Kinetic Limit for a Wave Equation in a Random Medium

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Scalar wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c(x)^2 \Delta u(x,t) \tag{1}$$

$u : \mathbb{R}_x^d \times \mathbb{R}_t \to \mathbb{C}$ in a medium with randomly perturbed wave speed

$$c(x) = 1 + \sqrt{\varepsilon}\xi(x), \varepsilon > 0$$

inhomogeneities distributed with a Poisson Point Process $(y_n)_{n\in\mathbb{N}}$

$$\xi(x) = \sum_{n\in\mathbb{N}} \phi(x - y_n) - \int_{\mathbb{R}^d} \phi(y)\mathrm{d}y.$$

Question. In case of weak ($\varepsilon \ll 1$) disorder, how does the solution of (1) behave on macroscopic ($\mathcal{O}(1/\varepsilon)$) space- and time scales (Kinetic limit)?

Previous results:

L. Erdös, H.-T. Yau ([1]): Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi = (-\frac{1}{2}\Delta + \sqrt{\varepsilon}V)\psi$$

V a Gaussian random field.

J. Lukkarinen, H. Spohn ([2]): Crystal lattice vibrations

$$m_y \frac{\mathrm{d}^2}{\mathrm{d}t^2} u_y = (\Delta_{\mathrm{lattice}} u)_y \,,$$

atom masses m_y have small fluctuations (different isotopes).

In both cases: Linear Boltzmann equation in the kinetic limit.

G. Bal, T. Komorowski, L. Ryzhik ([3]):
Wave equation, but with three scales

$$\text{wavelength} \ll \text{correlation length} \ll \text{observation scale.}$$

Result: Boltzmann equation, but with diffusion for wave vector k instead of jump process.

But so far no rigoros convergence result for full interaction between waves and impurities:

$$\mathcal{O}(1) = \text{wavelength} \approx \text{correlation length} \ll \text{observation scale} = \mathcal{O}(1/\varepsilon).$$

Energy conservation

$$E(u(t), \dot{u}(t)) = \frac{1}{2} \int_{\mathbb{R}^d} \left(|\nabla u(x,t)|^2 + \frac{|\dot{u}(x,t)|^2}{c(x)^2} \right) \mathrm{d}x = \text{const.}$$

Transform wave equation to a Hilbert space $\mathcal{H} = L^2\left(\mathbb{R}^d; \mathbb{C}^2\right)$ on which energy conservation corresponds to unitarity:

$$\Omega = \sqrt{-\Delta}$$

$$\mathcal{H} \ni \psi(t) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}(t) = \frac{1}{2} \begin{pmatrix} \Omega u(t) + i\dot{u}(t)/c \\ \Omega u(t) - i\dot{u}(t)/c \end{pmatrix}.$$

$$\Rightarrow \|\psi(t)\|_{\mathcal{H}}^2 = E(u(t), \dot{u}(t)).$$

Schrödinger type equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi^{\varepsilon}(t) = H_{\varepsilon}\psi^{\varepsilon}(t)$$

with generator

$$\begin{aligned} H_{\epsilon} &= H_0 + \sqrt{\epsilon}V \\ &= \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix} + \frac{\sqrt{\varepsilon}}{2}\begin{pmatrix} \Omega\xi + \xi\Omega & -\Omega\xi + \xi\Omega \\ \Omega\xi - \xi\Omega & -\Omega\xi - \xi\Omega \end{pmatrix}. \end{aligned}$$

with $\Omega = \sqrt{-\Delta}$.

