BESOV CAPACITY AND HAUSDORFF MEASURES IN METRIC MEASURE SPACES

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Abstract. This paper studies Besov $p$-capacities as well as their relationship to Hausdorff measures in Ahlfors regular metric spaces of dimension $Q$ for $1 < Q < p < \infty$. Lower estimates of the Besov $p$-capacities are obtained in terms of the Hausdorff content associated with gauge functions $h$ satisfying the decay condition $\int_0^1 h(t)^{1/(p-1)} \, \frac{dt}{t} < \infty$.

1. Introduction

In this paper $(X, d, \mu)$ is a proper (that is, closed bounded subsets of $X$ are compact) and unbounded metric space. In addition, it is Ahlfors $Q$-regular for some $Q > 1$. That is, there exists a constant $C = c_\mu$ such that, for each $x \in X$ and all $r > 0$,

$$C^{-1} r^Q \leq \mu(B(x, r)) \leq Cr^Q.$$ 

For $Q < p < \infty$ we define

$$B_p(X) = \{ u \in L^p(X) : ||u||_{B_p(X)} < \infty \},$$

where

$$||u||_{B_p(X)} = ||u||_{L^p(X)} + [u]_{B_p(X)}$$

with

$$[u]_{B_p(X)} = \left( \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x, y)^{2Q}} \, d\mu(x) \, d\mu(y) \right)^{1/p}.$$

The expressions $||u||_{B_p(X)}$ and $[u]_{B_p(X)}$ from (1) and (2) are called the Besov norm and the Besov seminorm of $u$ respectively. We have

$$[u]_{B_p(X)} = 0 \text{ if and only if } u \text{ is constant } \mu\text{-a.e.}$$

Besov spaces have recently been used in the study of quasiconformal mappings in metric spaces and in geometric group theory, see [Bou05] and [BP03].

Capacities associated with Besov spaces were studied by Netrusov in [Net92] and [Net96], and by Adams and Hurri-Syrjänen in [AHS03]. Bourdon in [Bou05] studied Besov $B_p$-capacity in the metric setting.

We develop a theory of Besov $B_p$-capacity on $X$ and prove that this capacity is a Choquet set function. We also relate Hausdorff measure and Besov capacity when $X$ is an Ahlfors $Q$-regular complete metric space with $Q > 1$ admitting a weak $(1, \tilde{p})$-Poincaré inequality, where $1 \leq \tilde{p} < Q < p < \infty$. Some of the ideas used here follow [KM96], [KM00], [BP03], and [Bou05].

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2. Preliminaries

In this section we present the standard notations to be used throughout this paper. Here and throughout this paper \( B(x, r) = \{ y \in X : d(x, y) < r \} \) is the open ball with center \( x \in X \) and radius \( r > 0 \), \( \overline{B}(x, r) = \{ y \in X : d(x, y) \leq r \} \) is the closed ball with center \( x \in X \) and radius \( r > 0 \), while \( S(x, r) = \{ y \in X : d(x, y) = r \} \) is the closed sphere with center \( x \in X \) and radius \( r > 0 \). For a positive number \( \lambda \), 
\[
\lambda B(a, r) = B(a, \lambda r) \quad \text{and} \quad \lambda \overline{B}(a, r) = \overline{B}(a, \lambda r).
\]

Throughout this paper, \( \mathcal{C} \) will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. \( \mathcal{C}(a, b, \ldots) \) is a constant that depends only on the parameters \( a, b, \ldots \). Here \( \Omega \) will denote a nonempty open subset of \( X \). For \( E \subset X \), the boundary, the closure, and the complement of \( E \) with respect to \( X \) will be denoted by \( \partial E, \overline{E}, \text{and } X \setminus E \), respectively; \( \text{diam } E \) is the diameter of \( E \) with respect to the metric \( d \) and \( E \subset\subset F \) means that \( E \) is a compact subset of \( F \).

For two sets \( A, B \subset X \), we define \( \text{dist}(A, B), \) the distance between \( A \) and \( B \), by
\[
\text{dist}(A, B) = \inf_{a \in A, b \in B} d(a, b).
\]

For \( \Omega \subset X \), \( C(\Omega) \) is the set of all continuous functions \( u : \Omega \rightarrow \mathbb{R} \). Moreover, for a measurable \( u : \Omega \rightarrow \mathbb{R} \), \( \text{supp } u \) is the smallest closed set such that \( u \) vanishes on the complement of \( \text{supp } u \). We also use the spaces
\[
\begin{align*}
C_0(\Omega) &= \{ \varphi \in C(\Omega) : \text{supp } \varphi \subset \subset \Omega \}, \\
\text{Lip}(\Omega) &= \{ \varphi : \Omega \rightarrow \mathbb{R} : \varphi \text{ is Lipschitz} \}, \\
\text{Lip}_{\text{loc}}(\Omega) &= \{ \varphi : \Omega \rightarrow \mathbb{R} : \varphi \text{ is locally Lipschitz} \}, \\
\text{Lip}_0(\Omega) &= \text{Lip}(\Omega) \cap C_0(\Omega).
\end{align*}
\]

Let \( f : \Omega \rightarrow \mathbb{R} \) be integrable. For \( E \subset \Omega \) measurable with \( 0 < \mu(E) < \infty \), we define
\[
f_E = \frac{1}{\mu(E)} \int_E f d\mu(x).
\]

We say that a locally integrable function \( u : X \rightarrow \mathbb{R} \) belongs to \( \text{BMO}(X) \), the space of functions of bounded mean oscillation, if
\[
[u]_{\text{BMO}(X)} = \sup_{a \in X} \sup_{r>0} \frac{1}{\mu(B(a, r))} \int_{B(a, r)} |u - u_{B(a, r)}| \, dx < \infty.
\]

3. Besov spaces

In this section we prove some basic properties of the Besov spaces \( B_p(X) \) and their closed subspaces \( B_p(\Omega) \) and \( B^0_p(\Omega) \), where \( \Omega \subset X \) is an open set. We also present standard lemmas needed for the proofs of our main results.

We know that in the Euclidean case \( B_p(\mathbb{R}^n) \) is a reflexive Banach space and moreover, \( \mathcal{S} \) is dense in \( B_p(\mathbb{R}^n) \) where \( \mathcal{S} = \mathcal{S}(\mathbb{R}^n) \) is the Schwartz class. See [AH96, Theorem 4.1.3] and [Pee76, Chapter 3]. We would like to prove similar results about reflexivity and density when \( (X, d, \mu) \) is an Ahlfors \( Q \)-regular metric space with \( Q > 1 \). It is easy to see that every Lipschitz function with compact support belongs to \( B_p(X) \) whenever \( X \) is proper and unbounded.

We have the following lemma regarding the reflexivity of \( B_p(X) \) when \( (X, d, \mu) \) is an Ahlfors \( Q \)-regular metric space with \( Q > 1 \).
Lemma 3.1. Suppose $1 < Q < p < \infty$ and that $X$ is an Ahlfors $Q$-regular metric space. Then $B_p(X)$ is a reflexive space.

Proof. Let $\nu$ be a measure on the product space $X \times X$ given by

$$d\nu(x, y) = d(x, y)^{-2Q}d\mu(x)d\mu(y).$$

We endow the product space $L^p(X, \mu) \times L^p(X \times X, \nu)$ with the product norm. Namely, for $(u, g) \in L^p(X, \mu) \times L^p(X \times X, \nu)$ we let

$$||(u, g)||_{L^p(X, \mu) \times L^p(X \times X, \nu)} = ||u||_{L^p(X, \mu)} + ||g||_{L^p(X \times X, \nu)}.$$

Clearly this product space is reflexive because it is a product of two reflexive spaces. Since $B_p(X)$ embeds isometrically into a closed subspace of this reflexive product space, we have that $B_p(X)$ is itself a reflexive space. This finishes the proof. \hfill \Box

Lemma 3.2. Suppose $1 < Q < p < \infty$ and that $X$ is an Ahlfors $Q$-regular metric space. There exists a constant $C = C(Q, p, c_\mu)$ such that $[u]_{BMO(X)} \leq C[u]_{B_p(X)}$ whenever $u \in L^1_{loc}(X)$.

