes et.al).
Less worst Solution:
Define the action with respect to a well choosen symmetric reference probability $\mathbb{P}_0$ such that $(\mathbb{P}, \mathbb{P}_0)$ are scalable

$$\frac{\mathrm{d}\mathbb{P}(X)}{\mathrm{d}\mathbb{P}_0(X)} = e^{-A(X)}$$

and

$$\mathbb{P}_0(\Theta X) = \mathbb{P}_0(X)$$

and then

$$\frac{\mathrm{d}\mathbb{P}(\Theta X)}{\mathrm{d}\mathbb{P}(X)} = e^{-A_a(X)}$$

with

$$A_a = A - A \circ \Theta.$$

Universal Fluctuation relations

Let f a regular observable on the finite space Ω,

$$\langle f(X)\rangle = \sum f(x)\mathbb{P}_X(x) = \langle e^{-A(X)} f(X)\rangle_0$$

with $\mathbb{P}(x) = \int \delta(X - x)\mathrm{d}\mathbb{P}(X)$.
and in the same way

$$\langle f(\Theta X)\rangle = \langle e^{-A_a(X)} f(X)\rangle$$

what is a first fluctuation relation (FT).

- taking $f(x) = \delta(S(x) - \sigma)$ imply

$$\frac{\mathbb{P}[A_a(X) = -\sigma]}{\mathbb{P}[A_a(X) = -\sigma]} = e^{-\sigma}$$

(Gallavotti-Cohen-like relation)

- $$1 = \sum \mathbb{P}(x) = \sum e^{-A(x)} \mathbb{P}_0(x) = \langle e^{-A} \rangle_0$$

(Jarzinsky like relation if conservation of mass).

- $\langle A(X) \rangle = \left\langle \log(\frac{\mathrm{d}\mathbb{P}(\Theta X)}{\mathrm{d}\mathbb{P}(X)} \right\rangle.$

The GC relation is valid for any anti-symmetric observable.

Dynamic structure

Let now add a dynamic structure. We look at $(X(t), \mathbb{P}_X, \mathbf{L}_0, \Omega \times \mathbb{R}^+)$. $\mathbf{L}_0$ is the generator of the flow of marginal or the hydrodynamic limit

- or $\partial_t \rho(t,x) = \mathbf{L}_0 \rho_0(t,x)$ with $u(x,t) = \langle u(X(t)\rangle_x$

More generally, let $(X_1, \cdots X_n)$ a process such that

$$\rho_n := \frac{1}{n} \sum \delta_{X_i(t)} \sim \rho \text{a.s}$$

is a weak solution of $\partial_t \rho(t,x) = \mathbf{L} \rho_0(t,x)$

Dynamic Symmetry

Let $\mathbf{L}^{\Theta}$ the generator of $\Theta X(t)$. Then the Process is said

- symmetric if
$$\mathbf{L}^{\Theta} = \mathbf{L}$$
.
- locally symmetric if there exists a well defined function h such that
$$e^{h}\mathbf{L}^{\Theta}e^{-h} = \mathbf{L}$$

Local-symmetry and Stochastic Mechanic

Idea of Girsanov, Onsager Machlup: LS has a mechanical meaning.
Principle: Let consider $\mathbf{L}^h = e^{-h}\mathbf{L}e^h$ for h in some functional space. Let, $\mathbb{P}_h^n$ and $\mathbb{P}_0^n$ the probability law of the associated empirical process ρ_n^h and ρ_n.
Let $\{\mathcal{H}(\rho, h), \mathcal{L}(\rho, \dot{\rho})\}$ well defined stochastic Hamitonian and Lagrangian associated

- $$\frac{1}{n}\log\left(\frac{\mathrm{d}\mathbb{P}_h^n}{\mathrm{d}\mathbb{P}_0^n}(\rho)\right) = \int_0^T \langle \dot{\rho}, h\rangle - \mathcal{H}(\rho, h)dt$$

  $\mathbb{P}_0^n$ almost surely.