Transformation to a (pseudo)density on classical phase space.
ε-scaled Wigner function

$$W^{\varepsilon}_{\sigma'\sigma}[\psi^{\varepsilon}](x,k) = \varepsilon^{-d}\int_{\mathbb{R}^d} \mathrm{d}y\, e^{i2\pi k\cdot y}\overline{\psi^{\varepsilon}_{\sigma'}\left(\frac{x}{\varepsilon}+\frac{y}{2}\right)}\psi^{\varepsilon}_{\sigma}\left(\frac{x}{\varepsilon}-\frac{y}{2}\right)$$
$$= \int_{\mathbb{R}^d} \mathrm{d}p\, e^{i2\pi x\cdot p}\overline{\widehat{\psi^{\varepsilon}_{\sigma'}}\left(k-\frac{\varepsilon p}{2}\right)}\widehat{\psi^{\varepsilon}_{\sigma}}\left(k+\frac{\varepsilon p}{2}\right),$$

$\sigma, \sigma' \in \{\pm 1\}$, $(x,k) \in \mathbb{R}^d \times \mathbb{R}^d$.
Basic properties of W^{ε}

$$\int_{\mathbb{R}^d} W^{\varepsilon}_{\sigma'\sigma}[\psi^{\varepsilon}](x,k)\mathrm{d}k = \varepsilon^{-d}\overline{\psi^{\varepsilon}_{\sigma'}\left(\frac{x}{\varepsilon}\right)}\psi^{\varepsilon}_{\sigma}\left(\frac{x}{\varepsilon}\right)$$
$$\int_{\mathbb{R}^d} W^{\varepsilon}_{\sigma'\sigma}[\psi^{\varepsilon}](x,k)\mathrm{d}x = \overline{\widehat{\psi^{\varepsilon}_{\sigma'}}(k)}\widehat{\psi^{\varepsilon}_{\sigma}}(k)\,.$$

Assumptions on initial conditions ψ^ε:

Bounded energy

$$\sup_{\epsilon} \|\psi^\varepsilon\|^2 = E_0 < \infty$$

Compactness argument $\Rightarrow W^\varepsilon_{\sigma\sigma}[\psi^\varepsilon] \xrightarrow{\mathcal{S}'} \mu_0^\sigma$ on a subsequence.

Avoid the singularity of the dispersion relation $\omega(k) = 2\pi|k|$
$\mu_0^+ \left(\mathbb{R}^d_x \times \{0\}\right) = \mu_0^- \left(\mathbb{R}^d_x \times \{0\}\right) = 0.$

Tight in momentum space
$\lim_{R\to\infty} \overline{\lim}_{\varepsilon\to 0} \int_{|k|>R} \mathrm{d}k \left(|\widehat{\psi^\varepsilon_+}(k)|^2 + |\widehat{\psi^\varepsilon_-}(k)|^2 \right) = 0.$

Tight on macroscopic position space
$\lim_{R\to\infty} \overline{\lim}_{\varepsilon\to 0} \int_{|x|>R/\varepsilon} \mathrm{d}x \left(|\psi^\varepsilon_+(x)|^2 + |\psi^\varepsilon_-(x)|^2 \right) = 0.$

Under those assumptions, in a homogeneous medium ($\xi \equiv 0$), for all macroscopic times t

$$W^{\varepsilon}_{++}[\psi^{\varepsilon}\left(\frac{t}{\varepsilon}\right)] \xrightharpoonup{\mathcal{S}'} \mu^{+}_{t}$$

and μ^{+}_{t} is a weak solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu^{+}_{t}(x,k) = -\widehat{k} \cdot \nabla_{x}\mu^{+}_{t}(x,k).$$

Transport equation for classical particles with Hamiltonian $H(x,k) = |k|$.

Theorem

Let all the above assumptions hold, fix a space dimension $d \geq 2$ and a macroscopic time $t > 0$. Then in the limit $\varepsilon \to 0$, the expectation of the Wigner function $\mathbb{E}\left(W^{\varepsilon}_{++}[\psi^{\varepsilon}(t/\varepsilon)]\right)$ converges to a measure μ_t as a distribution on phase space, and μ_t is the weak solution of the Boltzmann equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(x,k) = -\widehat{k} \cdot \nabla_x \mu_t(x,k) + \int_{\mathbb{R}^d} \nu_k(\mathrm{d}k')(\mu_t(x,k') - \mu_t(x,k)). \tag{2}$$

