Cascades in 2d turbulence, an overview of some recent theoretical results.

Paolo Muratore-Ginanneschi

Department of Mathematics and Statistics
University of Helsinki

FMI Helsinki, Sept. 2010

Velocity and Vorticity in 2*d*

The Navier Stokes equation

$$(\partial_t + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}})\boldsymbol{v} - \nu\, \partial^2 \boldsymbol{v} = -\partial_{\boldsymbol{x}} P + \boldsymbol{f}$$
$$\partial_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0$$

in 2 *d* is the transport equation for the vorticity pseudo-scalar

$$\partial_t \omega + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} \omega - \nu\, \partial^2 \omega = f_\omega$$
$$\omega := \epsilon_{\alpha\,\beta} \partial^\alpha v^\beta\,, \qquad \omega = \text{vorticity}$$
$$f_\omega := \epsilon_{\alpha\,\beta} \partial^\alpha f^\beta$$

Conservation of vorticity moments in the inviscid limit.

Velocity and Vorticity in 2d

The Navier Stokes equation

$$(\partial_t + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}})\boldsymbol{v} - \nu\, \partial^2 \boldsymbol{v} = -\partial_{\boldsymbol{x}} P + \boldsymbol{f}$$
$$\partial_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0$$

in 2 d is the transport equation for the vorticity pseudo-scalar

$$\partial_t \omega + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} \omega - \nu\, \partial^2 \omega = f_\omega$$
$$\omega := \epsilon_{\alpha\,\beta} \partial^\alpha\, v^\beta\,, \qquad \omega = \text{vorticity}$$
$$f_\omega := \epsilon_{\alpha\,\beta} \partial^\alpha \boldsymbol{f}^\beta$$

Conservation of vorticity moments in the inviscid limit.

Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x \ \| \boldsymbol{v} \|^2 = -2\,\nu \int d^2x\,\omega^2 \equiv -2\,\nu\,\Omega^2$$

$$\partial_t \int d^2x\,\omega^2 = -2\,\nu \int d^2x\,(\partial_\alpha \omega\, \partial^\alpha \omega)$$

Spectral densities and ensemble averages

$$E(k) = \int \frac{d^2p}{(2\,\pi)^2} \prec \| \boldsymbol{v} \|^2 (\boldsymbol{p}) \succ \delta(p-k) \qquad \text{Energy}$$

$$Z(k) = \int \frac{d^2p}{(2\,\pi)^2}\, p^2 \prec \| \boldsymbol{v} \|^2 (\boldsymbol{p}) \succ \delta(p-k) \qquad \text{Enstrophy}$$

Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x \; \| \boldsymbol{v} \|^2 = -2\,\nu \int d^2x\, \omega^2 \equiv -2\,\nu\,\Omega^2$$

$$\partial_t \int d^2x\, \omega^2 = -2\,\nu \int d^2x\, (\partial_\alpha \omega\, \partial^\alpha \omega)$$

Spectral densities and ensemble averages

$$E(k) = \int \frac{d^2p}{(2\,\pi)^2} \prec \| \boldsymbol{v} \|^2 \, (\boldsymbol{p}) \succ \delta(p-k) \qquad \text{Energy}$$

$$Z(k) = \int \frac{d^2p}{(2\,\pi)^2}\, p^2 \prec \| \boldsymbol{v} \|^2 \, (\boldsymbol{p}) \succ \delta(p-k) \qquad \text{Enstrophy}$$

Part II

Kraichnan's classical derivation

Kraichnan's 67 paper

THE PHYSICS OF FLUIDS VOLUME 10, NUMBER 7 JULY 1967

Inertial Ranges in Two-Dimensional Turbulence

Robert H. Kraichnan
Peterborough, New Hampshire
(Received 1 February 1967)

Two-dimensional turbulence has both kinetic energy and mean-square vorticity as inviscid constants of motion. Consequently it admits two formal inertial ranges, $E(k) \sim \epsilon^{2/3}k^{-5/3}$ and $E(k) \sim \eta^{2/3}k^{-3}$, where ϵ is the rate of cascade of kinetic energy per unit mass, η is the rate of cascade of mean-square vorticity, and the kinetic energy per unit mass is $\int_0^\infty E(k)\, dk$. The $-\frac{5}{3}$ range is found to entail backward energy cascade, from higher to lower wavenumbers k, together with zero-vorticity flow. The -3 range gives an upward vorticity flow and zero-energy flow. The paradox in these results is resolved by the irreducibly triangular nature of the elementary wavenumber interactions. The formal -3 range gives a nonlocal cascade and consequently must be modified by logarithmic factors. If energy is fed in at a constant rate to a band of wavenumbers $\sim k_i$ and the Reynolds number is large, it is conjectured that a quasi-steady-state results with a $-\frac{5}{3}$ range for $k \ll k_i$ and a -3 range for $k \gg k_i$, up to the viscous cutoff. The total kinetic energy increases steadily with time as the $-\frac{5}{3}$ range pushes to ever-lower k, until scales the size of the entire fluid are strongly excited. The rate of energy dissipation by viscosity decreases to zero if kinematic viscosity is decreased to zero with other parameters unchanged.