Proof. Indeed, let $u \in L^1_{loc}(X)$ be such that $[u]_{B_p(X)} < \infty$. Suppose that $B = B(a, R)$ is a ball in $X$. It is easy to see that there exists a constant $C = C(Q, p, c_\mu)$ such that

$$\frac{1}{\mu(B)} \int_B |u(x) - u_B|^p d\mu(x) \leq \frac{1}{\mu(B)^2} \int_B \int_B |u(x) - u(y)|^p d\mu(x)d\mu(y) \leq C \int_B \int_B |u(x) - u(y)|^p \frac{d\mu(x)d\mu(y)}{d(x, y)^{2Q}}.$$

Therefore,

$$[u]_{BMO(X)} \leq C(Q, p, c_\mu)[u]_{B_p(X)}$$

and the claim follows. \hfill \Box

For an open set $\Omega \subset X$ we define

$$B^0_p(\Omega) = \{u \in B_p(X) : u = 0 \ \mu\textrm{-a.e. in } X \setminus \Omega\}.$$

For a function $u \in B^0_p(\Omega)$ we let $||u||_{B^0_p(\Omega)} = ||u||_{B_p(X)}$.

We notice that $B^0_p(\Omega)$ is a closed subspace of $B_p(X)$ with respect to the Besov norm, hence it is itself a reflexive space.

We define $B^0_p(\Omega)$ as the closure of $\text{Lip}_0(\Omega)$ in $B_p(X)$. Since $\text{Lip}_0(\Omega) \subset B_p(\Omega)$, it follows that $B^0_p(\Omega) \subset B_p(\Omega)$, so we can say that $B^0_p(\Omega)$ is the closure of $\text{Lip}_0(\Omega)$ in $B_p(\Omega)$.

Lemma 3.3. $B^0_p(\Omega)$ is closed under truncations. In particular, bounded functions in $B^0_p(\Omega)$ are dense in $B_p(\Omega)$.

Proof. The proof is very similar to the proof of [Cos, Lemma 2.1] and omitted. \hfill \Box

For a measurable function $u : \Omega \to \mathbb{R}$, we let $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$.

Lemma 3.4. If $u_j \to u$ in $B^0_p(\Omega)$ and $v_j \to v$ in $B_p(\Omega)$, then $\min(u_j, v_j) \to \min(u, v)$ in $B_p(\Omega)$.

Proof. The proof is similar to the proof of [Cos, Lemma 2.2] and omitted. \hfill \Box

Next we show that the space $B^0_p(\Omega)$ is a lattice.
Lemma 3.5. If $u, v \in B^0_p(\Omega)$, then $\min(u, v)$ and $\max(u, v)$ are in $B^0_p(\Omega)$. Moreover, if $u \in B^0_p(\Omega)$ is nonnegative, then there is a sequence of nonnegative functions $\varphi_j \in Lip_0(\Omega)$ converging to $u$ in $B_p(\Omega)$.

Proof. It is enough to show, due to Lemma 3.4, that $u^+$ is in $B^0_p(\Omega)$ whenever $u$ is in $\text{Lip}_0(\Omega)$. But this is immediate, because $u^+ \in \text{Lip}_0(\Omega)$ whenever $u \in \text{Lip}_0(\Omega)$. This finishes the proof. □

Lemma 3.6. Let $\varphi$ be a Lipschitz function with compact support in $X$. If $u \in B_p(X)$, then $u\varphi \in B_p(X)$ with

$$\|u\varphi\|_{B_p(X)} \leq C \|u\|_{B_p(X)},$$

where $C$ depends on $Q, p, c_\mu$, the Lipschitz constant of $\varphi$, and the diameter of $\text{supp} \varphi$.

Proof. Let $R$ be the diameter of $\text{supp} \varphi$. We choose $x_0 \in \text{supp} \varphi$ such that $\text{supp} \varphi \subset \overline{B}$, where $B = B(x_0, R)$. Let $L > 0$ be a constant such that $|\varphi(x) - \varphi(y)| \leq L d(x, y)$ for every $x, y \in X$. Note that $||\varphi||_{L^\infty(X)} \leq LR$. We also notice that

$$\|u\varphi\|_{L^p(X)} \leq \|\varphi\|_{L^\infty(X)} \|u\|_{L^p(X)},$$

and $u\varphi \in L^p(X)$. Observe that

$$\int_X \int_X \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x, y)^{2q}} d\mu(x) \, d\mu(y) = I_1 + 2I_2,$$

where

$$(6) \quad I_1 = \int_{2B} \int_{2B} \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x, y)^{2q}} d\mu(x) \, d\mu(y)$$

and

$$(7) \quad I_2 = \int_{2B} \int_{X \setminus 2B} \frac{|u(x)\varphi(x) - u(y)\varphi(y)|^p}{d(x, y)^{2q}} d\mu(x) \, d\mu(y).$$

For every $x, y \in X$ we have

$$|u(x)\varphi(x) - u(y)\varphi(y)| \leq |u(x) - u(y)| |\varphi(x)| + |u(y)| |\varphi(x) - \varphi(y)|.$$

Therefore

$$(8) \quad I_1 \leq 2^p (||\varphi||_{L^\infty(X)}^p \|u\|_{B_p(X)}^p + I_{11}),$$

where

$$I_{11} = \int_{2B} \int_{2B} \frac{|u(y)|^p |\varphi(x) - \varphi(y)|^p}{d(x, y)^{2q}} d\mu(x) \, d\mu(y).$$

From the definition of $I_{11}$ we have, since $\varphi$ is Lipschitz with constant $L$,

$$(9) \quad I_{11} \leq \int_{2B} \int_{2B} \frac{L^p |u(y)|^p}{d(x, y)^{2Q-p}} d\mu(x) \, d\mu(y)$$

$$= L^p \int_{2B} |u(y)|^p \left( \int_{2B} \frac{d(x, y)^{p-2Q}}{d\mu(x)} \right) d\mu(y).$$

We have

$$(10) \quad \int_{2B} |x - y|^{p-2Q} d\mu(x) \leq C(Q, p, c_\mu) R^{p-Q}.$$
for every $y \in 2B$, where we recall that $R$ is the radius of $B$. From (9) and (10) we get

\begin{equation}
I_{11} \leq C(Q, p, c_\mu) L_p R^{p-Q} \int_{2B} |u(y)|^p d\mu(y)
\end{equation}

\[ \leq C(Q, p, c_\mu) L_p R^{p-Q} \|u\|_{L^p(X)}^p. \]

Since $\varphi$ is supported in $B$, it follows from the definition of $I_2$ that

\[ I_2 = \int_B \int_{X \setminus 2B} \frac{|u(y)|^p |\varphi(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y). \]

Hence

\[ I_2 \leq \|\varphi\|_{L^\infty(X)}^p \int_B \int_{X \setminus 2B} \frac{|u(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y) \]

and since $d(x, y) \geq \frac{d(x, x_0)}{2}$ whenever $x \in X \setminus 2B$ and $y \in B$, we get

\[ I_2 \leq 2^{2Q} \|\varphi\|_{L^\infty(X)}^p \int_B \int_{X \setminus 2B} \frac{|u(y)|^p}{d(x, y)^{2Q}} d\mu(x) \]

Hence

\begin{equation}
I_2 \leq C(Q, p, c_\mu) \|\varphi\|_{L^\infty(X)}^p R^{-Q} \int_B |u(y)|^p d\mu(y)
\end{equation}

\[ \leq C(Q, p, c_\mu) \|\varphi\|_{L^\infty(X)}^p R^{-Q} \|u\|_{L^p(X)}^p. \]

From (8), (11), (12), and the fact that $I = I_1 + 2I_2$, we get that $u\varphi \in B_p(X)$ with

\begin{equation}
\|u\varphi\|_{B_p(X)} \leq C\|u\|_{B_p(X)},
\end{equation}

where the constant $C$ is as required. This finishes the proof.

**Lemma 3.7.** Let $\varphi$ be a Lipschitz function with compact support in $X$. Suppose $u_k$ is a sequence in $B_p(X)$ converging to $u$ in $B_p(X)$. Then $u_k \varphi$ converges to $u \varphi$ in $B_p(X)$.

**Proof.** From Lemma 3.6, we have that $u_k \varphi \in B_p(X)$ for every $k \geq 1$ and $u \varphi \in B_p(X)$. Moreover, Lemma 3.6 implies

\begin{equation}
\|u_k \varphi - u \varphi\|_{B_p(X)} \leq C\|u_k - u\|_{B_p(X)}
\end{equation}

for every $k \geq 1$, and since $u_k \rightarrow u$ in $B_p(X)$, it follows that $u_k \varphi \rightarrow u \varphi$ in $B_p(X)$. This finishes the proof.

