

Analysis of eigenvalue ensembles using Ward identities

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Particle systems

A system $\{\zeta_j\}_1^n \in \mathbb{C}$ ("point charges") in external field nQ .

- **Energy:**

$$H_n = \sum_{j \neq k}^n \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^n Q(\zeta_j).$$

- **Boltzmann–Gibbs law:**

$$d\mathbf{P}_n(\zeta) = \frac{1}{Z_n^\beta} e^{-\beta H_n(\zeta)} d^{2n}\zeta, \quad \zeta = (\zeta_j)_1^n. \quad (1)$$

- **Assumptions.** $Q : \mathbb{C} \rightarrow \mathbb{R} \cup \{+\infty\}$ is l.s.c., C^ω -smooth, and

$$Q(\zeta) \gg \log |\zeta|, \quad (\zeta \rightarrow \infty).$$

A minimizer $\{\zeta_j\}_1^n$ of H_n is a *Fekete-configuration*.

Frostman's equilibrium measure

- Q -energy of a Borel p.m. μ on \mathbb{C}

$$I(\mu) := \iint \log \frac{1}{|\zeta - \eta|} d\mu(\zeta) d\mu(\eta) + \int Q d\mu.$$

The *equilibrium measure* σ minimizes $I(\mu)$ where $\mu(\mathbb{C}) = 1$.

- *Droplet*

$$S = S[Q] := \text{supp } \sigma. \quad (2)$$

Theorem

(Frostman)

$$d\sigma(z) = \chi_S(z) \Delta Q(z) dA(z).$$

(In particular $\Delta Q \geq 0$ on S .)

Conformal metric: $ds^2 = \Delta Q(\zeta) |d\zeta|^2$.

Abrikosov: Fekete configuration \leftrightarrow honeycomb lattice.

Nature of droplets

Lemma

Fix $p \in \partial S$. There is a nbh N of p and a "local Schwarz function" $s(\zeta)$ on N obeying

- s is analytic in $N \setminus S$,
- s is continuous on N and

$$\bar{\zeta} = s(\zeta), \quad \zeta \in (\partial S) \cap N.$$

Theorem

(Sakai, 1991) ∂S is a union of finitely many analytic curves. Possible singularities: cusps pointing out of S and double points.

Complement S^c is an *Unbounded Quadrature Domain* (in wide sense of Shapiro).

Droplets 1

Technical assumptions:

- Q is real-analytic in a nbh of S .
- $\Delta Q > 0$ in a nbh of ∂S .

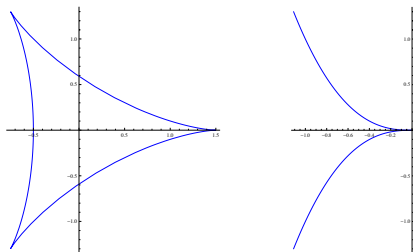


Figure: The Deltoid is not admissible; it has three maximal $3/2$ cusps. $5/2$ cusp is OK.

Droplets 2

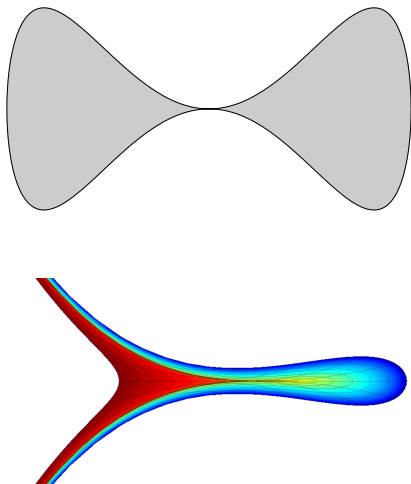


Figure: Double point and $5/2$ cusp under Hele-Shaw flow.

Joint intensities

Let $\{\zeta_j\}_1^n$ random sample. k -point function

$$\mathbf{R}_{n,1}(\eta) = \lim_{\epsilon \rightarrow 0} \frac{\mathbf{P}_n(D(\eta; \epsilon) \cap \{\zeta_j\}_1^n \neq \emptyset)}{\epsilon^2},$$

$$\mathbf{R}_{n,2}(\eta_1, \eta_2) = \lim_{\epsilon \rightarrow 0} \frac{\mathbf{P}_n(D(\eta_l; \epsilon) \cap \{\zeta_j\} \neq \emptyset, \quad l = 1, 2)}{\epsilon^4}, \text{ etc.}$$

Asymptotics as $n \rightarrow \infty$ should give CFT.