- $$\mathcal{L}(\rho, \dot{\rho}) = \sup_h \langle \dot{\rho}, h\rangle - \mathcal{H}(\rho, h)$$

(Donsker-Varadhan theory)

and then formally when $n \to \infty$

$$\mathbb{P}(\rho_n|_0^T \sim \rho|\rho_n(0) = \rho(0)) \asymp e^{-n\int_0^T \mathcal{L}(\rho,\dot{\rho})dt}$$

because of

$$\frac{1}{n}\log(\mathbb{P}_0^n(\rho_n \sim \rho_h|\rho(0)) = \frac{1}{n}\log(\int \mathbf{1}_{\rho_n\sim\rho_h}\frac{1}{n}\left(\frac{\mathrm{d}\mathbb{P}_0^n}{\mathrm{d}\mathbb{P}_h^n}(\rho)\right)\mathrm{d}\mathbb{P}_h^n(\rho)) \tag{0.1}$$

$$\asymp -\sup_h \int_0^T \langle\dot{\rho}, h\rangle - \mathcal{H}(\rho, h)dt \tag{0.2}$$

Cramer-Sanov Theory

Let ν be the unique stationary state of the process, i.e

$$\mathbf{L}(\nu) = 0$$

Then

$$\mathbb{P}_0^n(\rho_n(0) = \rho) \asymp e^{-n\mathcal{S}(\rho)}$$

with

$$\begin{aligned} \mathcal{S}(\rho) &= \int \log(\frac{\rho}{\nu})\rho dx \\ &= \int \log(\rho)\rho dx + \int V(x)\rho dx \end{aligned} \tag{0.3}$$

with $\nu = e^{-V(x)}$. and then

$$\mathbb{P}(\rho_n|_0^T \sim \rho) \asymp e^{-n(\mathcal{S}(\rho)+\int_0^T \mathcal{L}(\rho,\dot{\rho})dt)}$$

Local symmetry as Local equilibrium

The Girsanov-Onsager-Machlup theory translates fluctuation into mechanic symmetry. Let $\mathbf{L}$ a Locally symmetric generator with conjugate function h then

- $$\frac{\mathbb{P}(\rho_n^{\Theta}|_0^T \sim \rho)}{\mathbb{P}(\rho_n|_0^T \sim \rho)} \asymp e^{-n((\mathcal{S}(\rho^{\Theta})-\mathcal{S}(\rho)+\int_0^T \mathcal{L}(\rho^{\Theta},\dot{\rho})-\mathcal{L}(\rho,\dot{\rho})dt)}$$

We have now to look at the equations

$$\mathcal{S}(\rho^{\Theta}) - \mathcal{S}(\rho) = \int_0^T \mathcal{L}(\rho, \dot{\rho}) - \mathcal{L}(\rho^h, \dot{\rho})dt$$

Onsager phenomenology: physical decomposition of the flow

We have now to look at the structure of the generator following the usual phenomenology.

- $\mathbf{L}(\rho) = \nabla \cdot \mathcal{J}(\rho)$, with $\mathcal{J}$ the current.
- $\mathcal{J} = \mathcal{J}_{in} + \mathcal{J}_{out}$ with $\mathcal{J}_{in}(\rho) = \mathcal{D}(\rho)\nabla\rho$ (Fourier-Ohm-Fick law)
- $\mathcal{J}(\rho) = \chi(\rho)F(\rho)$ with χ the Onsager phenomenological coefficient and $F(\rho) = F_{in}(\rho) + F_{out}$
- Compatibility condtion Einstein relation: there exists f convex and with normalization conditions s.t

$$\mathcal{D}(\rho) = \chi(\rho)f''(\rho)$$

imply

$$\mathcal{J}_{in}(\rho) = \chi(\rho)\nabla f'(\rho) \text{ and } F_{in}(\rho) = \nabla f'(\rho) \text{ Gradient Force}$$

.

$\mathcal{S}(\rho) = \int f(\rho)dx$ is called entropy and $\mathrm{EP}(\rho) = \int J(\rho)F(\rho)dx$ is called Entropy production.