Energy transfer equation

$$(\partial_t + 2\,\nu\, k^2)\, E(k) = \frac{1}{2}\int_0^\infty dq \int_0^\infty dp\ T(k,p,q)$$

$$T(k,p,q) = T(k,q,p)$$

Inertial range relations

Inviscid conservation laws

$$\prod_{i=1}^{3} \int_0^\infty dk_i \, T(k_1, k_2, k_3) = 0$$ energy

$$\prod_{i=1}^{3} \int_0^\infty dk_i \, k_1^2 \, T(k_1, k_2, k_3) = 0$$ enstrophy

local sufficient condition

$$T(k_1, k_2, k_3) + T(k_2, k_3, k_1) + T(k_3, k_1, k_2) = 0$$ energy

$$k_1^2 \, T(k_1, k_2, k_3) + k_2^2 \, T(k_2, k_3, k_1) + k_3^2 \, T(k_3, k_1, k_2) = 0$$ enstrophy

Inertial range relations

Inviscid conservation laws

$$\prod_{i=1}^{3} \int_0^\infty dk_i \, T(k_1, k_2, k_3) = 0 \qquad \text{energy}$$

$$\prod_{i=1}^{3} \int_0^\infty dk_i \, k_1^2 \, T(k_1, k_2, k_3) = 0 \qquad \text{enstrophy}$$

local sufficient condition

$$T(k_1, k_2, k_3) + T(k_2, k_3, k_1) + T(k_3, k_1, k_2) = 0 \qquad \text{energy}$$

$$k_1^2 \, T(k_1, k_2, k_3) + k_2^2 \, T(k_2, k_3, k_1) + k_3^2 \, T(k_3, k_1, k_2) = 0 \qquad \text{enstrophy}$$

Transfer scaling relations

$$\mathcal{T}_v(q) = \int_q^\infty dk_1 \prod_{i=2}^3 \int_0^\infty dk_i \, T(k_1, k_2, k_3)$$

$$\mathcal{T}_\omega(q) = \int_q^\infty dk_1 \prod_{i=2}^3 \int_0^\infty dk_i \, k_1^2 \, T(k_1, k_2, k_3)$$

- Scaling Ansatz

$$T_\bullet(\lambda k_1, \lambda k_2, \lambda k_3) = \lambda^{-\zeta} \, T_\bullet(k_1, k_2, k_3)$$

- Special values

$$\mathcal{T}_v|_{\zeta=3} = \text{const.} \qquad \mathcal{T}_\omega|_{\zeta=3} = 0$$

$$\mathcal{T}_v|_{\zeta=5} = 0 \qquad \mathcal{T}_\omega|_{\zeta=5} = \text{const.}$$

- Dimensional analysis

$$[\mathcal{T}_v] = [E^{3/2} k^{-1/2}] \;\Rightarrow\; E(k) \propto k^{-\frac{2\zeta-1}{3}}$$

Transfer scaling relations

$$\mathcal{T}_v(q) = \int_q^\infty dk_1 \prod_{i=2}^3 \int_0^\infty dk_i \, T(k_1, k_2, k_3)$$

$$\mathcal{T}_\omega(q) = \int_q^\infty dk_1 \prod_{i=2}^3 \int_0^\infty dk_i \, k_1^2 \, T(k_1, k_2, k_3)$$

- Scaling Ansatz

$$T_\bullet(\lambda k_1, \lambda k_2, \lambda k_3) = \lambda^{-\zeta} \, T_\bullet(k_1, k_2, k_3)$$

- Special values

$$\mathcal{T}_v|_{\zeta=3} = \text{const.} \qquad \mathcal{T}_\omega|_{\zeta=3} = 0$$

$$\mathcal{T}_v|_{\zeta=5} = 0 \qquad \mathcal{T}_\omega|_{\zeta=5} = \text{const.}$$

- Dimensional analysis

$$[\mathcal{T}_v] = [E^{3/2} \, k^{-1/2}] \quad \Rightarrow \quad E(k) \propto k^{-\frac{2\zeta-1}{3}}$$

Transfer scaling relations

$$\mathcal{T}_v(q) = \int_q^\infty dk_1 \prod_{i=2}^3 \int_0^\infty dk_i \, T(k_1, k_2, k_3)$$

$$\mathcal{T}_\omega(q) = \int_q^\infty dk_1 \prod_{i=2}^3 \int_0^\infty dk_i \, k_1^2 \, T(k_1, k_2, k_3)$$

- Scaling Ansatz

$$T_\bullet(\lambda k_1, \lambda k_2, \lambda k_3) = \lambda^{-\zeta} \, T_\bullet(k_1, k_2, k_3)$$

- Special values

$$\mathcal{T}_v|_{\zeta=3} = \text{const.} \qquad \mathcal{T}_\omega|_{\zeta=3} = 0$$

$$\mathcal{T}_v|_{\zeta=5} = 0 \qquad \mathcal{T}_\omega|_{\zeta=5} = \text{const.}$$

- Dimensional analysis

$$[\mathcal{T}_v] = [E^{3/2} k^{-1/2}] \quad \Rightarrow \quad E(k) \propto k^{-\frac{2\zeta-1}{3}}$$

Transfer scaling relations

$$\mathcal{T}_v(q) = \int_q^\infty dk_1 \prod_{i=2}^3 \int_0^\infty dk_i \, T(k_1, k_2, k_3)$$

$$\mathcal{T}_\omega(q) = \int_q^\infty dk_1 \prod_{i=2}^3 \int_0^\infty dk_i \, k_1^2 \, T(k_1, k_2, k_3)$$

- Scaling Ansatz

$$T_\bullet(\lambda k_1, \lambda k_2, \lambda k_3) = \lambda^{-\zeta} \, T_\bullet(k_1, k_2, k_3)$$

- Special values

$$\mathcal{T}_v|_{\zeta=3} = \text{const.} \qquad \mathcal{T}_\omega|_{\zeta=3} = 0$$

$$\mathcal{T}_v|_{\zeta=5} = 0 \qquad \mathcal{T}_\omega|_{\zeta=5} = \text{const.}$$