**Remark 3.8.** Let $\Omega, \tilde{\Omega}$ be bounded and open subsets of $X$ with $\Omega \subset \subset \tilde{\Omega}$. Suppose that $\varphi$ is a function in $Lip_0(\tilde{\Omega})$ with Lipschitz constant $C(Q, c_\mu)/\text{dist}(\Omega, X \setminus \tilde{\Omega})$ such that

\begin{equation}
0 \leq \varphi \leq 1 \text{ and } \varphi = 1 \text{ in } \Omega.
\end{equation}

By an argument similar to the one from Lemma 3.6, one can show that $u \varphi \in B_p(\tilde{\Omega})$ whenever $u \in B_p(X)$ and $\varphi \in Lip_0(\tilde{\Omega})$ satisfies (15). Moreover, in this case

\[ \|u \varphi\|_{B_p(\tilde{\Omega})} \leq C\|u\|_{B_p(X)} \]

for all $u \in B_p(X)$ and the constant $C > 0$ can be chosen to depend only on $Q$, $p$, $c_\mu$, $\text{dist}(\Omega, X \setminus \tilde{\Omega})$, and the diameter of $\tilde{\Omega}$.
Remark 3.9. It is easy to see that \( u \varphi \in B_p(X) \) whenever \( u, \varphi \) are bounded functions in \( B_p(X) \). Moreover,

\[
\|u\varphi\|_{L^p(X)} \leq \min(\|u\|_{L^\infty(X)}\|\varphi\|_{L^p(X)}, \|\varphi\|_{L^\infty(X)}\|u\|_{L^p(X)})
\]

and

\[
[u\varphi]_{B_p(X)} \leq \|u\|_{L^\infty(X)}[\varphi]_{B_p(X)} + \|\varphi\|_{L^\infty(X)}[u]_{B_p(X)}.
\]

Lemma 3.10. Let \( B = B(x_0, R) \subset X \) and \( \eta \) be a \( C(c_\mu)/R \)-Lipschitz function supported in \( 2B \) such that \( 0 \leq \eta \leq 1 \). Then there exists a constant \( C = C(Q, p, c_\mu) \) such that

\[
[\eta(v - v_B)]_{B_p(X)} \leq C[v]_{B_p(X)}
\]

whenever \( v \in L^1_{loc}(X) \) with \([v]_{B_p(X)} < \infty\).

Proof. Let \( v \in L^1_{loc}(X) \) such that \([v]_{B_p(X)} < \infty\). Then \( v \in L^p_{loc}(X) \) and this implies, since \( \eta \in L^{p_0}(2B) \), that \( \eta(v - v_B) \in L^p(X) \). We repeat to some extent the argument of Lemma 3.6 with \( \varphi = \eta \) and \( u = v - v_B \). We can choose \( L = \frac{C(c_\mu)}{R} \) and we note that \( \|\eta\|_{L^\infty(X)} \leq 1 \). Hence

\[
\int_X \int_X \frac{|u(x)\eta(x) - u(y)\eta(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y) = I_1 + 2I_2,
\]

where

\[
I_1 = \int_{4B} \int_{4B} \frac{|u(x)\eta(x) - u(y)\eta(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y)
\]

and

\[
I_2 = \int_{4B} \int_{X \setminus 4B} \frac{|\eta(x)u(x) - \eta(y)u(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y)
\]

We notice that \( I_1 \leq 2^p(I_{10} + I_{11}) \), where

\[
I_{10} = \int_{4B} \int_{4B} \frac{|\eta(y)(u(x) - u(y))|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y)
\]

and

\[
I_{11} = \int_{4B} \int_{4B} \frac{|u(x)(\eta(x) - \eta(y))|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y).
\]

We have

\[
I_{10} \leq \int_{4B} \int_{4B} \frac{|u(x) - u(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y) \leq [v]_{B_p(X)}^p
\]

since \( \|\eta\|_{L^\infty(X)} \leq 1 \). As in (11) we get with \( L = \frac{C(c_\mu)}{R} \)

\[
I_{11} \leq C(Q, p, c_\mu) R^{-Q} \int_{4B} |v(y) - v_B|^p d\mu(y).
\]

Because \( \eta \) is supported in \( 2B \), it follows from the definition of \( I_2 \) that in fact

\[
I_2 = \int_{2B} \int_{X \setminus 4B} \frac{|v(y) - v_B|^p |\eta(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y).
\]

As in Lemma 3.6 we get

\[
I_2 \leq C(Q, p, c_\mu) R^{-Q} \int_{2B} |v(y) - v_B|^p d\mu(y).
\]
From (16), (17), (18), (19), and the fact that \( I_1 \leq 2^p(I_{10} + 2I_{11}) \), we have that
\[
\eta(v - v_B) \in B_p(X)
\]
with
\[
[v]_B^p(x) \leq C(Q, p, \mu) \int_{-\infty}^{\infty} |v(x) - v(y)|^p \frac{d\mu(x) d\mu(y)}{d(x, y)^{2Q}}
\]
for every \( x \), such that
\[
\sum_{i=1}^{\infty} \chi_{6B(x_i, \varepsilon)} < c_0 = c_0(Q, \mu).
\]

Now we choose a sequence of \( (20) \) bounded overlap, and form a \( \phi \) cover as in [KL02]. Here \( c_1 = c_1(c_\mu) \). More precisely, we choose a family of balls \( B(x_i, \varepsilon), i = 1, 2, \ldots \), such that
\[
X \subset \bigcup_{i=1}^{\infty} B(x_i, \varepsilon)
\]
and
\[
\sum_{i=1}^{\infty} \chi_{6B(x_i, \varepsilon)} < c_0 = c_0(Q, \mu).
\]

Now we choose a sequence of \( (21) \phi \) -Lipschitz functions \( \varphi_i, i = 1, 2, \ldots \), such that
\[
0 \leq \varphi_i \leq 1, \varphi_i = 0 \text{ on } X \setminus 6B(x_i, \varepsilon), \varphi_i \geq 1/c_0 \text{ on } 3B(x_i, \varepsilon),
\]
where \( c_0 \) is the constant from (20) and such that
\[
\sum_{i=1}^{\infty} \varphi_i = 1
\]
on \( X \). We define the approximation by setting
\[
u \varepsilon(x) = \sum_{i=1}^{\infty} \varphi_i(x) u_{3B(x_i, \varepsilon)}
\]
for every \( x \in X \). Then \( \nu \varepsilon \) is a locally Lipschitz function.

(i) We note that
\[
u \varepsilon(x) - u(x) = \sum_{i=1}^{\infty} \varphi_i(x) (u_{3B(x_i, \varepsilon)} - u(x))
\]
for every \( x \in X \). From this and (20) we obtain
\[
[u \varepsilon - u]_B^p(X) \leq (2c_0)^p \sum_{i=1}^{\infty} [\varphi_i(u_{3B(x_i, \varepsilon)} - u)]_B^p(X),
\]
where \( c_0 \) is the bounded overlap constant appearing in (20). However, from Lemma 3.10 there exists a constant \( C = C(Q, p, \mu) \) such that
\[
[\varphi_i(u_{3B(x_i, \varepsilon)} - u)]_B^p(X) \leq C \int_{12B(x_i, \varepsilon)} \int_{12B(x_i, \varepsilon)} |u(x) - u(y)|^p \frac{d\mu(x) d\mu(y)}{d(x, y)^{2Q}}
\]
for every $i = 1, 2, \ldots$. From this and (21) we obtain

\begin{equation}
[u_\varepsilon - u]^p_{L_p(X)} \leq C \sum_{i=1}^{\infty} \int_{12B(x_i,\varepsilon)} \int_{12B(x_i,\varepsilon)} \frac{|u(x) - u(y)|^p}{d(x,y)^{2q}} \, d\mu(x) \, d\mu(y),
\end{equation}

where $C = C(Q,p,c_\mu)$. If we denote

$$A_\varepsilon = \{(x,y) \in X \times X : d(x,y) < 24\varepsilon\},$$

we have from (20) and (22) that

$$[u_\varepsilon - u]^p_{L_p(X)} \leq C(Q,p,c_\mu) \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{2q}} \chi_{A_\varepsilon}(x,y) \, d\mu(x) \, d\mu(y).$$

An application of Lebesgue Dominated Convergence Theorem yields $[u_\varepsilon - u]_{L_p(X)} \to 0$ as $\varepsilon \to 0$. Moreover, we also notice that $[u_\varepsilon]_{L_p(X)} \leq C(Q,p,c_\mu)[u]_{L_p(X)}$ for every $\varepsilon > 0$.