If $\beta = 1$, the process is *determinantal*,

$$\mathbf{R}_{n,k}(\eta_1, \dots, \eta_k) = \det (\mathbf{K}_n(\eta_i, \eta_j))_{i,j=1}^k.$$

Here \mathbf{K}_n is a "correlation kernel" = reprokernel for

$$\mathcal{W}_n := \{q \cdot e^{-nQ/2}; \text{degree}(q) < n\} \subset L^2.$$

Note: $\mathbf{E}_n(f(\zeta_1, \dots, \zeta_k)) = \frac{(n-k)!}{n!} \int_{\mathbb{C}^k} f \cdot \mathbf{R}_{n,k}.$

"Classical" convergence result (for all β)

Random measure

$$\mu_n := \frac{1}{n} \sum_1^n \delta_{\zeta_j}.$$

For $f \in W^{1,2}$

$$\sigma_n(f) := \mathbf{E}_n(\mu_n(f)) = \frac{1}{n} \sum_1^n \mathbf{E}_n(f(\zeta_j)) = \frac{1}{n} \int_{\mathbb{C}} f \cdot \mathbf{R}_n.$$

Theorem

(MH) $\frac{1}{n} \mathbf{R}_n dA \rightarrow \sigma$ and

$$\sigma_n(f) \rightarrow \int f d\sigma, \quad (n \rightarrow \infty).$$

(Here $\mathbf{R}_n = \mathbf{R}_{n,1}$.)

Example: Ginibre ensemble ($\beta = 1$)

Let $Q(\zeta) = |\zeta|^2$. Then $S = \{|\zeta| \leq 1\}$ and $\sigma = \chi_S dA$.

The process $\{\zeta_i\}_1^n$ can be interpreted as eigenvalues of an $n \times n$ -matrix with i.i.d. centered complex Gaussian entries of variance $1/n$.

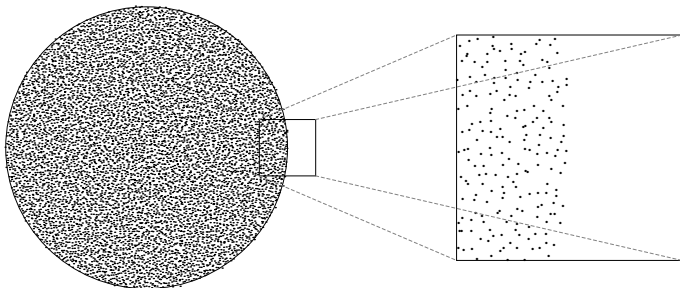


Figure: A sample of the Ginibre process for a large value of n . We will later look at the process near the boundary.

Fluctuation theorem ($\beta = 1$)

Random measures $\text{fluct}_n := n(\mu_n - \sigma) = \sum_1^n \delta_{\zeta_j} - n\sigma$.

Random variables on $(\mathbb{C}^n, \mathbf{P}_n)$

$$\text{fluct}_n(f) = \sum_1^n f(\zeta_j) - n\sigma(f), \quad (f \in C_b^\infty(\mathbb{C})).$$

Theorem

$\text{fluct}_n(f)$ converges in distribution to the normal $N(\mathbf{e}_f, \sigma_f^2)$, where

$$\mathbf{e}_f = \frac{1}{8\pi} \int_{\mathbb{C}} f \cdot \Delta(\chi_S + L^S), \quad \sigma_f^2 = \frac{1}{2} \int_{\mathbb{C}} |\nabla f^S|^2, \quad (L = \log \Delta Q).$$

Here f^S equals f in S and is harmonic and bounded in $\mathbb{C} \setminus S$.

Comments

- 1 There is no $(1/\sqrt{n})$ -normalization!
- 2 Theorem says that random distributions

$$\text{fluct}_n - \Delta(\chi_S + L^S)$$

converge to GFF on S with free boundary conditions.