Beyond the Onsager phenomenology

We then have as structure of the generator

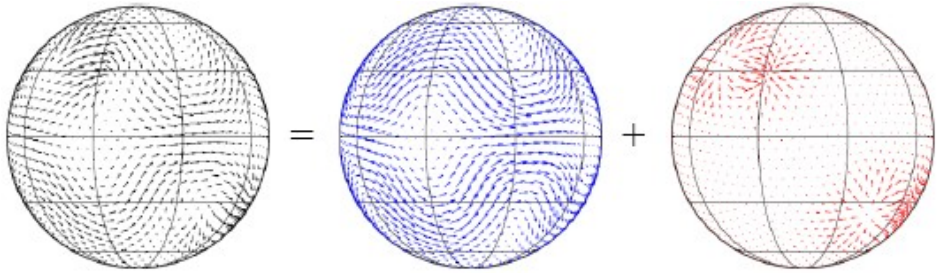
$$\mathbf{L}(\rho) = \operatorname{div}\left(\chi(\rho)(F_{in}(\rho) + F_{out})\right)$$

We now look for the geometrical decomposition

$$\mathbf{L}(\rho) = \operatorname{div}\left(\chi(\rho)(F_{grad}(\rho) \oplus_{\chi} F_{curl}(\rho))\right)$$

with $F_{grad} \perp_{\chi} F_{curl}$ in a natural metric.

- Onsager principle: We can compare two flows if they have same phenomenological structure



Relation with Local- time symmetry

Let Θ the time-symmetry, then there exists h such that

- F_{grad} is the local symmetric part or local balanced part
- F_{grad} is the local anti-symmetric part.

Meaning that the geometrical decomposition is the one that respect the fluctuation symmetry.
Main Point: The orthogonality provides to still get a generalized symmetry.
To sum up we need

- a generator $\mathbf{L}$,
- a phenomenological coefficient χ,
- A geometric decomposition .

Two main models

We first focus on two models of dynamic

- Fokker-Planck equation:

$$\partial_t \rho = \Delta \rho - \mathrm{div}(\rho F))$$

  in a distributional sense

- Discrete Master equation :

$$\partial_t \rho_t(i) = \sum_{j \neq i} c(i) k_{ij} \rho_j - c(j) k_{ji} \rho_i$$

in the continous case $\Omega \subset \mathbb{R}^n$ and F a smooth driving force. While in the discrete case Ω is a finite discrete space. k_{ij} is the transition of probability rate from the state i to the state j while c is called the escape rate and assumed positive.

The adjoint operator are then given in the two cases by

- $$\partial_t f = \Delta f + F \cdot \nabla f$$
- $$\partial_t f(i) = \sum_{j \neq i} c(i) k_{ij} (f_j - f_i)$$

The adjoint are very important in the sense that they are the time reversal of the original operator. The adjoint is the dynamic on the observable while the original is the dynamic on measure.
Finnally we assume that the two dynamic have a stationary state also called NESS. We note ν the FPE NESS and π the one of the ME. the existence and unicity of this state is equivalent to some ergodicity assumptions and is the key of the geometric decomposition.

Optimal transport geometry

We focus on the continous dynamic for which it is more natural to derive the Onsager geometry.

$$\partial_t \rho = \Delta \rho - \mathrm{div}(\rho F)$$

From this dynamic we would build the adapted geometry.
As discussed before the main point this to determine the Onsager coefficient.
In this case we have clearly $\chi(\rho) = \rho$.

Notation

- Ω is an open real set of of a Riemaniann manifold.
- $\mathrm{D}\mathcal{F}(\rho) := \frac{\delta \mathcal{F}}{\delta \rho}(\rho)$ The first Euler variation or functional derivative of the functional $\mathcal{F}$
- $\mathcal{P}_2(\mathbb{R}^d) :=$
  $\left\{\mu = \rho dx \text{ Borel probability measure s.t } \mu(x^2) < \infty\right\}$

First Steps

Let fix $\mu = \rho dx \in \mathcal{P}_2(\mathbb{R}^d)$,
we then have a couple of dual convex norm given by

$$\|\zeta\|_{1,\mu}^2 := \|\zeta\|_{H^1(\mu)}^2 = \int \nabla\zeta \cdot \rho\nabla\zeta dx, \forall \zeta \in C_c^\infty(\mathbb{R}^d)$$

and the dual one, for any $s \in \mathcal{D}'(\mathbb{R}^d)$,

$$\|s\|_{-1,\mu}^2 := \|s\|_{H_D^{-1}(\mu)}^2 = \sup_{\zeta \in C_c^\infty(\mathbb{R}^d)} \left\{2\langle s, \zeta\rangle - \|\zeta\|_{1,\mu}^2\right\}$$

We define the underlying dual weighted Sobolev space as their closure in $L^2(d\mu, \mathbb{R}^d)$.