- Dimensional analysis

$$[\mathcal{T}_v] = [E^{3/2} \, k^{-1/2}] \quad \Rightarrow \quad E(k) \propto k^{-\frac{2\zeta - 1}{3}}$$

Boltzmann equilibrium and cascades

Boltzmann equilibrium

$$P(\boldsymbol{v}) \sim \exp\left\{-\frac{1}{2}\int \frac{d^d p}{(2\,\pi)^d}\,(\beta+\gamma\,p^2)\parallel \boldsymbol{v}\parallel^2(\boldsymbol{p})\right\}$$

yields the spectra

$$E_B(p) = \frac{p\,\bar{U}}{1+\frac{\gamma\,p^2}{\beta}} \qquad \& \qquad Z_B(p) = \frac{p^3\,\bar{U}}{1+\frac{\gamma\,p^2}{\beta}}$$

to be contrasted with the constant flux spectra

$$E_K(p) \propto \begin{cases} p^{-\frac{5}{3}} \\ \\ p^{-3} \end{cases} \quad \& \quad Z_K(p) \propto \begin{cases} p^{\frac{1}{3}} & \zeta = -3 \\ \\ p^{-1} & \zeta = -5 \end{cases}$$

Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x \parallel \boldsymbol{v} \parallel^2 = -2\,\nu \int d^2x\,\omega^2 \equiv -2\,\nu\,\Omega^2$$

$$\partial_t \int d^2x\,\omega^2 = -2\,\nu \int d^2x\,(\partial_\alpha \omega \partial^\alpha \omega)$$

- Enstrophy Ω can only decrease.
- Energy dissipation vanishes as $\nu \downarrow 0$.
- Enstrophy flux strives to equilibrate the system

$$Z_B(p) \simeq p^3 < Z_K(p) \simeq p^{1/3} \qquad p \downarrow 0 \qquad \text{“eccess” of enstrophy}$$

$$Z_B(p) \simeq p > Z_K(p) \simeq p^{-1} \qquad p \uparrow \infty \qquad \text{“defect” of enstrophy}$$

- Energy conservation imposes a flux towards large scales (small wavevectors).

Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x \parallel \boldsymbol{v} \parallel^2 = -2\,\nu \int d^2x\,\omega^2 \equiv -2\,\nu\,\Omega^2$$

$$\partial_t \int d^2x\,\omega^2 = -2\,\nu \int d^2x\,(\partial_\alpha\omega\partial^\alpha\omega)$$

- Enstrophy Ω can only decrease.
- Energy dissipation vanishes as $\nu \downarrow 0$.
- Enstrophy flux strives to equilibrate the system

$$Z_B(p) \simeq p^3 < Z_K(p) \simeq p^{1/3} \qquad p \downarrow 0 \qquad \text{"eccess" of enstrophy}$$

$$Z_B(p) \simeq p > Z_K(p) \simeq p^{-1} \qquad p \uparrow \infty \qquad \text{"defect" of enstrophy}$$

- Energy conservation imposes a flux towards large scales (small wavevectors).

Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x \parallel \boldsymbol{v} \parallel^2 = -2\,\nu \int d^2x\,\omega^2 \equiv -2\,\nu\,\Omega^2$$

$$\partial_t \int d^2x\,\omega^2 = -2\,\nu \int d^2x\,(\partial_\alpha\omega\partial^\alpha\omega)$$

- Enstrophy Ω can only decrease.
- Energy dissipation vanishes as $\nu \downarrow 0$.
- Enstrophy flux strives to equilibrate the system

$$Z_B(p) \simeq p^3 < Z_K(p) \simeq p^{1/3} \qquad p \downarrow 0 \qquad \text{“eccess” of enstrophy}$$

$$Z_B(p) \simeq p > Z_K(p) \simeq p^{-1} \qquad p \uparrow \infty \qquad \text{“defect” of enstrophy}$$

- Energy conservation imposes a flux towards large scales (small wavevectors).

Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x \ \| \boldsymbol{v} \|^2 = -2\,\nu \int d^2x\,\omega^2 \equiv -2\,\nu\,\Omega^2$$

$$\partial_t \int d^2x\,\omega^2 = -2\,\nu \int d^2x\,(\partial_\alpha\omega\partial^\alpha\omega)$$

- Enstrophy Ω can only decrease.
- Energy dissipation vanishes as $\nu \downarrow 0$.
- Enstrophy flux strives to equilibrate the system

$$Z_B(p) \simeq p^3 < Z_K(p) \simeq p^{1/3} \qquad p \downarrow 0 \qquad \text{“eccess” of enstrophy}$$

$$Z_B(p) \simeq p > Z_K(p) \simeq p^{-1} \qquad p \uparrow \infty \qquad \text{“defect” of enstrophy}$$

- Energy conservation imposes a flux towards large scales (small wavevectors).

Energy and Enstrophy

Total Energy and Enstrophy equations

$$\partial_t \int d^2x \parallel \boldsymbol{v} \parallel^2 = -2\,\nu \int d^2x\,\omega^2 \equiv -2\,\nu\,\Omega^2$$

$$\partial_t \int d^2x\,\omega^2 = -2\,\nu \int d^2x\,(\partial_\alpha \omega \partial^\alpha \omega)$$

- Enstrophy Ω can only decrease.
- Energy dissipation vanishes as $\nu \downarrow 0$.
- Enstrophy flux strives to equilibrate the system

$$Z_B(p) \simeq p^3 < Z_K(p) \simeq p^{1/3} \qquad p \downarrow 0 \qquad \text{“eccess” of enstrophy}$$

$$Z_B(p) \simeq p > Z_K(p) \simeq p^{-1} \qquad p \uparrow \infty \qquad \text{“defect” of enstrophy}$$

- Energy conservation imposes a flux towards large scales (small wavevectors).

