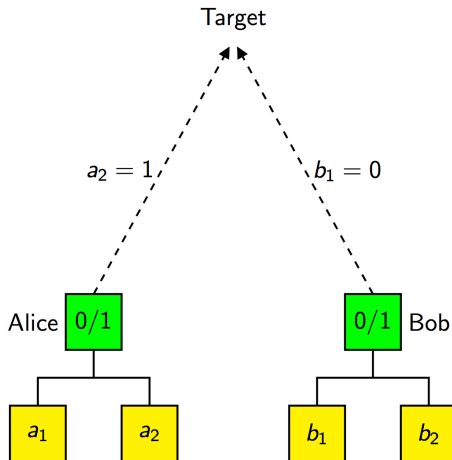
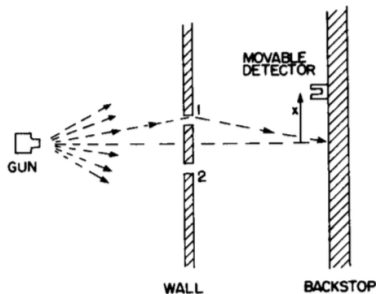


Quantum team logic and Bell's Inequalities

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Double-Slit Experiment



R. Feynman:

“Is it true, or is it not true, that the electron either goes through hole 1 or it goes through hole 2?”

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THE LOGIC OF QUANTUM MECHANICS

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1. Introduction. One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes. It asserts that even a complete mathematical description of a physical system \mathfrak{S} does not in general enable one to predict with certainty the result

16. The logical coherence of quantum mechanics. The above heuristic considerations suggest in particular that the physically significant statements in quantum mechanics actually constitute a sort of projective geometry, while the physically significant statements concerning a given system in classical dynamics constitute a Boolean algebra.

They suggest even more strongly that whereas in classical mechanics any propositional calculus involving more than two propositions can be decomposed into independent constituents (direct sums in the sense of modern algebra), quantum theory involves irreducible propositional calculi of unbounded complexity. This indicates that quantum mechanics has a *greater logical coherence*

- Propositional logic
- Atoms: "Gate is open", "Red light is on", "Spin at 60° is up"
- True/false
- Proposition symbols p_0, \dots, p_n
- Connectives $\neg, \wedge, \vee, \rightarrow$, e.g. $(p_0 \wedge p_1) \vee (\neg p_0 \wedge \neg p_1)$
- Valuation $v : \{p_0, \dots, p_n\} \rightarrow \{0, 1\}$
- Valuation v models (a version of) truth
- Truth-value $v(\phi) \in \{0, 1\}$

- **Multi**-team = an indexed set X of valuations

$$v : \{p_0, \dots, p_n\} \rightarrow \{0, 1\}$$

(repetitions allowed)

- X models “observations” about $\langle p_0, \dots, p_n \rangle$
- X models “probabilities” of different versions of truth about $\langle p_0, \dots, p_n \rangle$
- Can give meaning to “probability”

$$[\phi]_X$$

of ϕ in X : the number of valuations satisfying ϕ divided by the size of $|X|$ (in the finite case).

- Examples $[\phi \wedge \neg\phi]_X = 0$, $[\phi \vee \neg\phi]_X = 1$.

	p_0	p_1	p_2	p_3
0	1	1	0	1
1	1	1	1	1
2	1	1	1	1
3	1	1	1	0
4	0	0	1	1
5	0	0	0	0
6	0	0	0	0
7	0	0	0	0

A logic for arguing about probabilities

- Can build a formal language PTL for reasoning about statements of the form “the probability of ϕ (in the multi-team) is at least 0.125”.
- We shorten this to $\phi \geq 0.125$.
- We allow also more general statements $a_0\phi_0 + \dots + a_k\phi_k \geq c$, where the a_i are rationals.

Theorem (Fagin-Halpern-Megiddo 1990)

Let α be a formula of PTL. Then ϕ is provable in PTL if and only if ϕ is true in every multi-team. (“Completeness”)

Given arbitrary propositional formulas $\{\phi_j : j < k\}$, which are (i.e. the conjunction is) contradictory, then easily

$$\sum_{j < k} \phi_j \leq k - 1$$

is **provable** in PTL.

Observing entangled particles

- p_0 = "Alice measurement at 0° has outcome \uparrow .",
- p_1 = "Bob measurement at 180° has outcome \uparrow .",
- p_2 = "Alice measurement at 60° has outcome \uparrow .",
- p_3 = "Bob measurement at 120° has outcome \uparrow .",

Consider now the following propositional formulas:

$$\begin{aligned}\phi_0 &= (p_0 \wedge p_1) \vee (\neg p_0 \wedge \neg p_1) \\ \phi_1 &= (p_0 \wedge p_3) \vee (\neg p_0 \wedge \neg p_3) \\ \phi_2 &= (p_1 \wedge p_2) \vee (\neg p_1 \wedge \neg p_2) \\ \phi_3 &= (\neg p_2 \wedge p_3) \vee (p_2 \wedge \neg p_3)\end{aligned}$$

The multi-team approach is not the right approach!