The measure ν_k in the collision integral is given by

$$\nu_k(\mathrm{d}k') = |\hat{\phi}(k-k')|^2 \delta(|k| - |k'|) |2\pi k|^2 \mathrm{d}k'. \tag{3}$$

Lemma

To verify the convergence of $\mathbb{E}\left(W^{\varepsilon}_{++}[\psi^{\varepsilon}(t/\varepsilon)]\right)$*, it is enough to show*

$$\lim_{\varepsilon\to 0} F^{\varepsilon}_t(q,z) = \int_{\mathbb{R}^{2d}} \mu_t(\mathrm{d}x,\mathrm{d}k) e^{-2\pi i(q\cdot x - z\cdot k)} \quad \forall q,z \in \mathbb{R}^d \qquad (4)$$

with F^{ε} *the Fourier transform of the Wigner function*

$$F^{\varepsilon}_t(q,z) = \mathbb{E}\left[\int_{\mathbb{R}^d} \mathrm{d}k\, \overline{\widehat{\psi_+}(k-\varepsilon q/2; t/\varepsilon)}\widehat{\psi_+}(k+\varepsilon q/2; t/\varepsilon) e^{2\pi i z\cdot k}\right].$$

μ_t is the distribution of a jump process with rate $\sigma(k) = \nu_k\left(\mathbb{R}^d\right)$, starting at μ_0.

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}} \mu_t(\mathrm{d}x, \mathrm{d}k)\, e^{-i2\pi(q\cdot x - z\cdot k)} \\
&= \int_{\mathbb{R}^{2d}} \mu_0(\mathrm{d}x, \mathrm{d}k_0) \sum_{m=0}^{\infty} \int_{\mathbb{R}_+^{m+1}} \mathrm{d}r\, \delta\left(t - \sum_{j=0}^{m} r_j\right) \\
&\quad \times \int_{\mathbb{R}^d} \nu_{k_0}(\mathrm{d}k_1) \ldots \int_{\mathbb{R}^d} \nu_{k_{m-1}}(\mathrm{d}k_m) \prod_{j=0}^{m} e^{-r_j(\sigma(k_j)) + 2\pi i q\cdot \widehat{k_j})}\, e^{-i2\pi(q\cdot x - z\cdot k_m)}.
\end{aligned}
$$

Duhamel expansion

$$
\begin{aligned}
&\mathrm{e}^{-iH_\varepsilon t}\psi^\varepsilon \\
&= \sum_{N=0}^{N_0(\varepsilon)-1} \left(-i\sqrt{\varepsilon}\right)^N \int_{\mathbb{R}_+^{N+1}} \mathrm{d}s\delta\left(t - \sum_{l=1}^{N+1} s_l\right) \mathrm{e}^{-iH_0 s_{N+1}} V \ldots V \mathrm{e}^{-iH_0 s_1}\psi^\varepsilon \\
&\quad + R(\varepsilon, N_0(\varepsilon); t) \\
&= \sum_{N=0}^{N_0(\varepsilon)-1} F_N(t;\varepsilon)\psi^\varepsilon + R(\varepsilon, N_0(\varepsilon); t),
\end{aligned}
$$

with $N_0(\varepsilon) = \left\lfloor a_0(\phi)\frac{|\log\varepsilon|}{\log|\log\varepsilon|}\right\rfloor$.

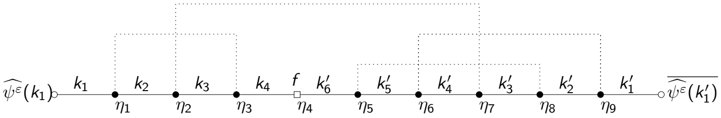
Contribution of each term to F^{ε}

$$\mathbb{E}\left[\langle F_{N_2}(t;\varepsilon)\psi^{\varepsilon}, B_{p,f} F_{N_1}(t;\varepsilon)\psi^{\varepsilon}\rangle\right] = \sum_{S\in\pi^*(I)} \mathcal{K}\left(t, S; (0, N_2), (0, N_1), \varepsilon, 0, p, f\right)$$