Numerical validation

256 G. Boffetta

G. Boffetta,

J. Fluid Mech.

589, 253 (2007).

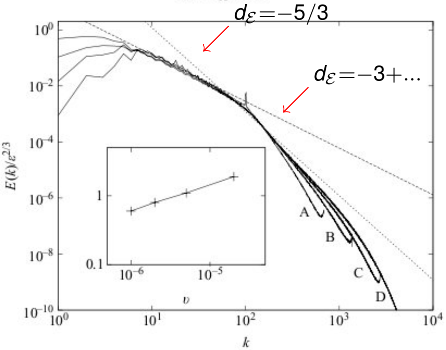


FIGURE 2. Energy spectra for the two simulations for the different resolutions (labels as in figure 1). Dashed and dotted lines represent the two predictions $Ck^{-5/3}$ with $C = 6$ and k^{-3} respectively. Inset: correction δ to the Kraichnan exponent -3 as a function of viscosity, computed by fitting the spectra with a power law $k^{-(3+\delta)}$ in the range $100 \leqslant k \leqslant 400$.

Part III

Kàrmàn-Howarth-Monin equation (Lindborg, Bernard)

Gaussian, zero average , time short-correlated and space translational invariant forcing

$$\prec \boldsymbol{f}(\boldsymbol{x},t)\cdot\boldsymbol{f}(\boldsymbol{y},s) \succ = \delta(t-s)\,F(\boldsymbol{x}-\boldsymbol{y},m)$$

For

$$\delta\boldsymbol{v}(\boldsymbol{x}) := \boldsymbol{v}(\boldsymbol{x},t)-\boldsymbol{v}(\boldsymbol{0},t)$$

the Kàrmàn-Howarth-Monin (KHM) equation (equal times) is

$$\frac{1}{2}\prec (\partial_{\boldsymbol{x}}\cdot\delta\boldsymbol{v})(\boldsymbol{x}) \parallel \delta\boldsymbol{v}\parallel^2(\boldsymbol{x}) \succ =$$
$$= \partial_t \prec \boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{0}) \succ - F(\boldsymbol{x},m) - 2\,\nu \prec (\partial_\alpha v_\beta)(\boldsymbol{x})(\partial^\alpha v^\beta)(\boldsymbol{0}) \succ$$

Gaussian, zero average , time short-correlated and space translational invariant forcing

$$\prec \boldsymbol{f}(\boldsymbol{x},t)\cdot\boldsymbol{f}(\boldsymbol{y},s) \succ = \delta(t-s)\,F(\boldsymbol{x}-\boldsymbol{y},m)$$

For

$$\delta\boldsymbol{v}(\boldsymbol{x}) := \boldsymbol{v}(\boldsymbol{x},t)-\boldsymbol{v}(\boldsymbol{0},t)$$

the Kàrmàn-Howarth-Monin (KHM) equation (equal times) is

$$\frac{1}{2}\prec (\partial_{\boldsymbol{x}}\cdot\delta\boldsymbol{v})(\boldsymbol{x}) \parallel \delta\boldsymbol{v}\parallel^2(\boldsymbol{x}) \succ =$$

$$= \partial_t \prec \boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{0}) \succ - F(\boldsymbol{x},m) - 2\nu \prec (\partial_\alpha v_\beta)(\boldsymbol{x})(\partial^\alpha v^\beta)(\boldsymbol{0}) \succ$$

Gaussian, zero average , time short-correlated and space translational invariant forcing

$$\prec \boldsymbol{f}(\boldsymbol{x},t)\cdot\boldsymbol{f}(\boldsymbol{y},s) \succ = \delta(t-s)\,F(\boldsymbol{x}-\boldsymbol{y},m)$$

For

$$\delta\boldsymbol{v}(\boldsymbol{x}) := \boldsymbol{v}(\boldsymbol{x},t)-\boldsymbol{v}(\boldsymbol{0},t)$$

the Kàrmàn-Howarth-Monin (KHM) equation (equal times) is

$$\frac{1}{2}\prec (\partial_{\boldsymbol{x}}\cdot\delta\boldsymbol{v})(\boldsymbol{x})\ \|\,\delta\boldsymbol{v}\,\|^2(\boldsymbol{x}) \succ =$$

$$= \partial_t \prec \boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{0}) \succ - F(\boldsymbol{x},m) - 2\,\nu \prec (\partial_\alpha v_\beta)(\boldsymbol{x})(\partial^\alpha v^\beta)(\boldsymbol{0}) \succ$$

Hypotheses encoding Kraichnan's theory:

i velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points:

$$\lim_{x \to 0} \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ = \| \boldsymbol{v} \|^2 (0) \qquad \nu > 0$$

ii Galilean invariant functions and in particular structure functions reach a steady state:

$$\lim_{t \uparrow \infty} \prec \delta v^{\mu}(\boldsymbol{x}) \, \| \delta \boldsymbol{v} \|^2 (\boldsymbol{x}) \succ = S_3^{\mu}(\boldsymbol{x})$$

iii No energy dissipative anomalies occur:

$$\left\{ \lim_{\nu \downarrow 0} \lim_{x \downarrow 0} - \lim_{x \downarrow 0} \lim_{\nu \downarrow 0} \right\} \nu \prec \partial_{\alpha} v^{\beta}(x, t) \partial^{\alpha} v_{\beta}(0, t) \succ = 0$$