(ii) By using (20) and the fact that $\varphi_i$ forms a partition of unity we obtain, via an argument similar to the one from Lemma 3.2

\begin{equation}
\left\| u_\varepsilon - u \right\|_{L^p(X)} \leq (c_0)^p \sum_{i=1}^{\infty} \left\| \varphi_i(u_{3B(x_i,\varepsilon)} - u) \right\|_{L^p(X)}^p
\end{equation}

\begin{equation}
\leq (c_0)^p \sum_{i=1}^{\infty} \int_{3B(x_i,\varepsilon)} \left| u(x) - u_{3B(x_i,\varepsilon)} \right|^p \, d\mu(x)
\end{equation}

\begin{equation}
\leq C(Q,p,c_\mu)\varepsilon^q \int_X \int_X \frac{|u(x) - u(y)|^p}{d(x,y)^{2q}} \, d\mu(x) \, d\mu(y),
\end{equation}

where $c_0$ is the constant from (20). This implies immediately that $\left\| u_\varepsilon - u \right\|_{L^p(X)} \to 0$ as $\varepsilon \to 0$. This finishes the proof.

\begin{proposition}
$Lip_0(X)$ is dense in $B_p(X)$.
\end{proposition}

\begin{proof}
Let $u \in B_p(X)$. Without loss of generality we can assume that $u$ is locally Lipschitz and in particular bounded. We fix $x_0 \in X$. For every integer $k \geq 2$, we define $\varphi_k : X \to \mathbb{R}$ by

$$\varphi_k(x) = \begin{cases} 
1 & \text{if } 0 \leq d(x,x_0) \leq k, \\
\frac{\ln \frac{\pi^2}{\pi^2 + x_0^2}}{\ln k} & \text{if } k < d(x,x_0) \leq k^2, \\
0 & \text{if } d(x,x_0) > k^2.
\end{cases}$$

Then $\varphi_k \in B_p(X)$ and moreover, $[\varphi_k]_{L_p(X)}^p \leq C(\ln k)^{1-p}$. (See (24).)

Let $u_k = u\varphi_k$. Then $u_k \in Lip_0(X)$ and

$$\left\| u - u_k \right\|_{L^p(X)} \leq \left\| u \chi_{X \setminus B(x_0,k)} \right\|_{L^p(X)} \to 0 \text{ as } k \to \infty.$$ 

We also have

$$[u - u_k]_{B_p(X)} \leq \left( \int_X \int_X \frac{(1 - \varphi_k(y))^p |u(x) - u(y)|^p}{d(x,y)^{2q}} \, d\mu(x) \, d\mu(y) \right)^{1/p}
\end{equation}

as $k \to \infty$. This finishes the proof.

\begin{lemma}
Let $v \in B_p(\Omega)$.
\begin{enumerate}
  \item If $\text{supp } v \subseteq \Omega$, then $v \in B^0_p(\Omega)$.
  \item If $u \in B^0_p(\Omega)$ and $0 \leq v \leq u$ in $X$, then $v \in B^0_p(\Omega)$.
\end{enumerate}
\end{lemma}
Proof. The proof is similar to the proof of [Cos, Lemma 2.10] and omitted. □

Lemma 3.14. Suppose that $\Omega \subset X$. Let $u \in B_p(\Omega)$ such that $u = 0$ on $X \setminus \Omega$ and $\lim_{\Omega \ni x \to y} u(x) = 0$ for all $y \in \partial \Omega$. Then $u \in B_p^0(\Omega)$.

Proof. The proof is similar to the proof of [Cos, Lemma 2.11] and omitted. □

4. RELATIVE BESOV CAPACITY

In this section, we establish a general theory of relative Besov capacity and study how this capacity is related to Hausdorff measures.

For $E \subset \Omega$ we define
\[ BA(E, \Omega) = \{ u \in B_p(\Omega); u \geq 1 \text{ on a neighborhood of } E \}. \]
We call $BA(E, \Omega)$ the set of admissible functions for the condenser $(E, \Omega)$. The relative Besov $p$-capacity of the pair $(E, \Omega)$ is denoted by
\[ \text{cap}_{B_p}(E, \Omega) = \inf \{ [u]_{B_p(\Omega)} : u \in BA(E, \Omega) \}. \]
If $BA(E, \Omega) = \emptyset$, we set $\text{cap}_{B_p}(E, \Omega) = \infty$.

Since $B_p^0(\Omega)$ is closed under truncations and the truncation does not increase the $B_p$-seminorm, we may restrict ourselves to those admissible functions $u$ for which $0 \leq u \leq 1$.

Remark 4.1. If $K$ is a compact subset of the bounded and open set $\Omega \subset X$, we get the same Besov $B_p$-capacity for $(K, \Omega)$ if we restrict ourselves to a smaller set of admissible functions, namely
\[ BW(K, \Omega) = \{ u \in \text{Lip}_0(\Omega); u = 1 \text{ in a neighborhood of } K \}. \]
Indeed, let $u \in BA(K, \Omega)$; we may clearly assume that $u = 1$ in a neighborhood $U \subset \subset \Omega$ of $K$. Then we choose a cut-off Lipschitz function $\eta$, $0 \leq \eta \leq 1$ such that $\eta = 1$ in $X \setminus U$ and $\eta = 0$ in a neighborhood $\tilde{U}$ of $K$, $\tilde{U} \subset \subset U$. Now, if $\varphi_j \in \text{Lip}_0(\Omega)$ is a sequence converging to $u$ in $B_p^0(\Omega)$, then $\psi_j = 1 - \eta(1 - \varphi_j)$ is a sequence belonging to $BW(K, \Omega)$ which converges to $1 - \eta(1 - u)$ in $B_p^0(\Omega)$. (See Lemma 3.7.) But $1 - \eta(1 - u) = u$. This establishes the assertion, since $BW(K, \Omega) \subset BA(K, \Omega)$. In fact, it is easy to see that if $K \subset \Omega$ is compact we get the same Besov $B_p$-capacity if we consider
\[ BW(K, \Omega) = \{ u \in \text{Lip}_0(\Omega); u = 1 \text{ on } K \}. \]
It is also useful to observe that if $\psi \in B_p^0(\Omega)$ is such that $\varphi - \psi \in B_p(\Omega \setminus K)$ for some $\varphi \in BW(K, \Omega)$, then
\[ \text{cap}_{B_p}(K, \Omega) \leq [\psi]_{B_p(\Omega)}^{p}. \]

4.1. Basic properties of the relative Besov capacity. A capacity is a monotone, subadditive set function. The following theorem expresses, among other things, that this is true for the relative Besov $p$-capacity.

Theorem 4.2. Suppose $(X, d, \mu)$ is a proper and unbounded Ahlfors $Q$-regular metric space with $1 < Q < p < \infty$. Let $\Omega \subset X$ be a bounded open set. The set function $E \mapsto \text{cap}_{B_p}(E, \Omega)$, $E \subset \Omega$, enjoys the following properties:
(i) If $E_1 \subset E_2$, then $\text{cap}_{B_p}(E_1, \Omega) \leq \text{cap}_{B_p}(E_2, \Omega)$.
(ii) If $\Omega_1 \subset \Omega_2$ are open, bounded, and $E \subset \Omega_1$, then
\[ \text{cap}_{B_p}(E, \Omega_2) \leq \text{cap}_{B_p}(E, \Omega_1). \]
Proof. The proof is very similar to the proof of [Cos, Theorem 3.1] and is therefore omitted. □

A set function that satisfies properties (i), (iv), (v) and (vi) is called a Choquet capacity (relative to Ω). We may thus invoke an important capacitability theorem of Choquet and state the following result. See [Doo84, Appendix II].

**Theorem 4.3.** Suppose $(X, d, \mu)$ is a metric measure space as in Theorem 4.2. Suppose that Ω is a bounded open set in X. The set function $E \mapsto \text{cap}_{B_p}(E, \Omega)$, $E \subset \Omega$, is a Choquet capacity. In particular, all Borel subsets (in fact, all analytic) subsets $E$ of $\Omega$ are capacitable, i.e.,

$$\text{cap}_{B_p}(E, \Omega) = \sup \{ \text{cap}_{B_p}(K, \Omega) : K \subset E \text{ compact} \}$$

whenever $E \subset \Omega$ is analytic.

4.2. Upper estimates for the relative Besov capacity. Next we derive some upper estimates for the relative Besov capacity. Similar estimates have been obtained earlier by Bourdon in [Bou05]. We follow his methods.