- $\langle\varphi, B_{p,f}\psi\rangle = \int_{\mathbb{R}^d} \mathrm{d}k\widehat{\varphi}(k - p/2) \cdot f(k)\widehat{\psi}(k + p/2)$
- in our case $p = \varepsilon q$, $f(k) = e^{2\pi i z\cdot k} P_{++}$
- $I = \{1, ..., N_1\} \cup \{N_1 + 2, ...N_1 + N_2 + 1\}$
- $\pi^*(I)$ partitions of I without isolated elements

$$
\begin{aligned}
&\mathcal{K}\left(t, S; (0, N_2), (0, N_1), \varepsilon, 0, p, f\right) \\
&= i^{-N_1+N_2} \varepsilon^{\overline{N}/2} e^{2\gamma t} \int_{R^d} \mathrm{d}\eta_0 \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\beta}{2\pi} e^{-it(\alpha+\beta)} \\
&\times \int_{\mathbb{R}^{(\overline{N}+1)d}} \mathrm{d}\eta \delta\left(\eta_{N_1+1} + p\right) \left(\prod_{A \in S} \delta\left(\sum_{l \in A} \eta_l \right) \right) \left(\prod_{l \in I} \widehat{\phi}(\eta_l) \right) \widehat{\psi^{\varepsilon}}(\eta_0 - p) \cdot \\
&\cdot \left[\left(\prod_{l=N_2}^{1} (\alpha + i\gamma + H(k_l'))^{-1} v(k_l', k_{l+1}') \right) i^{N_2+1} (\alpha + i\gamma + H(k_{N_2+1}'))^{-1} \right. \\
&\times f(k_{N_1+1} - \frac{p}{2}) i^{N_1+1} (\beta + i\gamma + -H(k_{N_1+1}))^{-1} \\
&\times \left. \left(\prod_{l=1}^{N_1} v(k_{l+1}, k_l)(\beta + i\gamma - H(k_l))^{-1} \right) \widehat{\psi^{\varepsilon}}(\eta_0) \right]
\end{aligned}
$$

Strategy of convergence proof:



- only partitions that are pairings contribute
- classify them into "nested", "crossing" and "simple" pairings
- all but the simple ones are negligible
- oscillatory integrals guarantee momentum conservation
- use similar methods to prove $\mathbb{E}\left[\|R(\varepsilon, N_0(\varepsilon); t/\varepsilon)\|^2\right] \longrightarrow 0$

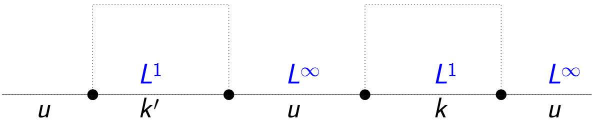
$A \in S$ a cluster $\Rightarrow |A| - 1$ "free", 1 "bound" variable.

- $|A| - 1$ resolvents with L^1 estimate $\sim \mathcal{O}\left(|\log \varepsilon|\right)$
- 1 resolvent with L^∞ estimate $\sim \mathcal{O}\left(\varepsilon^{-1}\right)$

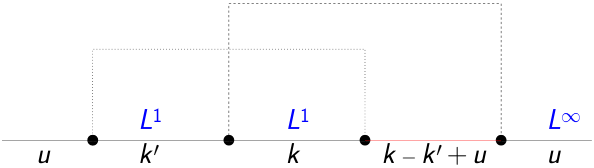
$$\mathcal{K} \sim C\left(\overline{N}\right) |\log \varepsilon|^{2\overline{N}+2} \varepsilon^{\overline{N}/2 - |S|}.$$