Hypotheses encoding Kraichnan's theory:

i velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points:

$$\lim_{x \to 0} \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ = \| \boldsymbol{v} \|^2 (0) \qquad \nu > 0$$

ii Galilean invariant functions and in particular structure functions reach a steady state:

$$\lim_{t \uparrow \infty} \prec \delta v^{\mu}(\boldsymbol{x}) \| \delta \boldsymbol{v} \|^2 (\boldsymbol{x}) \succ = S_3^{\mu}(\boldsymbol{x})$$

iii No energy dissipative anomalies occur:

$$\left\{ \lim_{\nu \downarrow 0} \lim_{x \downarrow 0} - \lim_{x \downarrow 0} \lim_{\nu \downarrow 0} \right\} \nu \prec \partial_{\alpha} v^{\beta}(\boldsymbol{x}, t) \partial^{\alpha} v_{\beta}(\boldsymbol{0}, t) \succ = 0$$

Hypotheses encoding Kraichnan's theory:

i velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points:

$$\lim_{x \to 0} \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ = \| \boldsymbol{v} \|^2 (0) \qquad \nu > 0$$

ii Galilean invariant functions and in particular structure functions reach a steady state:

$$\lim_{t \uparrow \infty} \prec \delta v^{\mu}(\boldsymbol{x}) \| \delta \boldsymbol{v} \|^2 (\boldsymbol{x}) \succ = S_3^{\mu}(\boldsymbol{x})$$

iii No energy dissipative anomalies occur:

$$\left\{ \lim_{\nu \downarrow 0} \lim_{x \downarrow 0} - \lim_{x \downarrow 0} \lim_{\nu \downarrow 0} \right\} \nu \prec \partial_{\alpha} v^{\beta}(\boldsymbol{x}, t) \partial^{\alpha} v_{\beta}(\boldsymbol{0}, t) \succ = 0$$

KHM equation and "mean field" scaling

$$\frac{1}{2}\partial_\mu S_3^\mu(\boldsymbol{x}) =$$

$$\partial_t \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\mathbf{0}) \succ - F(\boldsymbol{x}, m) - 2\,\nu \prec (\partial_\alpha v_\beta)(\boldsymbol{x})(\partial^\alpha v^\beta)(\mathbf{0}) \succ$$

KHM equation and "mean field" scaling

$$\frac{1}{2}\partial_\mu S_3^\mu(\boldsymbol{x}) = \partial_t \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ - F(\boldsymbol{x}, m) - 2\,\nu\, D_2(\boldsymbol{x})$$

KHM equation and "mean field" scaling

$$\frac{1}{2}\partial_\mu S_3^\mu = \underbrace{\partial_t \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ}_{=I_{\mathcal{E}}} - \underbrace{F(\boldsymbol{x}, m)}_{\overset{mx \gg 1}{\rightarrow} 0} - \underbrace{2\,\nu\, D_2(\boldsymbol{x})}_{\overset{\nu \downarrow 0}{\Rightarrow} 0}$$

KHM equation and "mean field" scaling

$$\frac{1}{2}\partial_\mu S_3^\mu = \underbrace{\partial_t \prec \mathbf{v}(\mathbf{x})\cdot\mathbf{v}(\mathbf{0}) \succ}_{=I_\varepsilon} - \underbrace{F(\mathbf{x}, m)}_{\overset{mx\gg 1}{\to} 0} - \underbrace{2\,\nu\, D_2(\mathbf{x})}_{\overset{\nu\downarrow 0}{\Rightarrow} 0}$$

Inverse Energy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta\, v_{||} \parallel v_\perp \parallel^2 \succ \overset{mx\gg 1}{=} \frac{3\, I_\varepsilon\, x}{2} \quad \overset{\text{mean field}}{\Rightarrow} \quad \delta v \sim x^{1/3}$$

KHM equation and "mean field" scaling

Inverse Energy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||} \parallel v_{\perp} \parallel^2 \succ \overset{mx \gg 1}{=} \frac{3\, I_{\varepsilon}\, x}{2} \quad \overset{\text{mean field}}{\Rightarrow} \quad \delta v \sim x^{1/3}$$

Direct Enstrophy Cascade

$$\frac{1}{2}\partial_{x^{\mu}} S_3^{\mu} = \underbrace{\partial_t \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ}_{=I_{\varepsilon}} - \underbrace{F(\boldsymbol{x}, m)}_{\overset{mx \ll 1}{\to} I_{\varepsilon} - I_{\Omega} x^2 + \ldots} - \underbrace{2\, \nu\, D_2(\boldsymbol{x})}_{\overset{\nu \downarrow 0}{\Rightarrow} 0}$$

KHM equation and "mean field" scaling

Inverse Energy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||} \parallel v_\perp \parallel^2 \succ \overset{mx \gg 1}{=} \frac{3\, I_\varepsilon\, x}{2} \quad \overset{\text{mean field}}{\Rightarrow} \quad \delta v \sim x^{1/3}$$

Direct Enstrophy Cascade

$$\frac{1}{2} \partial_{x^\mu} S_3^\mu = \underbrace{\partial_t \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ}_{= I_\varepsilon} - \underbrace{F(\boldsymbol{x}, m)}_{\overset{mx \ll 1}{\rightarrow} I_\varepsilon - I_\Omega x^2 + \dots} - \underbrace{2\, \nu\, D_2(\boldsymbol{x})}_{\overset{\nu \downarrow 0}{\Rightarrow} 0}$$

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||} \delta v_\perp^2 \succ \overset{\ell x \ll mx \ll 1}{=} \frac{I_\Omega\, x^3}{8} \quad \overset{\text{mean field}}{\Rightarrow} \quad \delta v \sim x$$