- 3 Test-class should be $f \in W^{1,2}(\mathbb{C})$: GFF is an isometry $\Phi : W^{1,2}(\mathbb{C}) \rightarrow L^2(P)$ such that $\Phi(f)$ is centered normal with variance $\int |f^S|^2 dA$, and (Wick's formalism)

$$\langle \Phi(f_1) \cdots \Phi(f_{2p}) \rangle = \sum \prod_{k=1}^p \langle f_{i_k}, f_{j_k} \rangle_{\nabla}.$$

- 4 The theorem is only proved for *connected* S with everywhere regular boundary.
- 5 There are "physical" results for arbitrary β ; also results by Johansson in dim 1. Our method "should" extend, but we need some estimates.

Field approximations: CFT_n

Let $(\zeta_j)_1^n$ and $(\zeta'_j)_1^n$ independent random samples, and put

$$\Phi_n(z) = 2 \sum_{j=1}^n (G(z, \zeta_j) - G(z, \zeta'_j))$$

where G is Green's function for S . In a sense Φ_n converges to the GFF Φ on S with Dirichlet boundary conditions, for example ($\beta = 1$)

$$\mathbf{E}_n [\Phi_n(z)\Phi_n(w)] = G(z, w) + o(1),$$

$$\mathbf{E}_n [\Phi_n(z)^2] = \log \sqrt{n} + (1 + \gamma)/2 + c(z) + o(1),$$

where $c(z) = \lim_{w \rightarrow z} (G(z, w) + \log |z - w|)$ is log-conformal radius. For suitable "finite parts", $\Phi_n^2 \rightarrow \Phi^{*2}$ (OPE square).

Similarly, one can obtain other constructions from CFT on a concrete level.

Ward's identity

Let $\{\zeta_j\}_1^n$ system. For smooth ψ define r.v.'s

$$A_\psi = \frac{1}{2} \sum_{j \neq k}^n \frac{\psi(\zeta_j) - \psi(\zeta_k)}{\zeta_j - \zeta_k}, \quad B_\psi = n \sum_1^n \partial Q(\zeta_j) \psi(\zeta_j), \quad C_\psi = \sum_1^n \partial \psi(\zeta_j).$$

Theorem

For all ψ

$$\mathbf{E}_n(\beta \cdot (A_\psi - B_\psi) + C_\psi) = 0.$$

This is an implicit relation between $\mathbf{R}_{n,1}$ and $\mathbf{R}_{n,2}$.

(Proof: reparametrization invariance of the partition function

$$Z_n := \int_{\mathbb{C}^n} e^{-\beta H_n} dV_n.$$

Rescaling

$\{\zeta_j\}_1^n$ random sample from \mathbf{P}_n . Fix $p_n \in \mathcal{S}$, $\theta_n \in \mathbb{R}$.

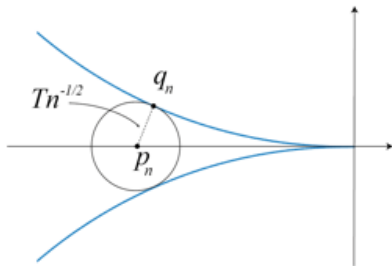
Mesoscopic scale: $r_n = r_n(p_n)$ satisfies:

$$n \cdot \int_{D(p_n, r_n)} \Delta Q(\zeta) dA(\zeta) = 1.$$

If $\Delta Q(p) > 0$ then $r_n \sim 1/\sqrt{n\Delta Q(p)}$.

Rescaled system:

$$z_j = e^{-i\theta_n} r_n^{-1} (\zeta_j - p_n).$$



Rescaled process ($\beta = 1$)

Rescaled process $\Theta_n := \{z_j\}_1^n$ has k -point function

$$R_{n,k}(z_1, \dots, z_k) = r_n^{2k} \mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k).$$

We have:

$$R_{n,k}(z_1, \dots, z_k) = \det(K_n(z_i, z_j))_{i,j=1}^k, \quad K_n(z, w) := r_n^2 \mathbf{K}_n(\zeta, \eta).$$

Known: if $p \in \text{Int } S$, then $R_{n,k}(z_1, \dots, z_k) \rightarrow \det(G(z_i, z_j))_{k \times k}$ where

$$G(z, w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2}.$$

is *Ginibre kernel*.