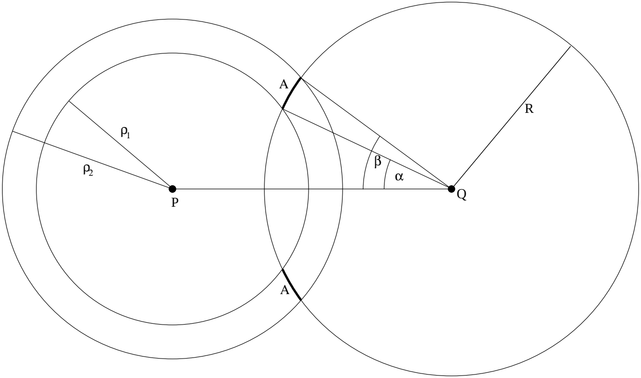
"Higher order partitions": $|S| < \overline{N}/2 \Rightarrow \mathcal{K} \rightarrow 0 \quad (\varepsilon \rightarrow 0)$.

Simple pairing:



Crossing pairing:

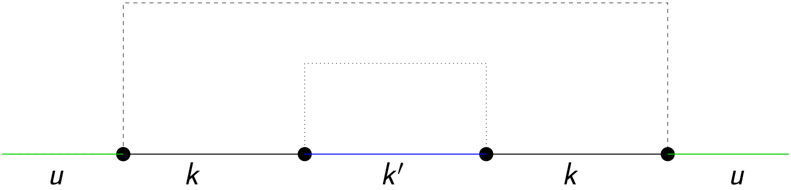




$$\int_{\mathbb{R}^{2d}} \frac{\mathrm{d}k_1 \mathrm{d}k_2}{\langle k_1\rangle^{2d} \langle k_2\rangle^{2d} |\alpha_1 - |k_1| + i\gamma| \, |\alpha_2 - |k_2| + i\gamma| \, |\alpha_3 - |k_1 - k_2 + u| + i\gamma|}$$

$$\leq C \frac{\langle \ln \gamma\rangle^2}{\langle \alpha_1\rangle \langle \alpha_2\rangle} \gamma^{-2/3} \ll \gamma^{-1}$$

Nested pairing:



Contribution from integrating out one "gate":

$$g(k,w) = \int_{\mathbb{R}^d} \mathrm{d}k' \, v(k,k') \frac{i \left| \hat{\phi}(k-k') \right|^2}{w - H(k')} v(k',k)$$

Nonconstant sign: Gain a factor $\gamma \ln \gamma$ by estimating

$$\left| \frac{1}{\beta + i\gamma - 2\pi|k|} \cdot \frac{1}{\beta + i\gamma + 2\pi|k|} \right| \leq \frac{1}{4\pi|k|} \left(\frac{1}{|\beta + i\gamma - 2\pi|k||} + \frac{1}{|\beta + i\gamma + 2\pi|k||} \right)$$

Constant sign: Represent resolvents by oscillatory integral

$$\int_{\mathbb{R}^d} \rho(k) e^{-i|k|s} \mathrm{d}k \sim |s|^{-\frac{Nd}{N+d}}$$

if $\partial^\alpha \rho$ is integrable for $|\alpha| \leq N$, gain a factor $\gamma^{1/5}$

We are left with the simple graphs $S_m(n, n')$

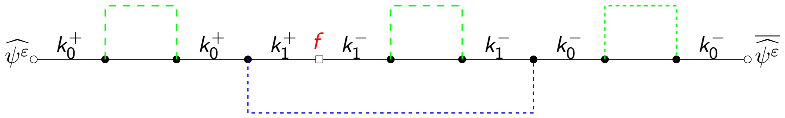


Figure: $m = 1,\ n = (1, 0),\ n' = (1, 1)$.

- $k_j^{\pm} = k_j \pm p/2 = k_j \pm \varepsilon q/2$
- blue lines: guarantee "particle" behavior, determine number of jumps.
- green gates: contribute g factors

$$\Theta_{\pm}(k) = \lim_{\gamma \searrow 0} g_{\pm\pm}(k, 2\pi\sigma|k| + i\gamma)$$

with

$$\Theta_{+}(k) + \Theta_{-}(k) = \sigma(k)$$

- only constant σ and σ' matter.
- replace $g_{\sigma\sigma}(k_j^{+}, \beta + i\gamma)$ by $\Theta_\sigma\left(|k_0|\widehat{k_j}\right)$ with $\mathcal{O}\left(\gamma^{1/5}\right)$ error.
- replace $\pi(\sigma|k_j^{\pm}| + \sigma|k_{j+1}^{\pm}|)$ by $|2\pi k_0|$ with $\mathcal{O}(\gamma) + \mathcal{O}(|p|)$ error.