KHM equation and "mean field" scaling

Inverse Energy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||} \| v_\perp \|^2 \succ \stackrel{mx \gg 1}{=} \frac{3\, I_\mathcal{E}\, x}{2} \quad \underset{\Rightarrow}{\text{mean field}} \quad \delta v \sim x^{1/3}$$

$\Downarrow$

$I_\mathcal{E} > 0$: sign opposite to d>2 case

Direct Enstrophy Cascade

$$\prec \delta v_{||}^3 \succ = \prec \delta v_{||} \delta v_\perp^2 \succ \stackrel{\ell x \ll mx \ll 1}{=} \frac{I_\Omega\, x^3}{8} \quad \underset{\Rightarrow}{\text{mean field}} \quad \delta v \sim x$$

Effect of Ekman dissipation (Bernard)

Steady state exists: in the inviscid limit $\nu \downarrow 0$

$$-\partial_{x^\mu} S^{\mu}_{(3,0)}(\boldsymbol{x}) + \frac{1}{\tau} \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ = F(\boldsymbol{x})$$

$$-\partial_{x^\mu} S^{\mu}_{(1,2)}(\boldsymbol{x}) + \nu \prec \partial_{x^\alpha} \omega(\boldsymbol{x})\, \partial_{x_\alpha} \omega(\boldsymbol{0}) \succ + \prec \frac{\omega(\boldsymbol{x})\omega(\boldsymbol{0})}{\tau} \succ = F_\omega(\boldsymbol{x})$$

Effect of Ekman dissipation (Bernard)

Steady state exists: in the inviscid limit $\nu \downarrow 0$

$$-\partial_{x^\mu} S^{\mu}_{(3,0)}(\boldsymbol{x}) + \tfrac{1}{\tau} \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ = F(\boldsymbol{x})$$

$$-\partial_{x^\mu} S^{\mu}_{(1,2)}(\boldsymbol{x}) + \nu \prec \partial_{x^\alpha}\omega(\boldsymbol{x})\,\partial_{x_\alpha}\omega(\boldsymbol{0}) \succ + \prec \tfrac{\omega(\boldsymbol{x})\omega(\boldsymbol{0})}{\tau} \succ = F_\omega(\boldsymbol{x})$$

$$\mathcal{S}^{\mu}_{(3,0)}(\boldsymbol{x}) := \prec [v^\mu(\boldsymbol{x},t) - v^\mu(0,t)]\,||\boldsymbol{v}(\boldsymbol{x},t) - \boldsymbol{v}(0,t)||^2 \succ$$

$$\mathcal{S}^{\mu}_{(1,2)}(\boldsymbol{x}) := \prec [v^\mu(\boldsymbol{x},t) - v^\mu(0,t)]\,\omega^2(\boldsymbol{x},t) \succ$$

Effect of Ekman dissipation (Bernard)

Steady state exists: in the inviscid limit $\nu \downarrow 0$

$$-\partial_{x^\mu} S^{\mu}_{(3,0)}(\boldsymbol{x}) + \frac{1}{\tau} \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ = F(\boldsymbol{x})$$

$$-\partial_{x^\mu} S^{\mu}_{(1,2)}(\boldsymbol{x}) + \nu \prec \partial_{x^\alpha}\omega(\boldsymbol{x})\,\partial_{x_\alpha}\omega(\boldsymbol{0}) \succ + \prec \frac{\omega(\boldsymbol{x})\omega(\boldsymbol{0})}{\tau} \succ = F_\omega(\boldsymbol{x})$$

Enstrophy dissipative anomaly

$$\epsilon^\star_\omega := \lim_{\nu\downarrow 0} \nu \prec \| \partial_{\boldsymbol{x}}\omega \|^2 \succ < F_\omega(\boldsymbol{0}) := \epsilon_\omega$$

Effect of Ekman dissipation (Bernard)

Steady state exists: in the inviscid limit $\nu \downarrow 0$

$$-\partial_{x^\mu} S^{\mu}_{(3,0)}(\boldsymbol{x}) + \tfrac{1}{\tau} \prec \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{0}) \succ = F(\boldsymbol{x})$$

$$-\partial_{x^\mu} S^{\mu}_{(1,2)}(\boldsymbol{x}) + \nu \prec \partial_{x^\alpha}\omega(\boldsymbol{x})\, \partial_{x_\alpha}\omega(\boldsymbol{0}) \succ + \prec \tfrac{\omega(\boldsymbol{x})\omega(\boldsymbol{0})}{\tau} \succ = F_\omega(\boldsymbol{x})$$

Enstrophy dissipative anomaly

$$\epsilon^{\star}_{\omega} := \lim_{\nu\downarrow 0} \nu \prec \| \partial_{\boldsymbol{x}}\omega \|^2 \succ < F_\omega(\boldsymbol{0}) := \epsilon_\omega$$

The enstrophy correlation is bounded by $\prec \omega^2(\boldsymbol{0}) \succ$

$$\lim_{\nu\downarrow 0} \prec \omega(\boldsymbol{x})\omega(\boldsymbol{0}) \succ = \tau\,(\epsilon_\omega - \epsilon^{\star}_{\omega}) - \tau\,\epsilon_\omega\, A \left(\frac{x}{L}\right)^{2\,\xi_2} + \dots$$

Effect of Ekman dissipation (Bernard)