Tangent space

We now want to define the tangent spaces on the space of probability measures as defined by [AS07]:

Definition

The tangent space on $\mathcal{P}_2(\mathbb{R}^d)$ are defined by

$$Tan_\mu := \overline{\{\nabla\phi,\ \phi \in\ C_c^\infty(\mathbb{R}^d)\}}^{L^2(\rho)} \tag{0.4}$$

$$Tan_\mu^\perp = \left\{\mathbf{w} \in\ L^2(\mu) : div(\mathbf{w}\rho) = 0\right\} \tag{0.5}$$

in particular we have, [**?**],

$$Tan_\mu \simeq H^{-1}(\mu).$$

First Steps

Noticing that we then get an orthogonal decomposition if we set Π_μ the orthogonal projector from $L^2(\mu)$ onto Tan_μ then we have that for any $\gamma \in L^2(\mu)$

$$\begin{cases} \gamma = \mathbf{v} + \mathbf{w} \\ \dot{\mu} + div(\mathbf{v}\mu) = 0 \\ div(\mathbf{w}\mu) = 0 \end{cases}$$

with $\mathbf{v} = \Pi_\mu(u)$ and $\mathbf{w} = (1 - \pi_\mu)(u)$. We point out that this decomposition will be physically meaningful for the problem under consideration.
We can get a variational characterization of the tangent space via

$$\mathbf{v} \in Tan_\mu \quad iif \quad \|\mathbf{v}\|^2_{\mu,-1} \leq \|\mathbf{v} + \mathbf{w}\|^2_{\mu,-1}, \quad \forall \mathbf{w} \in Tan^\perp_\mu, . \quad (0.6)$$

The Metric Associated to the Dynamic.

We then define very naturally the associated metric as the variational metric with metric tensor given by the dual weighted Sobolev norm, i.e the Wasserstein metric:

$$W(\mu_0,\mu_1) := \inf\left\{\int_0^1 \|\dot{\rho}_t\|^2_{H^{-1}(\rho)}dt : \rho \in \Gamma_t(\mu_0,\mu_1)\right\} \tag{0.7}$$

$$:= \inf\left\{\int_0^1 \|\mathbf{v}\|^2_{L^2(\mathbb{R}^d)}dt : (\mu,\mathbf{v}) \in EC(\mathbb{R}^d) \text{ and } \mu \in \Gamma_t(\mu_0,\mu_1)\right\}, \tag{0.8}$$

with

$$EC(\mathbb{R}^d) := \{(\rho.\mathbf{v}) : \dot{\rho} + \mathrm{div}(\rho\mathbf{v}) = 0\},$$

while

$$\Gamma_T(\mu|\nu) = \{\gamma : \gamma(0) = \mu \text{ and } \gamma(T) = \nu\} \subset \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$$

the time coupling space between two measures.

This is the so-called *Brenier dynamic formulation* of the Wasserstein metric. We then consider the absolutely continuous curve with respect to this metric:

$$AC_W(I, \mathcal{P}_2(\mathbb{R}^d)) \tag{0.9}$$

$$:= \left\{ \mu_t : I \to \mathcal{P}_2(\mathbb{R}^d) : \exists m \in L^1(I) \text{ s.t } W(\mu_t, \mu_s) \leq \int_s^t m(r)dr, \forall t, s \in I \right\} \tag{0.10}$$

Metric Gradient Flow

Let the Onsager operator

$$\mathbb{K}(\rho) \cdot f := \nabla \cdot \chi(\rho) \nabla f$$

Then the metric reads

$$\langle \zeta_1, \zeta_2 \rangle_{1,\mu}^2 = -\langle \zeta \cdot \mathbb{K}(\rho) \zeta \rangle, \forall \zeta \in C_c^\infty(\mathbb{R}^d)$$