$$
\begin{aligned}
&\mathcal{K}^{(main)}\left(t, S_m(n,n'), 0; \varepsilon, 0, p, f\right) \\
&= \varepsilon^m \sum_{\sigma',\sigma\in\{\pm 1\}} \int_{\left(\mathbb{R}^d\right)^{I'_m}} \mathrm{d}k \overline{\widehat{\psi^{\varepsilon}_{\sigma'}}(k_0^-)} \widehat{\psi^{\varepsilon}_{\sigma}}(k_0^+) f_{\sigma'\sigma}(k_m) |2\pi k_0|^{2m} \\
&\times \prod_{j=0}^{m} \left((-\varepsilon\Theta_\sigma(|k_0|\hat{k}_j))^{n_j} (-\varepsilon\Theta_{-\sigma'}(|k_0|\hat{k}_j))^{n'_j} \right) \prod_{j=0}^{m-1} \left(|\hat{\phi}(k_j - k_{j+1})|^2 \right) \\
&\times K_{m+n'+1}(t, -\sigma'|2\pi k^-|) K_{m+n+1}(t, \sigma|2\pi k^+|)
\end{aligned}
$$

- $f(k) = e^{2\pi i y\cdot k} P_{++}$
- $p = \varepsilon q$
- $t = \overline{t}/\varepsilon$

Phase factor

$$\varepsilon^m \delta\left(\frac{\bar{t}}{\varepsilon} - \sum_{j=0}^{m} s_j\right) \delta\left(\frac{\bar{t}}{\varepsilon} - \sum_{j=0}^{m} s_j'\right) \prod_{j=0}^{m} e^{-2\pi i(|k_j^+|s_j - |k_j^-|s_j')} \mathrm{d}s \mathrm{d}s'$$

Change to macroscopic times

$$r_j = \varepsilon \frac{s_j + s_j'}{2}, \qquad b_j = s_j - s_j'$$

Transport term and $|k|$-conservation

$$\delta\left(\bar{t} - \sum_{j=0}^{m} r_j\right) \prod_{j=0}^{m} e^{-2\pi i r_j(|k_j^+| - |k_j^-|)/\varepsilon} \prod_{j=1}^{m} e^{-\pi i b_j\left(|k_0^+| + |k_0^-| - |k_j^+| - |k_j^-|\right)} \mathrm{d}r \mathrm{d}b$$

$$\lim_{\varepsilon\to 0} \sum_{\substack{m+2n<N_0(\varepsilon)\\ m+2n'<N_0(\varepsilon)}} \mathcal{K}^{(main)}\left(\overline{t}/\varepsilon, S_m(n,n'), 0; \varepsilon, 0, \varepsilon q, e_y P_{++}\right)$$

$$= \int_{\mathbb{R}^{2d}} \mu_{\overline{t}}\,(\mathrm{d}x, \mathrm{d}k)\, e^{-i2\pi(q\cdot x - y\cdot k)}.$$

Thank you for your attention!

- Erdös, L., Yau, H.-T.: Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation. *Comm. Pure Appl. Math.* **53**, 667-735 (2000)
- Lukkarinen, J., Spohn, H.: Kinetic Limit for Wave Propagation in a Random Medium. *Arch. Rational Mech. Anal.* **183**, 93-162 (2007)
- Bal, G., Komorowski, T., Ryzhik, L.: Self-averaging of Wigner transforms in random media. *Comm. Math. Phys.* **242**, 81-135 (2003)
- Ryzhik, L., Papanicolaou, G., Keller, J. B.: Transport equations for elastic and other waves in random media. *Wave motion* **24**, 327-370 (1996)