**Theorem 4.4.** Let $(X, d, \mu)$ be a metric measure space as in Theorem 4.2. There exists a constant $C = C(Q, p, c_{\mu}) > 0$ depending only on $Q$, $p$ and $c_{\mu}$ such that

$$\text{cap}_{B_p}(B(x_0, r), B(x_0, R)) \leq C \left( \ln \frac{R}{r} \right)^{1-p}$$

for every $0 < r < \frac{R}{2}$ and every $x_0 \in X$.

Proof. We use the function $u : X \to \mathbb{R}$,

$$u(x) = \begin{cases} 
1 & \text{if } 0 \leq d(x, x_0) \leq r, \\
\frac{\ln \frac{d(x, x_0)}{r}}{\ln \frac{R}{r}} & \text{if } r < d(x, x_0) < R, \\
0 & \text{if } d(x, x_0) \geq R.
\end{cases}$$

Then $u \in B_p(X)$ because it is Lipschitz with compact support. Since $u$ is continuous on $X$ and 0 outside $B(x_0, R)$, we have in fact from Lemma 3.14 that $u \in B^0_p(B(x_0, R))$. In fact $u \in BA(B(x_0, r), B(x_0, R))$ since $u = 1$ on $B(x_0, r)$. Let $v(x) = \ln \frac{R}{r} u(x)$. We will get an upper bound for $[v]_{B_p(B(x_0, R))}$. Let $k \geq 3$ be the smallest integer such that $2^{k-1}r \geq R$. For $i = 1, \ldots, k$ we define $B_i = B(x_0, 2^i r) \setminus \overline{B}(x_0, 2^{i-1}r)$. We also define $B_0 = B(x_0, r)$ and $B_{k+1} = X \setminus B(x_0, 2^k r)$. We have

$$[v]_{B_p(B(x_0, R))} = \sum_{0 \leq i,j \leq k+1} I_{i,j} = \sum_{0 \leq i,j \leq k+1} \int_{B_i} \int_{B_j} \frac{|v(x) - v(y)|^p}{d(x, y)^{2Q}} d\mu(x) d\mu(y).$$
Obviously we have $I_{i,j} = I_{j,i}$. We majorize $I_{i,j}$ by distinguishing a few cases. For $j \leq k$ and $0 \leq i \leq j - 2$ we have from the definition of $v$ that $|v(x) - v(y)| \leq j - i + 1$ whenever $x \in B_i$ and $y \in B_j$, hence

$$I_{i,j} \leq C_0 (j - i + 1)^p (2^r)^{-2Q} (2^r)^Q (2^r)^Q,$$

that is $I_{i,j} \leq C_1 (j - i)^p 2^{(i-j)Q}$. For $0 \leq i \leq j \leq k$ we notice, since $v$ is $\frac{1}{2^r}$-Lipschitz on $\bigcup_{j \geq 1} B_j$ that

$$I_{i,j} \leq (2^{i-1}r)^{-p} \int_{B_i} \int_{B_j} \frac{1}{d(x,y)^{2Q-p}} d\mu(x) d\mu(y).$$

Moreover, we have

$$\int_{B_i} \frac{1}{d(x,y)^{2Q-p}} d\mu(x) \leq C_2 (\text{diam } B_j)^{p-Q}$$

for every $y \in B(x_0, 2^r)$, where $C_2$ depends only on $p$, $Q$ and $c_\mu$. Hence for $0 \leq i \leq j \leq k$ we have

$$I_{i,j} \leq C_3 (2^{i-1}r)^{-p} (2^r)^Q (2^r)^p \leq C_4 2^{(i-j)(p-Q)}.$$

In particular, for $j - 1 \leq i \leq j \leq k$, the integral $I_{i,j}$ is bounded by a constant that depends only on $p$, $Q$ and $c_\mu$. Now we have to bound $I_{i,j}$ when $j = k + 1$. Since $v$ is constant on $B_k \cup B_{k+1}$, we have $I_{i,k+1} = 0$ for $i \in \{k, k+1\}$. For $0 \leq i \leq k - 1$ we have

$$I_{i,k+1} \leq (k - i + 1)^p \int_{B_i} \int_{B_{k+1}} \frac{1}{d(x,y)^{2Q}} d\mu(x) d\mu(y).$$

But there exists $C_5 > 0$ such that

$$\int_{B_{k+1}} \frac{1}{d(x,y)^{2Q}} d\mu(x) \leq C_5 (2^{k+1}r)^{-Q}$$

for every $y \in X$ with $d(y, x_0) \leq 2^{k+1}r$. Hence $I_{i,k+1} \leq C_6 (k - i + 1)^p 2^{(i-k-1)Q}$. Finally we have

$$[v]_{B_p(B(x_0,R))}^p \leq C_7 k + C_8 \sum_{0 \leq i \leq j \leq k+1} (j - i)^p 2^{(i-j)Q}.$$ 

The last sum is equal to

$$\sum_{l=1}^{k+1} (k + 1 - l) l^{p-2} 2^{-lQ}.$$

But $k + 1 - l \leq k + 1$ and there exists $a > 1$ such that $l^{p-2} 2^{-lQ} \leq C_9 a^{-l}$ for $l \geq 1$. Hence

$$[v]_{B_p(B(x_0,R))}^p \leq C_{10} \ln \frac{R}{r}$$

and

$$[u]_{B_p(B(x_0,R))}^p \leq C_{10} \left( \ln \frac{R}{r} \right)^{1-p}.$$ 

The claim follows with $C = C_{10}$. \hfill $\square$

For a fixed $r > 0$ we construct the dyadic partition of $X$ as in [Chr90, Theorem11]. That is, a family of open sets $\mathcal{D}_r = \{K_{m,r}^\alpha : m \in \mathbb{Z}, \alpha \in I_m \}$ such that

(i) $\mu(X \setminus \bigcup_n K_{m,r}^\alpha) = 0, \forall m$. 
(ii) If $l \geq m$ then either $K_{l,r}^\alpha \subset K_{m,r}^\alpha$ or $K_{l,r}^\alpha \cap K_{m,r}^\alpha = \emptyset$. 
(iii) For each $(m, \alpha)$ and each $l < m$ there is a unique $\beta$ such that $K_{m,r}^\alpha \subset K_{l,r}^\beta$.
(iv) For every \((m, \alpha)\) there exists a ball \(B^\alpha_{m,r} = B(x^\alpha_{m,r}, 10^{-m}r)\) such that
\[
\frac{1}{10} B^\alpha_{m,r} \subset K^\alpha_{m,r} \subset 3B^\alpha_{m,r}.
\]
We call these open sets “dyadic cubes”.

Two distinct dyadic cubes \(K, K'\) in \(D_r\) are \textit{adjacent} if there exists an integer \(k\) such that either
(i) \(K, K'\) are in generation \(k\) and \(K \cap K' \neq \emptyset\), or
(ii) one of the cubes \(K, K'\) is in generation \(k\), the other one is in generation \(k + 1\) the one in generation \(k\) contains the other one.

Similarly, if \(K_0 \subset X\) is a dyadic cube in \(D_r\), we denote by \(D_r(K_0)\) the dyadic subcubes of \(K_0\).

For two adjacent cubes \(K, K' \in D_r\) we have
\[
|f_K - f_{K'}|^p = \left| \frac{1}{\mu(K)} \int_K f(x) \, d\mu(x) - \frac{1}{\mu(K')} \int_{K'} f(y) \, d\mu(y) \right|^p
\]
\[
= \frac{1}{\mu(K) \mu(K')} \int_K \int_{K'} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
\leq C \int_K \int_{K'} \frac{|f(x) - f(y)|^p}{d(x,y)^{2Q}} \, d\mu(x) \, d\mu(y),
\]
where \(C\) is a constant that depends only on the Ahlfors regularity of \(X\).

For the following lemma see [BP03, Lemma 3.5].

\textbf{Lemma 4.5.} There exists a constant \(C\) depending only on the Ahlfors regularity of \(X\) such that
\[
C^{-1} |\eta - \zeta|^{-2Q} \leq \sum_{K,K' \in D_r \text{ adjacent}} \frac{\chi_K(\eta)\chi_{K'}(\zeta)}{\mu(K)\mu(K')} \leq C |\eta - \zeta|^{-2Q}
\]
for \(\mu\)-a.e. \(\eta, \zeta \in X\).