Compactness and analyticity

A *cocycle* is a function $c(z, w) = g(z)\bar{g}(w)$ where g is continuous and unimodular. Correlation kernels are only determined up to cocycles.

Theorem

There are cocycles c_n such that (on subsequences)

$$c_{n_k} K_{n_k} \rightarrow K, \quad \text{where} \quad K(z, w) = G(z, w)\Psi(z, w).$$

Here $G(z, w) = e^{-|z|^2/2 - |w|^2/2 + z\bar{w}}$ and $\Psi(z, w)$ is Hermitian entire.

(Proof: Taylor's formula + normal families.)

A limit point $K = G\Psi$ is called a *limiting kernel* at (moving) point p .

Limiting point fields

A limiting kernel K is correlation kernel of a limiting point field $\{\zeta_j\}_1^\infty$ with k -point function $R_k(\eta_1, \dots, \eta_k) = \det(K(\eta_i, \eta_j))_{i,j=1}^k$.

Limiting 1-pt function

$$R(z) := K(z, z) = \Psi(z, z).$$

R determines Ψ by polarization and $K = G\Psi$ so

R determines all k -point functions.

Forrester–Honner's formula

Theorem

Let $Q = |\zeta|^2$ and rescale about a boundary point in the outer normal direction. Then $R(z) = F(z + \bar{z})$ where F is "free boundary plasma function"

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-(z-t)^2/2} dt.$$

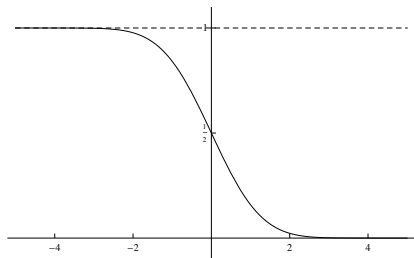


Figure: F is the analytic continuation to \mathbb{C} of the d.f. of the standard normal.

Rescaled Ward identity

Suppose $R(z) = \Psi(z, z) \neq 0$. Put

$$B(z, w) := \frac{|K(z, w)|^2}{R(z)} = e^{-|z-w|^2} \frac{|\Psi(z, w)|^2}{\Psi(z, z)}, \quad C(z) := \int \frac{B(z, w)}{z-w} dA(w).$$

Theorem

$R > 0$ everywhere, $C(z)$ is smooth, and we have Ward's equation

$$\bar{\partial}C(z) = R(z) - 1 - \Delta \log R(z).$$

Since $R \mapsto \Psi$ by polarization, this is an equation for R !

Note: Ward's equation holds at any (moving) point s.t. $R \neq 0$. To fix R uniquely, we need side-conditions. These depend on the nature of the point we're zooming on (bulk point, regular boundary pt, singular boundary pt).

Apriori estimates (side conditions)

Rescale about a regular boundary point in outer normal direction. Let $R(z)$ a limiting 1-point function.

1 Exterior estimate:

$$R(z) \leq Ce^{-2x^2}, \quad (x \geq 0).$$

2 1/8-formula:

$$\int_{-\infty}^{+\infty} t \cdot (R(t) - \chi_{(-\infty,0)}(t)) dt = \frac{1}{8}.$$

(Proof: (1) by potential theory; (2) fluctuation theorem.)

Complementarity

A limiting kernel $K = G\Psi$ is a positive matrix in Aronszajn's sense,

$$\sum_{j,k=1}^N \alpha_j \bar{\alpha}_k K(z_j, z_k) \geq 0.$$

(Because $\det(K(z_j, z_k))_{N \times N} = R_N(z_1, \dots, z_N) \geq 0$.)

Theorem

The complementary kernel

$$\tilde{K}(z, w) = G(z, w)(1 - \Psi(z, w))$$

is also a positive matrix. In particular $R(z) = \Psi(z, z) \leq 1$.

Warning: \tilde{K} does not solve Ward, in general.

Translation Invariance (T.I.)

