Singular Integrals: Blow-up for Transport Equations

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$$\boxed{\int_{\mathbb{R}^n} \frac{(Rf(0) - Rf(x)) \cdot \nabla f(x)}{|x|^{n+\alpha}} dx \geq C_\alpha \int_{\mathbb{R}^n} \frac{(f(0) - f(x))^2}{|x|^{n+\alpha+1}} dx}$$

where

$$\begin{cases} c_\alpha > 0\,, \quad 0 \leq \alpha(n) < \alpha < 1\,, \\ f \text{ has an extremum at } x = 0. \end{cases}$$

$$Rf(x) = (R_1 f(x), \ldots, R_n f(x))$$

$$R_j f(x) = c_n \text{ P.V.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x-y) dy$$

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{f}(\xi)$$

Blow-up

$$\left.\begin{array}{l}\dfrac{\partial\theta}{\partial t} = R(\theta)\cdot\nabla_x\theta \\ \theta(x,0) = \theta_0(x)\end{array}\right\}$$

active scalar θ
non-linear transport equation
non-local velocity: $u = -R(\theta)$

For every smooth initial datum $\theta_0 \geq 0$ ($\not\equiv 0$) with compact support, we have that

$$\|\nabla_x\theta(\cdot,t)\|_{L^\infty}$$

blows up at finite time.

Particle Dynamics

$$\frac{dx}{dt} = u(x(t), t) = -R\theta(x(t), t)$$

$$\begin{cases} \frac{d}{dt}\big(\theta(x(t), t)\big) = \theta_t + \dot{x}(t) \cdot \nabla_x \theta = 0 \\ \sup_x \theta(x, t) = \sup_x \theta_0(x) \\ \inf_x \theta(x, t) = \inf_x \theta_0(x) \\ \theta(x, t) \geq 0 \quad \text{if } \theta_0(x) \geq 0. \end{cases}$$

Well Posed

$$\left.\begin{aligned} \frac{\partial\theta}{\partial t} &= R(\theta)\cdot\nabla_x\theta \\ \theta(x,0) &= \theta_0(x) \end{aligned}\right\}$$

Proposition

If $\theta_0 \in H^m(\mathbb{R}^n)$, $m > \frac{n}{2}+1$, *then* $\exists\varepsilon > 0$, $T > 0$ *and a unique solution* $\theta \in C^{1,\varepsilon}(\mathbb{R}^n \times (0,T))$.

Well Posed

Energy Estimate

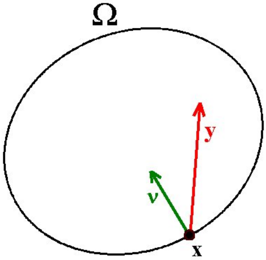
$$\begin{cases} \Lambda^m = (-\Delta)^{m/2} \\ \frac{d}{dt}\left[\|\theta\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2\right] << \left[\|\theta\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2\right]^{3/2} \end{cases}$$

Proposition (Kato-Ponce Inequality)

$$\|\Lambda^\beta(f\cdot g) - f\Lambda^\beta g\|_2 << \|\nabla f\|_\infty \cdot \|\Lambda^{\beta-1}g\|_2 + \|g\|_\infty \cdot \|\Lambda^\beta f\|_2\,,$$

$$\beta > n/2$$

Compact support



Ω convex

$\delta\Omega$ smooth

$0 \leq \theta \in C^\alpha(\mathbb{R}^n)$

$\mathrm{supp}(\theta) \subset \Omega$

$$-R(\theta)(x) \cdot \nu(x) = c_n \text{ p.v.} \int_\Omega \frac{(y-x)\cdot \nu(x)}{|x-y|^{n+1}} \theta(y)dy \geq 0$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= -R\theta(x(t),t) \\ u &= -R\theta \end{aligned}}$$

$$\theta(x_M(t), t) = \max \theta_0$$
$$y = x - x_M(t)$$
$$f(y, t) = \theta(x_M(t), t) - \theta(x, t)$$
$$\operatorname{supp} \nabla_y f(\cdot, t) \subset B_r$$