We also have (see [BP03, Theorem 3.4]):

\textbf{Lemma 4.6.} There exists a constant \(C\) depending only on \(p\) and on the Ahlfors regularity of \(X\) such that
\[
C^{-1} [f]_{B^p(X)}^p \leq \sum_{K,K' \in D_r \text{ adjacent}} \frac{1}{\mu(K) \mu(K')} \int_K \int_{K'} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
\leq C [f]_{B^p(X)}^p
\]
for every \(f \in B^p(X)\).

This implies (see [BP03, Lemma 3.5]):

\textbf{Lemma 4.7.} There exists a constant \(C\) depending only on \(p\) and on the Ahlfors regularity of \(X\) such that
\[
\sum_{K,K' \in D_r \text{ adjacent}} |f_K - f_{K'}|^p \leq C [f]_{B^p(X)}^p
\]
for every \(f \in B^p(X)\).
4.3. Hausdorff measure and relative Besov capacity. Now we examine the relationship between Hausdorff measures and the $B_p$-capacity. Let $h$ be a real-valued and increasing function on $[0, \infty)$ such that $\lim_{t \to 0} h(t) = h(0) = 0$ and $\lim_{t \to \infty} h(t) = \infty$. Such a function $h$ is called a measure function. Let $0 < \delta \leq \infty$. Suppose $\Omega \subset X$ is open. For $E \subset \overline{\Omega}$ we define

$$\Lambda^\delta_{\bar{h},\bar{\Omega}}(E) = \inf \sum_i h(r_i),$$

where the infimum is taken over all coverings of $E$ by open sets $G_i$ in $\overline{\Omega}$ with diameter $r_i$ not exceeding $\delta$. The set function $\Lambda^\infty_{\bar{h},\bar{\Omega}}$ is called the $h$-Hausdorff content relative to $\Omega$. Clearly $\Lambda^\delta_{\bar{h},\bar{\Omega}}$ is an outer measure for every $\delta \in (0, \infty]$ and every open set $\Omega \subset X$. We write $\Lambda^\delta_{\bar{h}}(E)$ for $\Lambda^\delta_{\bar{h},\bar{\Omega}}(E)$.

Moreover, for every $E \subset \overline{\Omega}$, there exists a Borel set $\tilde{E}$ such that $E \subset \tilde{E} \subset \overline{\Omega}$ and $\Lambda^\delta_{\bar{h},\bar{\Omega}}(E) = \Lambda^\delta_{\bar{h},\bar{\Omega}}(\tilde{E})$. Clearly $\Lambda^\delta_{\bar{h},\bar{\Omega}}(E)$ is a decreasing function of $\delta$. It is easy to see that $\Lambda^\delta_{\bar{h},\bar{\Omega}_2}(E) \leq \Lambda^\delta_{\bar{h},\bar{\Omega}_1}(E)$ for every $\delta \in (0, \infty]$ whenever $\Omega_1$ and $\Omega_2$ are open sets in $X$ such that $E \subset \overline{\Omega}_1 \subset \overline{\Omega}_2$. This allows us to define the $h$-Hausdorff measure relative to $\overline{\Omega}$ of $E \subset \overline{\Omega}$ by

$$\Lambda_{\bar{h},\bar{\Omega}}(E) = \sup_{\delta > 0} \Lambda^\delta_{\bar{h},\bar{\Omega}}(E) = \lim_{\delta \to 0} \Lambda^\delta_{\bar{h},\bar{\Omega}}(E).$$

The measure $\Lambda_{\bar{h},\bar{\Omega}}$ is Borel regular; that is, it is an additive measure on Borel sets of $\overline{\Omega}$ and for each $E \subset \overline{\Omega}$ there is a Borel set $G$ such that $E \subset G \subset \overline{\Omega}$ and $\Lambda_{\bar{h},\bar{\Omega}}(E) = \Lambda_{\bar{h},\bar{\Omega}}(G)$. (See [Fed69, p. 170] and [Mat95, Chapter 4].) If $h(t) = t^s$, we write $\Lambda_s$ for $\Lambda_{t^s,\bar{\Omega}}$. It is immediate from the definition that $\Lambda_u(E) < \infty$ implies $\Lambda_u(E) = 0$ for all $u > s$. The smallest $s \geq 0$ that satisfies $\Lambda_s(E) = 0$ for all $u > s$ is called the Hausdorff dimension of $E$.

For $\Omega \subset X$ open and $\delta > 0$ the set function $\Lambda^\delta_{\bar{h},\bar{\Omega}}$ has the following property:

(i) If $K_i$ is a decreasing sequence of compact sets in $\overline{\Omega}$, then

$$\Lambda^\delta_{\bar{h},\bar{\Omega}}(\bigcap_{i=1}^\infty K_i) = \lim_{i \to \infty} \Lambda^\delta_{\bar{h},\bar{\Omega}}(K_i).$$

Moreover, if $\Omega \subset \subset X$ and $h$ is a continuous measure function, then $\Lambda^\delta_{\bar{h},\bar{\Omega}}$ satisfies the following additional properties:

(ii) If $E_i$ is an increasing sequence of arbitrary sets in $\overline{\Omega}$, then

$$\Lambda^\delta_{\bar{h},\bar{\Omega}}(\bigcup_{i=1}^\infty E_i) = \lim_{i \to \infty} \Lambda^\delta_{\bar{h},\bar{\Omega}}(E_i).$$

(iii) $\Lambda^\delta_{\bar{h},\bar{\Omega}}(E) = \sup \{ \Lambda^\delta_{\bar{h},\bar{\Omega}}(K) : K \subset E \text{ compact} \}$ whenever $E \subset \overline{\Omega}$ is a Borel set. (See [Rog70, Chapter 2:6].)

We have the following proposition:

**Proposition 4.8.** Suppose $(X, d, \mu)$ is an Ahlfors $Q$-regular metric space with $Q > 1$. Let $h : [0, \infty) \to [0, \infty)$ be a measure function.

(a) If $\lim_{t \to 0} h(t)t^{-Q} = 0$, then $\Lambda^\delta_{\bar{h}}(X) = 0$.

(b) If $\lim_{t \to \infty} h(t)t^{-Q} > 0$, then there is an increasing function $h^* : [0, \infty) \to [0, \infty)$ such that $h^*(0) = 0$, $h^*$ is continuous, $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$ is decreasing and there
exists a constant $C = C(Q, c_\mu)$ such that for all $E \subset X$ and all $\delta > 0$

$$C^{-1} \Lambda_h^\delta(E) \leq \Lambda_h^{\delta, s}(E) \leq C \Lambda_h^\delta(E).$$

**Proof.** The proof is similar to the proof of [AH96, Proposition 5.1.8] and omitted. □

If $h : [0, \infty) \to [0, \infty)$ is a continuous increasing measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$ is decreasing, we know that $\Lambda_h(E) = 0$ if and only if $\Lambda_h^\infty(E) = 0$. (See [AH96, Proposition 5.1.5].) If $h(t) = t^s$, $0 < s < \infty$, we write $\Lambda_h^s$ for $\Lambda_h^{\delta, s}$.

**Theorem 4.9.** Suppose $1 \leq \tilde{p} < p < \infty$. Let $(X, d, \mu)$ be a complete and unbounded Ahlfors $Q$-regular metric space that supports a weak $(1, \tilde{p})$-Poincaré inequality. Suppose $h : [0, \infty) \to [0, \infty)$ is a continuous increasing measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$ is decreasing. Let $K_{0,r} \in D_r$ be a dyadic cube of generation 0 and let $x_0 \in X$ be such that $B(x_0, r/10) \subset K_{0,r}$. There exists a positive constant $C'_1 = C'_1(Q, p, c_\mu)$ such that

$$\int_0^{t_0^{10^{-kr}}} h(t)t^{p' - 1} dt < \infty$$

for every $E \subset X$, every $k > 1$, $r > 0$, and for every $K_{k,r} \in D_r(K_{0,r})$ cube of generation $k$ such that $B(x_0, 10^{-kr}) \cap K_{k,r} \neq \emptyset$.

**Proof.** We fix $r > 0$ and $k > 1$. Suppose $K_{k,r} \in D_r(K_{0,r})$ is a dyadic subcube of $K_{0,r}$ of generation $k$ such that $\overline{K}_{k,r} \cap B(x_0, 10^{-kr}) \neq \emptyset$.

Let $E \subset X$. From the fact that there exists a Borel set $\tilde{E}$ such that $E \subset \tilde{E} \subset X$ and $\cap_{B_{p^*}}(E \cap \overline{K}_{k,r}, B(x_0, r/10)) = \cap_{B_{p^*}}(\tilde{E} \cap \overline{K}_{k,r}, B(x_0, r/10))$, we can assume that $E$ is a Borel set. Moreover, from the discussion before Proposition 4.8 and the fact that $\cap_{B_{p^*}}(\cdot, B(x_0, r/10))$ is a Choquet capacity, we can assume without loss of generality that $E$ is compact.