Steady state exists: in the inviscid limit $\nu \downarrow 0$

$$-\partial_{x^\mu} S^{\mu}_{(3,0)}(\boldsymbol{x}) \simeq a_1 \epsilon^{\star}_{\omega}\, x^2 + a_2 x^{2\,(1+\xi_2)}$$

$$-\partial_{x^\mu} S^{\mu}_{(1,2)}(\boldsymbol{x}) + \nu \prec \partial_{x^\alpha}\omega(\boldsymbol{x})\,\partial_{x_\alpha}\omega(\boldsymbol{0}) \succ + \prec \frac{\omega(\boldsymbol{x})\omega(\boldsymbol{0})}{\tau} \succ = F_\omega(\boldsymbol{x})$$

Enstrophy dissipative anomaly

$$\epsilon^{\star}_{\omega} := \lim_{\nu\downarrow 0} \nu \prec \| \partial_{\boldsymbol{x}}\omega \|^2 \succ < F_\omega(\boldsymbol{0}) := \epsilon_\omega$$

$$\lim_{\nu\downarrow 0} \prec \omega(\boldsymbol{x})\omega(\boldsymbol{0}) \succ = \tau\,(\epsilon_\omega - \epsilon^{\star}_{\omega}) - \tau\,\epsilon_\omega\, A \left(\tfrac{x}{L}\right)^{2\,\xi_2} + \dots$$

$$\lim_{\nu\downarrow 0} \prec \| \delta \boldsymbol{v} \|^2 (\boldsymbol{x}) \succ = \frac{\tau\,(\epsilon_\omega - \epsilon^{\star}_{\omega})\, x^2}{2} - \tau\,\epsilon_\omega\, \tilde{A} \left(\tfrac{x}{L}\right)^{2\,(1+\xi_2)} + \dots$$

Part IV

Non universality of the direct cascade

Energy Spectrum

Absence of enstrophy dissipative anomaly

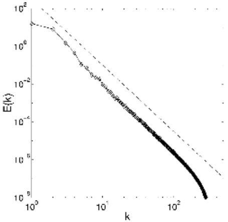
$$-\partial_{x^\mu} S^{\mu}_{(3,0)}(\boldsymbol{x}) \simeq a_1 \epsilon^{\star}_{\omega}\, x^2 + a_2 x^{2\,(1+\xi_2)}$$

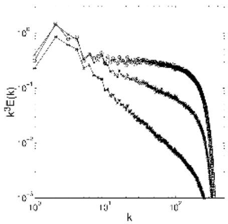
implies

$$S^{\mu}_{(3,0)}(\boldsymbol{x}) \sim x^{3\,+2\,\xi_2}$$

The anomalous exponent should then depend upon the Ekman friction τ

VOLUME 84, NUMBER 22 PHYSICAL REVIEW LETTERS 29 MAY 2000





K. Nam, E. Ott, T. M. Antonsen and P.N. Guzdar

PRL 84 5134

FIG. 1. Energy spectrum with a drag force coefficient ν_0 =

FIG. 2. Energy spectrum with various drag coefficients ν_0.

Lagrangian interpretation

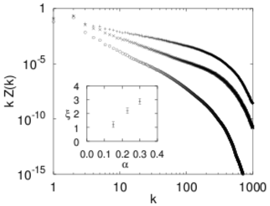


FIG. 2. The vorticity spectrum $Z(k) \sim k^{-1-\xi}$ steepens by increasing the Ekman coefficient α. Here $\alpha = 0.15$ (+), $\alpha = 0.23$ (×), $\alpha = 0.30$ (⊙). In the inset, the exponent ξ as a function of α.

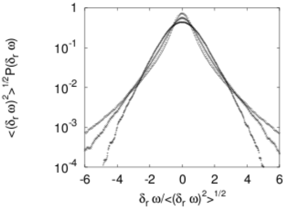


FIG. 3. Probability density functions of normalized vorticity increments $\delta_r\omega/\langle(\delta_r\omega)^2\rangle^{1/2}$. Here, $r = 0.20$ (+), $r = 0.07$ (×), $r = 0.02$ (▽). For large separations the statistics is close to Gaussian, becoming increasingly intermittent for smaller r.

G. Boffetta,
A. Celani,
S. Musacchio and
M. Vergassola
PRE 66 026304

$$\omega(\boldsymbol{x}', t) - \omega(\boldsymbol{x}, t) = \int_{-\infty}^{t} ds \left[f\left(\xi_s^{\boldsymbol{x}'}, s\right) - f\left(\xi_s^{\boldsymbol{x}}, s\right)\right] e^{-\alpha\,(t-s)}$$

Lagrangian interpretation

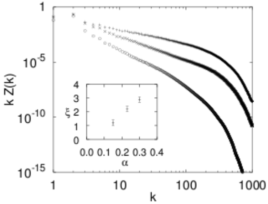


FIG. 2. The vorticity spectrum $Z(k) \sim k^{-1-\xi}$ steepens by increasing the Ekman coefficient α. Here $\alpha = 0.15$ (+), $\alpha = 0.23$ (×), $\alpha = 0.30$ (⊙). In the inset, the exponent ξ as a function of α.

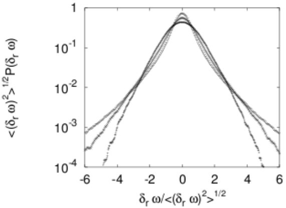


FIG. 3. Probability density functions of normalized vorticity increments $\delta_r\omega/\langle(\delta_r\omega)^2\rangle^{1/2}$. Here, $r = 0.20$ (+), $r = 0.07$ (×), $r = 0.02$ (▽). For large separations the statistics is close to Gaussian, becoming increasingly intermittent for smaller r.