$R(z) = \Psi(z, z)$ is called *t.i.* if $\Psi(z + it, w + it) = \Psi(z, w)$, $t \in \mathbf{R}$.
Equivalently,

$$\Psi(z, w) = \Phi(z + \bar{w})$$

where Φ is entire.

Gaussian representation

Theorem

If $K(z, w) = G(z, w)\Phi(z + \bar{w})$ is a t.i. limiting kernel, then there exists a Borel function f on \mathbf{R} with $0 \leq f \leq 1$ such that

$$\Phi(z) = \gamma * f(z) = \int_{-\infty}^{+\infty} \gamma(z - t)f(t) dt,$$

where γ is the Gaussian kernel

$$\gamma(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Examples:

- Bulk: $\Phi \equiv 1 = \gamma * \chi_{\mathbf{R}}$
- The plasma kernel: $F = \gamma * \chi_{(-\infty, 0)}$

Proof: Uses Bochner's theorem on positive definite functions (also for the complementary kernel).

T.i. solutions to Ward's equation

Theorem

Let $R(z) = \Phi(z + \bar{z})$ a t.i. limiting 1-point function with $R \not\equiv 0$. Then R solves Ward's equation iff there is an interval $I \subset \mathbb{R}$ such that

$$\Phi = \gamma * \chi_I.$$

Corollary

If $R(z) = \Phi(z + \bar{z})$ is rescaled about a regular boundary point and R is t.i. then

$$\Phi = F = \gamma * \chi_{(-\infty, 0)}.$$

Comments:

- 1 **Conjecture:** An arbitrary limiting kernel is t.i (exception: bulk singularities, degenerate boundary singularities).
- 2 In the general (non-t.i.) case, Ward is a twisted convolution equation. The physically relevant ones "should" be the above.

Consequences

For radial $Q(\zeta) = Q(|\zeta|)$ and for "ellipse potential" $Q = |\zeta|^2 - t\text{Re}(\zeta^2)$ we know that any limiting 1-point function R is t.i.

Corollary

If Q is one of the types above, rescale about a boundary point. The rescaled systems $\{z_j\}_1^n$ converges to the field with kernel

$$K(z, w) = G(z, w) \cdot F(z + \bar{w}).$$

- This should be true for a general potential at a regular boundary point.
- Lee and Riser obtained this for the ellipse potential $Q(\zeta) = |\zeta|^2 - t\text{Re}(\zeta^2)$ using orthogonal polynomials.

Singular boundary points

Now assume that p is a cusp or a double point. If p is a cusp, we assume it has type $(\nu, 2)$ where $\nu > 3$, i.e. it resembles

$$x^\nu = y^2.$$

Theorem

Rescale according to $z_j = \sqrt{n\Delta Q(p)}(\zeta_j - p)$. Then any limiting kernel is trivial: $R = 0$.

(Proof: exterior estimate (suitable coord system))

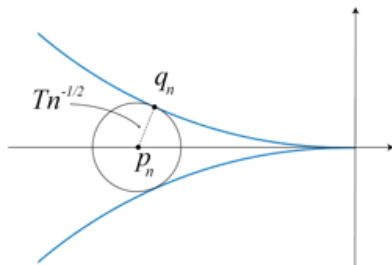
$$R(z) = \Psi(z, z) \leq Ce^{-2x^2}.$$

Since $L(z, w) = e^{z\bar{w}}\Psi(z, w)$ is Hermitian-entire and positive definite, $\log L(z, z)$ is subharmonic. This gives $\Psi = 0$, by the maximum principle.)

Cusps; moving points

Assume S has a $(\nu, 2)$ cusp at p and fix $T > 0$.

Let $p_n \in S$ be the point of distance $T/\sqrt{n\Delta Q(p)}$ from the boundary which is closest to p .



Rescale about p_n

$$z_j = e^{-i\theta_n} r_n^{-2} (\zeta_j - p_n), \quad j = 1, \dots, n$$

where θ_n is chosen so that $e^{-i\theta_n}(p - p_n)$ is positive imaginary.

