The Muskat problem in 3D

Helsinki
May 2011

References

(A.C., D. Córdoba and F. Gancedo)

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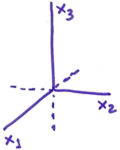
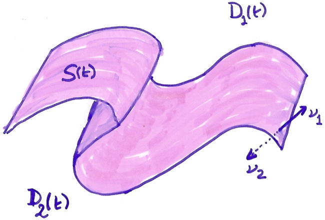
Setting





Two fluids
velocity u^j
pressure p^j
viscosity $\mu^1 \neq \mu^2$
density $\rho^1 \neq \rho^2$

- Porous media: porosity $\kappa > 0$ (homogeneous, isotropic).

Laws

- Incompressibility (mass conservation):

$$\mathrm{div}(u) = 0\,, \qquad u = u^1\,\chi_{D_1} + u^2\,\chi_{D_2}$$

- Darcy's Law (conservation of momentum):

$$\frac{\mu^j}{\kappa} u^j = -\nabla p^j - (0, 0, \rho^j g) = -\nabla(p^j + g\,\rho^j x_3) \qquad (g = \text{gravity}).$$

Immediate consequences

- $\mathrm{div}(u) = 0 \quad \Longrightarrow \quad (u^1 - u^2) \cdot \nu_j = 0.$
- $\mathrm{curl}(u) = \Omega \, dS$, Ω is a tangent vector field at S.
- $p^1\big|_S = p^2\big|_S.$
- $\exists$ velocity potentials: $u^j = \nabla \phi^j$

$$\boxed{\frac{\mu^j}{\kappa}\,\phi^j = -p^j - \rho^j\, g\, x_3}$$

Interface motion

$$u(x) = \int_S \frac{x - z}{|x - z|^3} \wedge \Omega(z)\, dS(z)\,, \qquad x \notin S \quad \text{(Biôt-Savart).}$$

Taking limits in each domain:

$$u^j(x) = p.v. \int_S \frac{x - z}{|x - z|^3} \wedge \Omega(z)\, dS(z) + \text{tangential terms}, \quad x \in S.$$

Choosing a parametrization (isothermal coordinates)

$$X(\alpha) = X(\alpha_1, \alpha_2) \qquad \begin{cases} X_{\alpha_1} \cdot X_{\alpha_2} = 0 \\ \|X_{\alpha_1}\| = \|X_{\alpha_2}\| \end{cases}$$

Two scenarios:

- Periodic in the horizontal variables;
- asymptotically flat.

Equation of motion

$$\frac{dX(\alpha)}{dt} = p.v. \int \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \wedge \omega(\alpha)\, d\alpha$$

$$+ C_1(X(\alpha,t),t)X_{\alpha_1} + C_2(X(\alpha,t),t)X_{\alpha_2}$$

$$= BR(X,\omega) + C_1 X_{\alpha_1} + C_2 X(\alpha_2).$$

where

$$\boxed{\mathrm{curl}(u) = \omega(\alpha)\ \delta_{X - X(\alpha,t)}}$$

Darcy's law:

$$\frac{\mu^j}{\kappa}\phi^j = -p^j - \rho^j g\, x_3, \qquad u^j = \nabla\phi^j$$

We define

$$\Pi(\alpha) = \phi^1(X(\alpha, t), t) - \phi^2(X(\alpha, t), t).$$

Then we have

- $\omega(\alpha) = \dfrac{\partial \Pi}{\partial \alpha_2} X_{\alpha_1} - \dfrac{\partial \Pi}{\partial \alpha_1} X_{\alpha_2}$
- $\Pi(\alpha) + \dfrac{\mu^2 - \mu^1}{\mu^2 + \mu^1}\mathcal{D}(\Pi)(\alpha) = g\,\dfrac{\rho^2 - \rho^1}{\mu^2 + \mu^1}\, X_3(\alpha)$

where

$$\mathcal{D}(\Pi)(\alpha) = \int_S \frac{X(\alpha) - X(\beta)}{|X(\alpha) - X(\beta)|^3} \; N(\beta)\; \Pi(\beta)\; d\beta$$

and

$$N(\beta) = X_{\alpha_1} \wedge X_{\alpha_2}.$$

It is then crucial to estimate

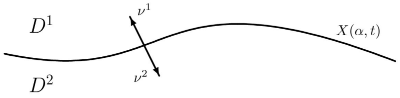
$$\|(I + \lambda\mathcal{D})^{-1}\|\,, \qquad |\lambda| = \left|\frac{\mu^2 - \mu^1}{\mu^2 + \mu^1}\right| < 1.$$

Equations

The problem

$$\begin{cases} \dfrac{dX}{dt} & = BR(X,\omega) + C_1\, X_{\alpha_1} + C_2\, X_{\alpha_2} \\ \omega & = \frac{\partial \Pi}{\partial \alpha_2} X_{\alpha_1} - \frac{\partial \Pi}{\partial \alpha_1} X_{\alpha_2} \\ \Pi(\alpha) & = \left(I + \frac{\mu^2-\mu^1}{\mu^2+\mu^1} \mathcal{D} \right)^{-1} \left(g\, \frac{\rho^2-\rho^2}{\mu^2+\mu^1}\; X_3(\alpha,t) \right) \end{cases}$$

is not well-possed.



Rayleigh-Taylor:

$$\begin{aligned} \sigma(\alpha,t) &= (\nabla p^2 - \nabla p^1) \cdot (\nu^2 - \nu^1) \\ &= (\mu^2 - \mu^1) BR(X,\omega) \cdot (X_{\alpha_1} \wedge X_{\alpha_2}) + (\rho^2 - \rho^1)(X_{\alpha_1} \wedge X_{\alpha_2})_3 \\ &> 0 \end{aligned}$$

Geometric constraint: not self-intersecting surface

$$F(x) = \sup_{\alpha \neq \beta} \frac{\|\alpha - \beta\|}{\|X(\alpha) - X(\beta)\|} < \infty.$$

Theorem

If

- $X_0(\alpha) \in H^4(\mathbb{R}^2)$;
- $\sigma(\alpha, 0) > 0$;
- $F(X_0) < \infty$;

then $\exists T > 0$ *and a unique solution* $X \in C^1([0, T]; H^4)$.

Energy estimates

$$\frac{d}{dt}\|X\|^2_{H^k} \leq P\big(\|X\|_{H^k} + \|F(X)\|_{L^\infty}\big)$$
$$- \sum_{j=0}^{k} \frac{2^{3/2}}{\mu^1+\mu^2} \int \frac{\sigma(\alpha)}{|\nabla X(\alpha)|^3} \frac{\partial^{k-j}}{\partial\alpha_1}\frac{\partial^j}{\partial\alpha_2}X(\alpha)$$
$$\cdot \Lambda\Big(\frac{\partial^{k-j}}{\partial\alpha_1}\frac{\partial^j}{\partial\alpha_2}X(\alpha)\Big)\, d\alpha$$

where

$$\boxed{\Lambda = \sqrt{-\Delta}}$$

Then we use the pointwise inequality

$$\theta(x)\cdot\Lambda\theta(x)\geq\frac{1}{2}\Lambda\theta^2(x)$$

to get a polynomial estimate in the norm H^k.

However, we need to control the evolution of all the involved quantities, namely,

- $\sigma > 0$.
- $F < \infty$.
- C_j (isothermal).
- $\|I+\lambda\mathcal{D}\|^{-1}, \quad |\lambda| < 1$.