$$\frac{d}{dt} \int_{B_r} \frac{f(y,t)}{|y|^{n+\alpha}} = + \int_{B_r} \frac{(Rf(0,t) - Rf(y,t)) \cdot \nabla_y f}{|y|^{n+\alpha}} dy$$
$$\geq C_\alpha \int_{B_r} \frac{f^2}{|y|^{n+\alpha+1}} dy \geq \widetilde{C}_\alpha \left[\int_{B_r} \frac{f(y,t)}{|y|^{n+\alpha}} \right]^2$$

$$\int_{B_r} \frac{f(y,t)}{|y|^{n+\alpha}} \leq \widetilde{\widetilde{C}}_\alpha \|\nabla \theta\|_{L^\infty}$$

$n = 1$, Hf, Hilbert Transform

$$\int_{-\infty}^{\infty} \frac{[Hf(0) - Hf(x)] \cdot f'(x)}{|x|^{1+\alpha}} dx >> \int_{-\infty}^{\infty} \frac{f^2(x)}{|x|^{2+\alpha}} dx$$

$$f = f_{\mathrm{even}} + f_{\mathrm{odd}}$$

$$\int_{0}^{\infty} \frac{[Hf(0) - Hf(x)]}{|x|^{\alpha/2+1/2}} \frac{f'(x)}{|x|^{\alpha/2-1/2}} \frac{dx}{x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{A(\lambda)} B(\lambda) d\lambda$$

$n = 1$, Hf, Hilbert Transform

$$\begin{cases} A(\lambda) = \displaystyle\int_0^\infty x^{i\lambda} \frac{f'(x)}{x^{\alpha/2-1/2}} \frac{dx}{x} = M\left(\frac{f'}{x^{\alpha/2-1/2}}\right)(\lambda) \\ B(\lambda) = \displaystyle\int_0^\infty x^{i\lambda} \frac{Hf(0) - Hf(x)}{x^{\alpha/2+1/2}} \frac{dx}{x} = M\left(\frac{Hf(0) - Hf(x)}{x^{\alpha/2+1/2}}\right)(\lambda) \end{cases}$$

Even case

$$A(\lambda) = a(\lambda)\ M\left(\frac{f}{x^{1/2+\alpha/2}}\right)(\lambda)$$

$$B(\lambda) = b(\lambda)\ M\left(\frac{f}{x^{1/2+\alpha/2}}\right)(\lambda)$$

$$a(\lambda) = -\left(i\lambda - \frac{1}{2} - \frac{\alpha}{2}\right)$$

$$b(\lambda) = -\frac{-1 + \cos\left[(-i\lambda + \frac{1}{2} + \frac{\alpha}{2})\pi\right]}{\sin\left[(-i\lambda + \frac{1}{2} + \frac{\alpha}{2})\pi\right]}$$

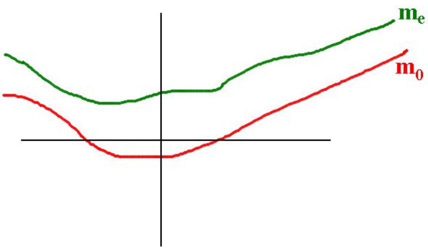
Even case

$$\frac{1}{2\pi}\int \overline{A(\lambda)}\, B(\lambda) d\lambda = \frac{1}{2\pi}\int \overline{a(\lambda)} b(\lambda) \left| M\left(\frac{f}{x^{1/2+\alpha/2}}\right)\right|^2 d\lambda$$

$$m_e(\lambda) = \mathrm{Re}\big(\overline{a(\lambda)} b(\lambda)\big) > 0\,, \qquad 0 < \alpha < 1\,.$$

Odd case

$$m_0(\lambda) \geq -\varepsilon(\alpha)$$



$$\inf_\lambda m_e(\lambda) + \inf_\lambda m_0(\lambda) > 0$$

f of constant sign $\Longrightarrow \left|f_{\text{even}}(x)\right| \geq \left|f_{\text{odd}}(x)\right|$