Existence theorem

Theorem

If T is sufficiently large, then each limiting 1-point function $R(z) = K(z, z)$ is positive, satisfies Ward's equation, and the estimate

$$R(z) \leq Ce^{-2(|x|-T)^2}. \quad (3)$$

- Estimate (3) shows that R is associated with a "new" determinantal point field.
- After the rescaling, the droplet looks like the strip

$$\Sigma_T : \quad -T \leq \operatorname{Re} z \leq T,$$

so it is natural to assume that the field is t.i.

Natural candidates 1

- For $s > 0$ let

$$\Phi_s(z) = \gamma * \chi_{(-s,s)}(z) = \frac{1}{\sqrt{2\pi}} \int_{-s}^s e^{-(z-t)^2/2} dt.$$

If $s \leq 2T$ then $R_s(z) = \Phi(z + \bar{z})$ satisfies $R_s(x) \leq Ce^{-2(|x|-T)^2}$ and Ward's equation.

- How should we choose s ?
- In regular case, we used 1/8-formula:

$$\int_{\mathbb{R}} t \cdot (R(t) - \chi_{(-\infty,0)}(t)) dt = \frac{1}{8}.$$

Something similar should hold at cusps.

Conclusion. We must extend the boundary fluctuation theorem to domains with cusps.

Natural candidates 2

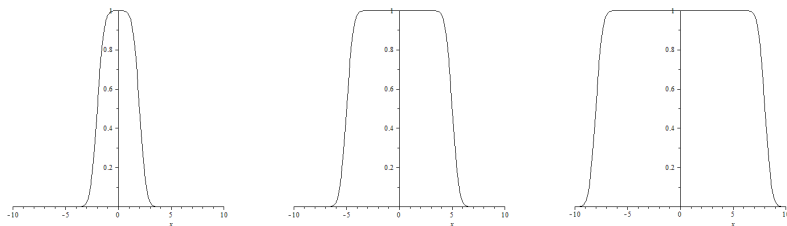


Figure: The graphs of $R_T(x) := \Phi_{T/2}(2x)$ for $T = 2, 5, 8$.

The hard edge (Neumann B.C.'s)

Let Q be a potential. Define $Q^S = Q$ on S and $Q^S = +\infty$ otherwise. Let $\{\zeta_j\}$ be a random sample from \mathbf{P}_n and rescale about a *regular* boundary point p to obtain $\{z_j\}_1^n$.

Theorem

For u.t.i. potentials, the processes $\{z_j\}_1^n$ converge to a unique point field with correlation kernel

$$K(z, w) = G(z, w)H(z + \bar{w})\chi_{\mathbb{L}}(z)\chi_{\mathbb{L}}(w), \quad \mathbb{L} = \{\operatorname{Re} z < 0\},$$

where H is the hard edge plasma function,

$$H(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \frac{e^{-(z-t)^2/2}}{F(t)} dt,$$

(F = free boundary plasma function).

Hard edge plasma function

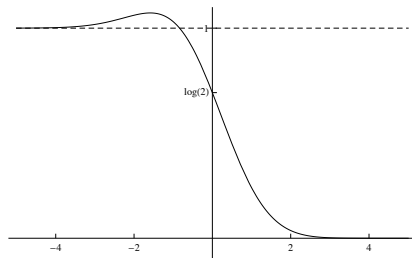


Figure: The graph of H .

The hard-edge theory is parallel to the free boundary; we can obtain existence of new hard edge fields near cusps, and so on.

Remaining problem. We need a fluctuation theorem for the hard edge to completely characterize the limiting kernels. (We can now do it up to a constant.)

Natural candidates 3: hard edge near a singular point

For $T > 0$ define

$$H_T(z) := \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} \frac{e^{-(z-t)^2/2}}{F_T(t)} dt, \quad (F_T = \gamma * \chi_{(-2T,2T)}).$$

The "1-point function" is then $R_T^h(z) := H_T(z + \bar{z})\chi_{(-T,T)}(\operatorname{Re} z)$.