Let $\mathcal{F}$ a regular and convex functional, if it exists $grad_\chi[\mathcal{F}]$ is the unique element such that for any absolute continuous curve $\mu_t = \rho_t dx$ the chain rule holds

$$\partial_t \mathcal{F}(\mu_t) = \langle grad_\chi[\mathcal{F}], \partial_t \rho_t \rangle_{-1,\mu}$$

and then by construction we have

$$grad[\mathcal{F}](\mu) = \mathbb{K}(\rho) \mathrm{D}\mathcal{F}(\mu)$$

Geodesic convexity

One important point of the differential calculus developed by [AS07] is the notion of λ-convex displacement.
A couple $(\mathcal{F}, \mathbb{K})$ is said λ-geodesic convex iif there exists for any $\rho_1, \rho_2 \in \mathcal{P}_2 \cap Dom(\mathcal{F})$ a unique geodesic ρ_s s.t

$$\Phi(\rho_s) \leq (1-s)\mathcal{F}(\rho_1) + s\mathcal{F}(\rho_2) - \lambda(1-s)sW_\chi(\rho_1, \rho_2)$$

The λ-convexity was introduced by Mccain to generalize the so called interpolation displacement. Under general condition it is well known that

$$(\mathcal{F}, \mathbb{K}) \text{ is } \lambda - \text{geodesic convex} \Rightarrow W(\mu_t, \nu) \leq e^{-\lambda t} W(\mu_0, \nu).$$

geodesic convexity $\sim$ fast return to equilibrium.
In particular this is also equivalent to a lower bound on the Ricci curvature meaning that is a geometrical feature.

Associated Mechanic

We define the dual mechanical functional $(\mathcal{L}_{\mathcal{V}}, \mathcal{H}_{\mathcal{V}})$ of the system as

$$\begin{cases} \mathcal{L}_{\mathcal{V}}(\mu, s) & := \frac{1}{2}\|s\|^2_{H^{-1}(\mu)} - \mathcal{V}(\mu) \\ \mathcal{H}_{\mathcal{V}}(\mu, \zeta) & := sup_s \left\{\langle \zeta, s\rangle - \mathcal{L}_{\mathcal{V}}(\mu, s)\right\} \end{cases}$$

In particular, $\mathcal{H}_{\mathcal{V}}(\mu, \zeta) = \frac{1}{2}\|s\|^2_{H^1(\mu)} + \mathcal{V}(\mu)$.

Associated Mechanic

We can then now introduce naturally the action functional and dissipation functional.
We define

i) *the Action functional* as the time integrated sum of kinetic energy and potential one:

$$\mathcal{A}_{\mathcal{V}}(\mu, T) := \int_0^T \mathcal{L}_{\mathcal{V}}(\mu, \dot{\mu}) dt \tag{0.11}$$

ii) *Dissipation functional*:

$$\mathcal{D}_{\mathcal{V}}(\mu \| \nu : T) = \inf \{\mathcal{A}_T(\hat{\mu}) : \hat{\mu} \in \Gamma_T(\mu, \nu)\} \tag{0.12}$$

Associated Mechanic

Let the general Value function given

$$\mathbb{V}(\rho, T) = \inf \left\{ \mathcal{G}(\sigma(0)) + \int_0^T \|\dot{\sigma}(t)\|^2_{H_D^{-1}(\rho)} - \mathcal{V}(\sigma(t))dt : \sigma(T) = \rho \right\} \tag{0.13}$$

Then

i) $\mathbb{V}$ is countinuous in $\mathcal{P}_2(\mathbb{R}^d_D)$.

ii) $\lim_{t\to 0^+} \mathcal{V}(\rho) = \mathcal{G}(\rho)$.

iii) $\mathbb{V}$ is a viscosity solution of the reduced HJB equation

$$\partial_t \mathbb{V} + \mathcal{H}(\mu, D\mathbb{V}) = 0 \tag{0.14}$$

Now using the so called *dynamic programming principle* we can get by a straightforward generalization of [FK13] Assume that $\mathcal{V}(\rho)$ is a regular potential, then a minimization path for the generalized value function $\mathcal{V}_{\mathcal{F}}$, σ with minimal tangent vector $\mathbf{v}$ is solution of the Euler-Poisson equation

$$\begin{cases} \partial_s \sigma + \nabla \cdot (\sigma \mathbf{v}) = 0 \\ \partial_s (\sigma \mathbf{v}) + \nabla \cdot (\sigma \mathbf{v} \otimes \mathbf{v}) = -\sigma \nabla V'(\sigma). \end{cases}$$

and finally we get the Newton formulation

$$\nabla^{\mathcal{W}}_{|\dot{\mu}|_t} \left(|\dot{\mu_t}| \right) = -grad_{\mathcal{W}}[\mathcal{V}](\mu) \tag{0.15}$$

with $\nabla^{\mathcal{W}}_{|\dot{\mu}|_t} \left(|\dot{\mu_t}| \right)$ the covariant derivative or parralel transport. See Lott and Gigli.