G. Boffetta,

A. Celani,

S. Musacchio and

M. Vergassola

PRE 66 026304

$$\omega(\boldsymbol{x}', t) - \omega(\boldsymbol{x}, t) \sim \Omega\, e^{-\alpha\, T_L(\|\boldsymbol{x}' - \boldsymbol{x}\|)}$$

Lagrangian interpretation

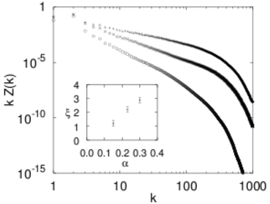


FIG. 2. The vorticity spectrum $Z(k) \sim k^{-1-\xi}$ steepens by increasing the Ekman coefficient α. Here $\alpha = 0.15$ (+), $\alpha = 0.23$ (×), $\alpha = 0.30$ (⊙). In the inset, the exponent ξ as a function of α.

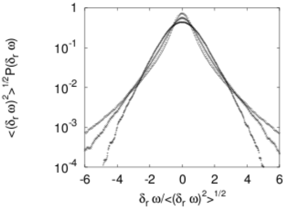


FIG. 3. Probability density functions of normalized vorticity increments $\delta_r\omega/\langle(\delta_r\omega)^2\rangle^{1/2}$. Here, $r = 0.20$ (+), $r = 0.07$ (×), $r = 0.02$ (▽). For large separations the statistics is close to Gaussian, becoming increasingly intermittent for smaller r.

G. Boffetta,
A. Celani,
S. Musacchio and
M. Vergassola

PRE 66 026304

$$\omega(\boldsymbol{x}', t) - \omega(\boldsymbol{x}, t) \sim \Omega\, e^{-\alpha\, T_L(\|\boldsymbol{x}' - \boldsymbol{x}\|)}$$

$T_L(\| \boldsymbol{x}' - \boldsymbol{x} \|)$ = time (backward from t) to separate to a scale L

Intermittence of the direct cascade

- Particle separation for a smooth velocity field are esponential
- Statistics is described by finite-time Lyapunov exponent γ
- For $t \uparrow \infty$ γ

$$P(\gamma\, t) \sim t^{1/2}\, e^{-G(\gamma)\, t}$$

- Lyapunov exponents and exit-time L are related by

$$L \sim \parallel \boldsymbol{x}' - \boldsymbol{x} \parallel\; e^{\gamma\, T_L}$$

- For $\parallel \boldsymbol{x}' - \boldsymbol{x} \parallel \ll L$ a large deviation estimate predicts intermittence of the direct “cascade”:

$$S_{(0,p)}(\boldsymbol{x}) \sim \boldsymbol{x}^{\zeta_p}$$

Intermittence of the direct cascade

- Particle separation for a smooth velocity field are esponential
- Statistics is described by finite-time Lyapunov exponent γ
- For $t \uparrow \infty$ γ

$$P(\gamma\, t) \sim t^{1/2}\, e^{-G(\gamma)\, t}$$

- Lyapunov exponents and exit-time L are related by

$$L \sim \| \boldsymbol{x}' - \boldsymbol{x} \| \; e^{\gamma\, T_L}$$

- For $\| \boldsymbol{x}' - \boldsymbol{x} \| \ll L$ a large deviation estimate predicts intermittence of the direct "cascade":

$$S_{(0,p)}(\boldsymbol{x}) \sim \boldsymbol{x}^{\zeta_p}$$

Intermittence of the direct cascade

- Particle separation for a smooth velocity field are esponential
- Statistics is described by finite-time Lyapunov exponent γ
- For $t \uparrow \infty$ γ

$$P(\gamma\, t) \sim t^{1/2}\, e^{-G(\gamma)\, t}$$

- Lyapunov exponents and exit-time L are related by

$$L \sim \| \boldsymbol{x}' - \boldsymbol{x} \| \; e^{\gamma\, T_L}$$

- For $\| \boldsymbol{x}' - \boldsymbol{x} \| \ll L$ a large deviation estimate predicts intermittence of the direct "cascade":

$$S_{(0,p)}(\boldsymbol{x}) \sim \boldsymbol{x}^{\zeta_p}$$

Intermittence of the direct cascade

- Particle separation for a smooth velocity field are esponential
- Statistics is described by finite-time Lyapunov exponent γ
- For $t \uparrow \infty$ γ

$$P(\gamma\, t) \sim t^{1/2}\, e^{-G(\gamma)\, t}$$

- Lyapunov exponents and exit-time L are related by

$$L \sim \parallel \boldsymbol{x}' - \boldsymbol{x} \parallel \; e^{\gamma\, T_L}$$

- For $\parallel \boldsymbol{x}' - \boldsymbol{x} \parallel \ll L$ a large deviation estimate predicts intermittence of the direct "cascade":

$$S_{(0,p)}(\boldsymbol{x}) \sim \boldsymbol{x}^{\zeta_p}$$

Intermittence of the direct cascade

- Particle separation for a smooth velocity field are esponential
- Statistics is described by finite-time Lyapunov exponent γ
- For $t \uparrow \infty$ γ

$$P(\gamma\, t) \sim t^{1/2}\, e^{-G(\gamma)\, t}$$

- Lyapunov exponents and exit-time L are related by

$$L \sim \parallel \boldsymbol{x}' - \boldsymbol{x} \parallel \; e^{\gamma\, T_L}$$

- For $\parallel \boldsymbol{x}' - \boldsymbol{x} \parallel \ll L$ a large deviation estimate predicts intermittence of the direct “cascade”:

$$S_{(0,p)}(\boldsymbol{x}) \sim \boldsymbol{x}^{\zeta_p}$$