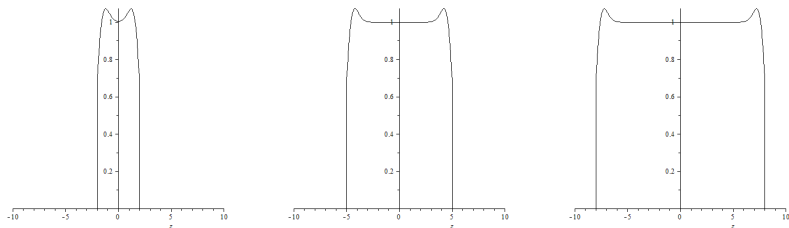


Figure: The graph of R_T^h restricted to the reals, for $T = 2$, $T = 5$, and $T = 8$.

Bulk singularities 1

Suppose that $0 \in \text{Int } S$ and $\Delta Q(0) = 0$. Taylor expansion

$$Q(x + iy) = Q_0(x + iy) + \text{Re } H(z) + \dots$$

where $Q_0(x + iy) = \sum_{j=0}^{2k} a_j x^j y^{2k-j}$ is homogeneous of degree $2k$.
Mesoscopic scale r_n satisfies $n\sigma(D(0, r_n)) = 1$, i.e.

$$r_n \sim n^{-1/2k}.$$

Limiting rescaled kernels take the form

$$K(z, w) = L(z, w) e^{-Q_0(\tau_0 z)/2 - Q_0(\tau_0 w)/2}.$$

Let $L_0(z, w)$ be the Bergman kernel of $L_a^2(\mu_0)$, $d\mu_0 = e^{-Q_0(\tau_0 \cdot)} dA$ and
 $R_0(z) = L_0(z, z) e^{-Q_0(\tau_0 z)}$.

Bulk Singularities 2

Under suitable assumptions on Q we have $R = R_0$ for each limiting 1-point function. If $Q_0 = |\zeta|^{2k}$, $R_0(z) = M_k(|z|^2) e^{-Q_0(z)}$ where M_k is a Mittag-Leffler function.

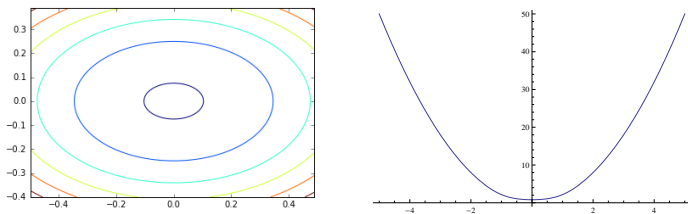


Figure: R_0 for $Q_0 = |z|^4 - |z|^2 \operatorname{Re}(z^2)/2$ and the graph of the Mittag-Leffler function $M_2(x^2)$

Bulk singularities 3

Berezin kernels $B(z, w) = |L_0(z, w)|^2 e^{-Q_0(\tau_0 w)} / L_0(z, z)$ corresponding to $Q = |\zeta|^4$.

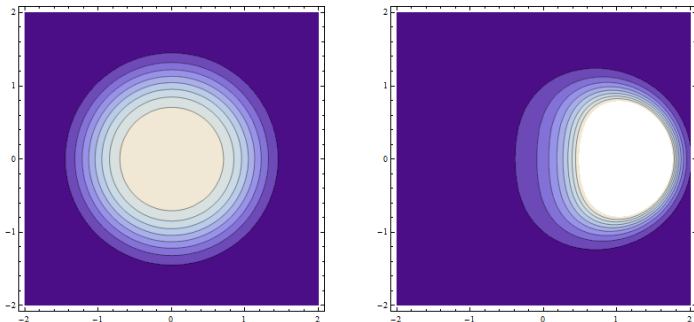


Figure: Berezin kernels rooted at 0 and at 1

Ward: $\bar{\partial} C = R - \Delta[Q_0(\tau_0 \cdot)] - \Delta \log R$.

Logarithmic singularities

Potential $Q(\zeta) = c_1|\zeta|^{2\lambda} + 2c_2n^{-1} \log |\zeta| + \dots$ has mesoscopic scale r_n satisfying $c_2 + c_1'nr_n^{2\lambda} = 1$. Rescaling on the scale r_n we find that limiting 1-point functions are of the type $R_{\lambda,\mu}(z) = cE_{\lambda,\mu}(|z|^2)e^{-Q_0(\tau_0z)}$ where

$$Q_0(z) = |z|^{2\lambda} + 2(1 - \lambda/\mu) \log |z|$$

and $E_{\lambda,\mu}(z) = \sum \frac{z^j}{\Gamma(\lambda j + \mu)}$ a Mittag-Leffler function. The case $c_2 < 0$ corresponds to conical singularities, $c_2 > 0$ corresponds to branch-points on the Riemann surface associated with Q .