Consequences

The unique solution to the stationary action problem

$$\mathcal{F}(\rho(T)) = \mathcal{F}(\rho(0)) + \int_0^T \frac{1}{2}\|s\|^2_{H^{-1}(\mu)} + \frac{1}{2}\|\mathrm{D}\mathcal{F}(\rho)\|^2_{1,\mu}$$

is given by the gradient flow

$$\partial_t \rho = \mathrm{grad}[\mathcal{F}](\rho)$$

Then $\mathcal{F}$ is a Lyapunov functional for the flow i.e $\partial_t \mathcal{F}(\rho) \geq 0$ along the flow.
In particular ρ is a gradient flow iif

$$\int_0^T \|\dot{\rho} - \mathrm{grad}[\mathcal{F}](\rho)\|^2_{-1,\mu} dt = 0$$

What is the condition in order $\mathcal{F}$ is still a Lyapunov functional for the flow? Let write the generator $L = \mathrm{grad}[\mathcal{F}](\rho) + L_0$ then

$$\int_0^T \|\dot{\rho} - L\|^2_{-1,\mu} = 0$$

is equivalent to

$$\mathcal{F}(\rho(T)) = \mathcal{F}(\rho(0)) + \int_0^T \frac{1}{2}\|\dot{\rho} - L_0\|^2_{H^{-1}(\mu)} + \frac{1}{2}\|\mathrm{D}\mathcal{F}(\rho)\|^2_{1,\mu} + \langle \mathrm{grad}[\mathcal{F}], L_0\rangle_{-1,\mu} \tag{0.16}$$

then $\mathcal{F}$ is still a Lyapunov functional if

$$\langle \mathrm{grad}[\mathcal{F}], L_0\rangle_{-1,\mu} = 0$$

and in particular we regognize the Dissipation functional introduced by Bertini et al. being a genralization of the Onsager one:

$$\Phi(\rho, \dot{\rho}) := \partial_t \mathcal{F} + \frac{1}{2} \|\dot{\rho} - L_0\|^2_{H^{-1}(\mu)}$$

At least formally we have

$$\mathrm{D}_{\dot{\rho}} \Phi(\rho, \dot{\rho}) = 0 \sim \dot{\rho} = L(\rho).$$

Application to the canonical models

The main ideas:

- We want to project the flow with respect to the Onsager metric on the optimal gradient part.
- Use the principle : Local-time symmetry $\sim$ gradient force.

Let (L, ν) a generator and its stationary state then we have equivalence

- L is Local time symmetric
- $L = \nu L^* \nu^{-1}$
- $\mathcal{J}(\nu) = 0$

We will begin with the simplest case that is the linear diffusion irreversible: Let

$$L_b(\rho_t) := \Delta\rho_t - \mathrm{div}(\rho_t b)$$

such that L is not locally time symmetric i.e $b \neq \nabla f$,
Result : There exists a uplet $(\mathbb{K}, \mathbb{M}, \mathcal{S}_\nu, F_s)$ such that the irreversible linear diffusion Dynamic :

$$\partial\mu_t = L_b(\mu_t) \tag{0.17}$$

can be decomposed as

$$\partial\mu_t = \mathbb{K}(\mu)\cdot \mathrm{D}\mathcal{S}_\nu(\mu_t) + \mathbb{M}(\mu)\cdot F_s. \tag{0.18}$$

$$= -\mathrm{grad}[\mathcal{S}_\nu](\rho) + \mathrm{div}(\rho F_s) \tag{0.19}$$

With

- $\mathcal{S}_\nu(\rho) = \int \rho\log(\frac{\rho}{\nu})\rho dx$ the free energy or relative entropy
- $F_s = \frac{J(\nu)}{\nu}$ the stationary force

Let

$$\begin{cases} L_s = \mathbb{K}(\mu) \cdot \mathrm{D}\mathcal{S}_\nu & (\textit{ Symmetric Dynamic }) \\ L_a = \mathbb{M}(\mu) \cdot \mathrm{D}\mathcal{Q}_\nu & (\textit{ Asymmetric Dynamic }) \end{cases}$$

then

- $\langle L_s, L_a \rangle_{-1.\mu_t} = 0.$
- $\partial_t \mathcal{S}(\mu_t) \leq 0$ Lyapunov function.