- For ϕ_0, ϕ_1, ϕ_2 and ϕ_3 as above:

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 \leq 3 \quad (1)$$

is provable in PTL.

- But our experiment (and QM) supports

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 > 3.$$

Thus PTL is not the **right** “quantum” logic.

- Therefore, the multi-team (probabilistic) approach is not the right approach to describing the logic of quantum phenomena.

- Suppose $(Q_i)_{i \in \Omega}$ is a finite sequence of finite non-empty sets of proposition symbols.
- A *quantum team* on $(Q_i)_{i \in \Omega}$ is an indexed set X of valuations $v(i)$ such that $v(i)$ is a truth-value assignment to the proposition symbols in Q_i for $i \in \Omega$.

	p_0	p_1	p_2	p_3
0	1	1	—	—
1	1	1	—	—
2	1	1	—	—
3	1	1	—	—
4	0	0	—	—
5	0	0	—	—
6	0	0	—	—
7	0	0	—	—
8	1	—	—	1
9	1	—	—	1
10	1	—	—	1
11	0	—	—	1
12	1	—	—	0
13	0	—	—	0
14	0	—	—	0
15	0	—	—	0
16	—	1	1	—
17	—	1	1	—
18	—	1	1	—
19	—	0	1	—
20	—	1	0	—
21	—	0	0	—
22	—	0	0	—
23	—	0	0	—
24	—	—	1	1
25	—	—	1	0
26	—	—	1	0
27	—	—	1	0
28	—	—	0	1
29	—	—	0	1
30	—	—	0	1
31	—	—	0	0

- Given a finite set U of proposition symbols and a quantum team X on $(Q_i)_{i \in \Omega}$, we let $\Omega_U = \{i \in \Omega \mid U \subseteq Q_i\}$. We use this notation only if $\Omega_U \neq \emptyset$.
- We can define a new quantum team X_U by letting $v_U(i) = v(i) \upharpoonright U$ for $i \in \Omega_U$.
- For each valuation w on U we define

$$P_X^U(w) = \frac{|\{i \in \Omega_U : v_U(i) = w\}|}{|\Omega_U|}.$$

- This extends canonically to a definition of the probability of a propositional formula ϕ with its proposition symbols in U such that $\Omega_U \neq \emptyset$:

$$[\phi]_X^U = P_X^U(\{i \in \Omega_U \mid v_U(i)(\phi) = 1\}).$$

For the above quantum team, formulas ϕ_j , and an obvious choice of $(V_j)_{j < 4}$

$$\sum_{j < 4} [\phi_j]_{\mathcal{X}}^{V_j} = 3 + \frac{1}{4}.$$

Hence the “false” Bell’s Inequality is **not** true in this quantum team.

A logic for arguing about probabilities in quantum teams

- Can build a formal language QTL for reasoning about statements of the form “the probability of ϕ , with propositional variables in V , is at least 0.125”.
- We shorten this to $\phi^V \geq 0.125$.
- We allow more general statement $a_0\phi_0^{V_0} + \dots + a_{k-1}\phi_{k-1}^{V_{k-1}} \geq c$.

The Main Result

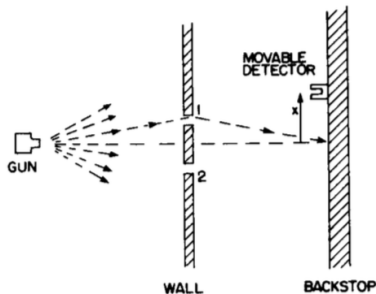
Theorem (Completeness)

Let ϕ be a formula of QTL. Then ϕ is provable in QTL if and only if it is true in every quantum team.

Hence the “false” Bell’s Inequalities are **not** provable in QTL.

- Can we find axioms such that ϕ is provable if and only if ϕ is true in every **real** quantum team, i.e. a quantum team that can be physically realised by an experiment (or by QM)?

Double-Slit Experiment



R. Feynman:

“Is it true, or is it not true, that the electron either goes through hole 1 or it goes through hole 2?”

Let

- $p_0 =$ "we see the flash at hole 1".
- $p_1 =$ "we see the flash at hole 2".
- $q_i =$ "the detector got the electron at distance i from the center", for $i \in \mathbb{Z}$.

Let ϕ be the sentence $(p_0 \vee p_1) \wedge \neg(p_0 \wedge p_1)$. It is easy to construct a quantum team X so that:

- 1 $X \models \phi$.
- 2 $X \models (\phi \wedge q_i) \neq q_i$.

- **Multi-teams** model probabilistic propositional logic.
- Validities in multi-teams can be completely axiomatized.
- From the point of view of quantum phenomena this approach is not satisfactory, as “false” Bell’s Inequalities are provable.
- **Quantum teams** generalise multi-teams by allowing the circumstance that some attributes cannot be simultaneously measured.
- Validities in quantum teams can be completely axiomatised.
- “False” Bell’s Inequalities are not provable.
- Is this the right “Quantum Logic”?

Thank you!