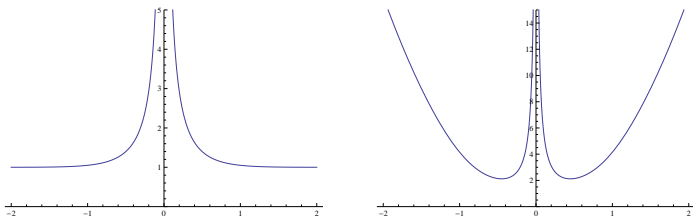


Figure: The 1-point functions $R_{1,2}$ and $R_{2,4}$.

Fekete points

Let p be a regular boundary point, and let $\mathcal{F}_n = \{\zeta_{jn}\}_{j=1}^n$ be an n -Fekete configuration, $n = 1, 2, \dots$. Denote

$$d_n(\zeta_{nj}) = \sqrt{n\Delta Q(\zeta_{nj})} \cdot \min_{k \neq j} |\zeta_{nj} - \zeta_{nk}|,$$

and ("asymptotic separation constant")

$$\Delta(\mathcal{F}) = \liminf_{n \rightarrow \infty} \min_{j=1, \dots, n} \{d_n(\zeta_{nj})\}.$$

Theorem

$$\Delta(\mathcal{F}) \geq 1/\sqrt{e}.$$

We believe that $\Delta(\mathcal{F}) = \sqrt{2/\sqrt{3}}$.

(This comes close to Abrikosov's conjecture.)

Comparison: the one-dimensional case

If Q is real-analytic on \mathbb{R} and $Q = +\infty$ outside \mathbb{R} , then S is a finite union of compact intervals. Let p be a boundary point. Rescale by

$$x_j = cn^{2/3}(\xi_j - p).$$

Theorem

In free boundary case, there is a c such that $\{x_j\}_1^n$ converges to the Airy process with kernel

$$K(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}.$$

In hard edge case we get a Bessel process with kernel

$$K(x, y) = \frac{J_0(\sqrt{x})\sqrt{y} J_0'(\sqrt{y}) - \sqrt{x} J_0'(\sqrt{x}) J_0(\sqrt{y})}{2(x - y)}.$$

These are reprokernels for "de Branges spaces". The two-dimensional plasma kernels are quite different.

β -ensembles 1

We don't now have a kernel, but we put

$$B(z, w) = \frac{R(z)R(w) - R_2(z, w)}{R(z)}$$

and

$$C(z) = \int_{\mathbb{C}} \frac{B(z, w)}{z - w} dA(w).$$

Ward's equation is

$$\bar{\partial}C = R - 1 - \frac{1}{\beta} \Delta \log R.$$

The equation needs to be "closed". When $\beta = 1$ we used the extra structure of existence of a kernel $K = G\Psi$.

β -ensembles 2

- In the "regular bulk case", Jancovici obtained the expansion $B^\beta(z, w) = e^{-|z-w|^2} + (\beta - 1)f(|z - w|) + O((\beta - 1)^2)$ with a certain explicit f .
- At a regular boundary point, Wiegmann et al proves $B^\beta(z, w) = e^{-|z-w|^2} |F(z + \bar{w})|^2 / F(z + \bar{z}) + (\beta - 1)f(z, w) + \dots$

Problems

- 1 What does Ward's equation say about the correction f ?
- 2 What is the correction in bulk singular case?

Degenerate boundary singularities

There can be points at the boundary where $\Delta Q = 0$.

Example: $Q = |\zeta|^4 - \sqrt{2}\operatorname{Re}(\zeta^2)$ gives a "figure 8" at 0.

- 1 Can we obtain new non-trivial point-fields by zooming at a suitable moving point approaching 0?
- 2 Can one characterize the degenerate boundary singularities a la Sakai?