Moreover, the Entropy Production of the system can be decomposed as

$$EP(L_b) = \mathcal{I}_\nu + \mathcal{Q}_\nu \tag{0.20}$$

with $\mathcal{I}$ the Fisher information and $\mathcal{Q}_\nu$ the so called *keeping warm house*.

Correspondence with local symmetry

We have

- $$\frac{L_b(\rho_t) + \nu \cdot L_b^*(\frac{\rho_t}{\nu})}{2} = -\mathrm{grad}[\mathcal{S}_\nu](\rho). \tag{0.21}$$

  While,

- $$\frac{L_b(\rho_t) - \nu \cdot L_b^*(\frac{\rho_t}{\nu})}{2} = \mathbb{M}(\rho) \cdot F_s. \tag{0.22}$$

Consistent with the physical picture.

relation with the stochastic symmetries

We call Onsager adjoint the flow generated by the uplet $(\mathbb{K}, -\mathbb{M}, \mathcal{S}_\nu, F_s)$.
We note $L^\sharp$ the associated generator. The Onsager adjonction is a isometry for the associated norm by orthogonality.
we note the Lagrangian of the forward dynamic by

$$\mathcal{L}(\mu, s) := \frac{1}{2}\|s - L_b(\mu)\|^2_{-1.\mu}$$

Symmetries for gradient flow:

$$\mathcal{L}(\mu, s) - \mathcal{L}(\mu, -s) = 2\langle s, \mathrm{D}\mathcal{S}_\nu\rangle \tag{0.23}$$

and by duality

$$\mathcal{H}(\mu, v + \mathrm{D}\mathcal{S}(\rho)) = \mathcal{H}(\mu, -v + \mathrm{D}\mathcal{S}(\rho)) \tag{0.24}$$

giving the stochastic symmetries.

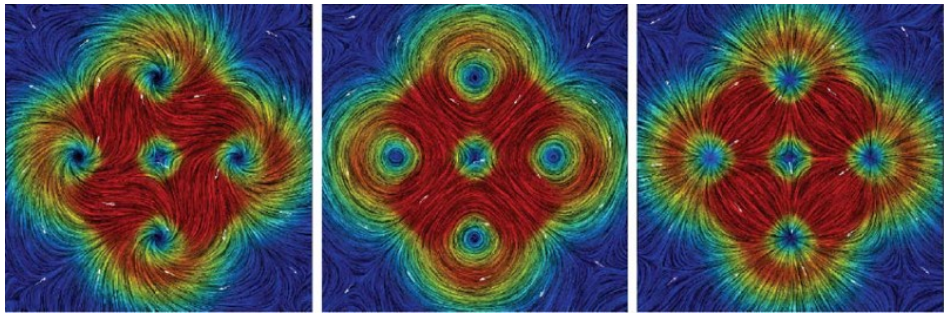
Generalized symmetries

In our case the orthogonal decomposition provides us to get

$$\mathcal{L}(\mu, s) - \mathcal{L}^\sharp(\mu, -s) = 2\langle s, \mathrm{D}\mathcal{S}_\nu \rangle \tag{0.25}$$

and by duality

$$\mathcal{H}(\mu, v + \mathrm{D}\mathcal{S}(\rho)) = \mathcal{H}^\sharp(\mu, -v + \mathrm{D}\mathcal{S}(\rho)) \tag{0.26}$$

Onsager-Machlup principle:

$$\mathcal{A}(\rho, T) = \mathcal{S}(\rho(T)) - \mathcal{S}(\rho(0)) \sim \mathcal{L}^\sharp(\rho, -\rho) = 0$$