There is nothing to prove if either $\Lambda_h^\infty(E \cap \overline{K}_{k,r}) = 0$ or if $\int_0^{t_0^{10^{-kr}}} h(t)t^{p' - 1} dt = \infty$. So we can assume without loss of generality that $\alpha = \Lambda_h^\infty(E \cap \overline{K}_{k,r}) > 0$ and that $\int_0^{t_0^{10^{-kr}}} h(t)t^{p' - 1} dt < \infty$.

For every $\zeta \in S(x_0, r/10)$ there exists a decreasing sequence $(K_{s,\zeta})_{s \leq 0}$ of dyadic subcubes of $K_{0,r}$ such that $K_{s,\zeta}$ is a cube of generation $s$ for every integer $s \leq 0$ and

$$\bigcap_{s \leq 0} \overline{K}_{s,\zeta} = \{\zeta\}.$$

We denote by $s^0_\zeta$ the sequence $(\overline{K}_{s,\zeta})_{s \leq 0}$.

Similarly, for every $\eta \in \overline{K}_{k,r}$ there exists a decreasing sequence $(K_{s+k,\eta})_{s \geq 0}$ of dyadic subcubes of $K_{k,r}$ such that $K_{s+k,\eta}$ is of generation $s + k$ for every $s \geq 0$ and

$$\bigcap_{s \geq 0} \overline{K}_{s+k,\eta} = \{\eta\}.$$

We denote by $s^1_\eta$ the sequence $(\overline{K}_{s+k,\eta})_{s \geq 0}$. Let $I = \{K_{0,r}, \ldots, K_{k,r}\}$ be a shortest sequence of pairwise adjacent cubes connecting $K_{0,r}$ and $K_{k,r}$.

For $(\zeta, \eta) \in S(x_0, r/10) \times \overline{K}_{k,r}$ we define $\gamma_{\zeta, \eta} = (\overline{K}_{s,\zeta, \eta})_{s \in \mathbb{Z}}$, where

$$K_{s, \zeta, \eta} = \begin{cases} K_{s, \zeta} & \text{if } s \leq 0 \\ K_{s, \eta} & \text{if } 0 \leq s \leq k \\ K_{s, \eta} & \text{if } s \geq k. \end{cases}$$
For $K, K' \in D_r$ we define
\[ C(K, K') = \{(\zeta, \eta) \in S(x_0, \frac{r}{10}) \times K_{k,r} : K = K_{s,\zeta,\eta}, K' = K_{s+1,\zeta,\eta} \text{ for some } s \in \mathbb{Z} \}. \]
We notice that $C(K, K') = \emptyset$ if $K, K'$ are not adjacent or if they are adjacent but of the same generation.

Since $X$ is an Ahlfors $Q$-regular complete metric space that satisfies a weak $(1, \tilde{p})$-Poincaré inequality with $1 \leq \tilde{p} < Q$, there exists (see [Kor07, Theorem 4.2]) a constant $C$ depending only on $\tilde{p}$ and on the data of $X$ such that
\[ C^{-1} \lambda^{Q-\tilde{p}} \leq \Lambda_0 \chi_0(S(x, t)) \leq C \lambda^{Q-\tilde{p}} \]
for all closed spheres $S(x, t)$ of radius $t$ in $X$. We also have $\alpha = \Lambda_0 \chi_0(E \cap \bar{K}_{k,r}) > 0$.

Therefore, by applying Frostman’s lemma (see [Mat95, Theorem 8.8]), there exists a constant $C > 0$ and probability measures $\nu_0$ on $S(x_0, r/10)$ and $\nu_1$ on $E \cap \bar{K}_{k,r}$ such that for every ball $B(x, t)$ of radius $t$ in $X$ we have
\[ \nu_0(B(x, t)) \leq C \left( \frac{t}{r} \right)^{Q-\tilde{p}} \] and $\nu_1(B(x, t)) \leq C \frac{h(t)}{\alpha}$.

For $K, K' \in D_r$ we define
\[ m(K, K') = \nu_0 \times \nu_1(C(K, K')). \]
We notice that $m(K, K') m(K', K) = 0$ for every pair of cubes $K, K' \in D_r$. Moreover, if $m(K, K') \neq 0$, then this implies that $K$ and $K'$ are adjacent but of different generations.

Let $f$ be in $BW(E, B(x_0, r/10))$. Then, since $f$ is continuous, we have that
\[ f_{K_v} \to f(y) \]
for every $y \in X$ for every nested sequence $K_v$ of $r$-dyadic cubes containing $y$ and converging to $y$. It follows that
\[ 1 \leq f(\eta) - f(\zeta) \leq \sum_{s \in \mathbb{Z}} (f_{\frac{K_{s+1,\zeta,\eta}}{r}} - f_{\frac{K_{s,\zeta,\eta}}{r}}) \]
whenever $\eta \in E \cap K_{k,r}$ and $\zeta \in S(x_0, r/10)$.

We obtain with the definition of $m(K, K')$ and by Hölder’s inequality, that
\[ 1 \leq \frac{1}{S(x_0, r/10)} \int_{E \cap \bar{K}_{k,r}} \sum_{s \in \mathbb{Z}} (f_{\frac{K_{s+1,\zeta,\eta}}{r}} - f_{\frac{K_{s,\zeta,\eta}}{r}}) d\nu_0(\zeta) d\nu_1(\eta) \leq \frac{1}{S(x_0, r/10)} \int_{E \cap \bar{K}_{k,r}} \sum_{s \in \mathbb{Z}} |f_{\frac{K_{s+1,\zeta,\eta}}{r}} - f_{\frac{K_{s,\zeta,\eta}}{r}}| d\nu_0(\zeta) d\nu_1(\eta) \]
\[ = \sum_{K, K' \in D_r, \text{ adjacent}} |f_K - f_{K'}| m(K, K') \]
\[ \leq \left( \sum_{K, K' \in D_r, \text{ adjacent}} |f_K - f_{K'}|^p \right)^{1/p} \left( \sum_{K, K' \in D_r, \text{ adjacent}} m(K, K')^{p'} \right)^{1/p'} \]
\[ \leq C[f]_{B_p(X)} \left( \sum_{K, K' \in D_r, \text{ adjacent}} m(K, K')^{p'} \right)^{1/p'}, \]
where we used (25) for the last inequality. Here the constant \( C \) depends only on \( p \) and on the Ahlfors regularity of \( X \). For a nonnegative integer \( s \) we let

\[
E_{0,s} = \{(K, K') \in \mathcal{D}_r \times \mathcal{D}_r : K = K_{-s-1, \zeta}, K' = K_{-s, \zeta} \text{ for some } \zeta \in S(x_0, r/10)\}
\]

and similarly

\[
E_{1,s} = \{(K, K') \in \mathcal{D}_r \times \mathcal{D}_r : K = K_{s+k, \eta}, K' = K_{s+k+1, \eta} \text{ for some } \eta \in \mathcal{K}_{k,r}\}.\]

We notice that we can break \( \sum = \sum_{(K, K') \in E_0,s} m(K, K') p' \) into 3 parts, namely

\[
\sum = \sum_{s=0}^{\infty} \sum_{(K, K') \in E_0,s} m(K, K') p' + \sum_{K, K' \in I} m(K, K') p' + \sum_{s=0}^{\infty} \sum_{(K, K') \in E_1,s} m(K, K') p'.
\]

We recall that \( I = \{K_0, \ldots, K_{k,r}\} \) is a shortest sequence of pairwise adjacent cubes in \( \mathcal{D}_r \), connecting \( K_0 \) and \( K_{k,r} \). Thus, the sum in the middle is exactly \( k \). We get upper bounds for the first and the third term in the sum. We notice that for every \( s \geq 0 \) we have

\[
\sum_{(K, K') \in E_0,s} m(K, K') = 1
\]

since \( \nu_0 \times \nu_1 \) is a probability measure. On the other hand, there exists a constant \( C' \) depending only on \( p \) and on the Hausdorff dimension of \( X \) such that

\[
m(K, K') \leq C' \frac{h(10^{-s-k}r)}{\alpha} \text{ for every } (K, K') \in E_1,s
\]

for every integer \( s \geq 0 \) and

\[
m(K, K') \leq C'10^{(p'-Q)s} \text{ for every } (K, K') \in E_0,s
\]

for every integer \( s \geq 0 \).

Therefore

\[
\sum_{s=0}^{\infty} \sum_{(K, K') \in E_1,s} m(K, K') p' = \sum_{s=0}^{\infty} \sum_{(K, K') \in E_1,s} m(K, K') p' - 1 \sum_{s=0}^{\infty} \sum_{(K, K') \in E_0,s} m(K, K')
\]

\[
\leq C \alpha^{1-p'} \sum_{s=0}^{\infty} h(10^{-s-k}r)^{p' - 1} \left( \sum_{(K, K') \in E_1,s} m(K, K') \right).
\]

But there exists a constant \( C_0 = C_0(Q, p) > 1 \) such that

\[
\frac{1}{C_0} \int_0^{10^{-k}r} h(t)^{p' - 1} \frac{dt}{t} \leq \sum_{s=0}^{\infty} h(10^{-k-s}r)^{p' - 1} \leq C_0 \int_0^{10^{-k}r} h(t)^{p' - 1} \frac{dt}{t}
\]

for every \( r > 0 \), every integer \( k > 1 \) and every continuous increasing measure function \( h : [0, \infty) \to [0, \infty) \) such that \( t \mapsto h(t) t^{-Q} \), \( 0 < t < \infty \), is decreasing. Hence

\[
\sum_{s=0}^{\infty} \sum_{(K, K') \in E_1,s} m(K, K') p' \leq C \alpha^{1-p'} \int_0^{10^{-k}r} h(t)^{p' - 1} \frac{dt}{t}.
\]

From a similar computation we get

\[
\sum_{s=0}^{\infty} \sum_{(K, K') \in E_0,s} m(K, K') p' = \sum_{s=0}^{\infty} \sum_{(K, K') \in E_0,s} m(K, K') p' - 1 \sum_{s=0}^{\infty} \sum_{(K, K') \in E_0,s} m(K, K')
\]

\[
\leq C \sum_{s=0}^{\infty} 10^{-(p' - 1)(Q - p)} \left( \sum_{(K, K') \in E_0,s} m(K, K') \right) = C.
\]

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So we get
\[ \sum \leq C' \left( \alpha^{1-p'} \int_0^{\frac{10^{-k_r}}{r}} h(t)^{p'-1} \frac{dt}{t} + k + 1 \right). \]
It is easy to see that there exists a constant $C$ depending only on $p$ and on the Hausdorff dimension of $X$ such that
\[ \frac{\Lambda_h^\infty(K_{k,r})}{\left( \int_0^{10^{-k_r}} h(t)^{p'-1} \frac{dt}{t} \right)^{p-1}} \leq C. \]
for every $r > 0$, every integer $k > 1$ and every continuous increasing measure function $h : [0, \infty) \to [0, \infty)$ such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. Hence
\[ \sum \leq Ck \alpha^{1-p'} \int_0^{10^{-k_r}} h(t)^{p'-1} \frac{dt}{t}. \]
Therefore we obtain
\[ 1 \leq C[f]_{B_p(B(x_0,r/10))} \left( k \alpha^{1-p'} \int_0^{10^{-k_r}} h(t)^{p'-1} \frac{dt}{t} \right)^{1/p'} \]
for every integer $k > 1$ and for every $f \in BW(E \cap K_{k,r}, B(x_0,r/10))$. This implies that there exists a constant $C'_1$ depending only on $p$ and on the Hausdorff dimension of $X$ such that
\[ \frac{\Lambda_h^\infty(E \cap K_{k,r})}{\left( \int_0^{10^{-k_r}} h(t)^{p'-1} \frac{dt}{t} \right)^{p-1}} k^{1-p} \leq C'_1 \text{cap}_{B_p}(E \cap K_{k,r}, B(x_0,r/10)). \]
This finishes the proof. \hfill $\square$

As a consequence of Theorem 4.9, we obtain the following theorem.

**Theorem 4.10.** Suppose $1 \leq \tilde{p} < Q < p < \infty$. Let $(X,d,\mu)$ be a complete and unbounded Ahlfors $Q$-regular metric space as in Theorem 4.9. Suppose $h : [0, \infty) \to [0, \infty)$ is a continuous increasing measure function such that $t \mapsto h(t)t^{-Q}$, $0 < t < \infty$, is decreasing. There exists a positive constant $C_1 = C_1(Q, p, c_\mu)$ such that
\[ \frac{\Lambda_h^\infty(E \cap B(x,r))}{\left( \int_0^{10^{-k_r}} h(t)^{p'-1} \frac{dt}{t} \right)^{p-1}} \leq C_1 \left( \ln \frac{R}{r} \right)^{p-1} \text{cap}_{B_p}(E \cap B(x,r), B(x,R)) \]
for every $E \subset X$, every $x \in X$, and every pair of positive numbers $r, R$ such that $r < \frac{R}{2}$.

**Proof.** Fix $x \in X$ and $r, R$ such that $0 < r < \frac{R}{2}$. Without loss of generality we can assume that $B(x, 100R) \subset K_{0,1000R}$. We choose $k \geq 3$ integer such that $10^{2-k}R \leq r < 10^{3-k}R$. From the construction of the dyadic cubes and the fact that $X$ is a $Q$-Ahlfors regular space with $Q > 1$, it follows that there exists a constant $C = C(Q, c_\mu)$ independent of $k$ such that every ball of radius $10^{2-k}R$ intersects with at most $C$ dyadic subcubes of $K_{0,1000R}$ from the $k$th generation. We leave the rest of the details to the reader. \hfill $\square$

It follows easily that if $X$ is a complete and unbounded Ahlfors $Q$-regular metric space as in Theorem 4.10, then there exists a constant $C = C(Q, p, \tilde{p}, c_\mu)$ such that
\[ \frac{\Lambda_h^\infty(E \cap B(a,R))}{R} \leq C \text{cap}_{B_p}(E \cap B(a,R), B(a,2R)) \]
(28)

\[ \frac{\Lambda^\infty(E \cap B(a,R))}{R} \leq C \text{cap}_{B_p}(E \cap B(a,R), B(a,2R)) \]
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whenever \( E \subset X, R > 0, \) and \( a \in X. \)

As a corollary we have the following.

**Corollary 4.11.** Suppose \( X \) is a complete and unbounded Ahlfors \( Q \)-regular metric space as in Theorem 4.10. There exists a positive constant \( C_2 = C_2(Q, p, \tilde{p}, c_\mu) \) such that

\[
C_2 \left( \ln \frac{R}{r} \right)^{1-p} \leq \text{cap}_{B_p}(B(x, r), B(x, R))
\]

for every \( x \in X \) and every pair of positive numbers \( r, R \) such that \( r < \frac{R}{2}. \)

**Proof.** We apply Theorem 4.10 for \( h(t) = t^{Q-\tilde{p}}. \) We notice (see [Kor07, Theorem 4.2]) that there exists a constant \( C_2' = C_2'(Q, p, \tilde{p}, c_\mu) \) such that

\[
\frac{1}{C_2'} \leq \frac{\Lambda_{Q-\tilde{p}}(B(x, r))}{(\int_0^r t^{(p'-1)(Q-\tilde{p})} \frac{dt}{t})^{p-1}} \leq C_2'
\]

for every \( x \in X \) and every \( r > 0. \) The rest is routine. \( \square \)

Theorem 4.4 and Corollary 4.11 easily yield the following theorem, (cf. [Bou05]).

**Theorem 4.12.** Suppose \( X \) is a complete and unbounded Ahlfors \( Q \)-regular metric space as in Theorem 4.10. There exists \( C_0 = C_0(Q, p, c_\mu) > 0 \) such that

\[
\frac{1}{C_0} \left( \ln \frac{R}{r} \right)^{1-p} \leq \text{cap}_{B_p}(B(x, r), B(x, R)) \leq C_0 \left( \ln \frac{R}{r} \right)^{1-p}
\]

for every \( x \in X \) and every pair of positive numbers \( r, R \) such that \( r < \frac{R}{2}. \)

A set \( E \subset X \) is said to be of Besov \( B_p \)-capacity zero if \( \text{cap}_{B_p}(E \cap \Omega, \Omega) = 0 \) for all open and bounded \( \Omega \subset X. \) In this case we write \( \text{cap}_{B_p}(E) = 0. \) The following lemma is obvious.

**Lemma 4.13.** A countable union of sets of Besov \( B_p \)-capacity zero has Besov \( B_p \)-capacity zero.

The next lemma shows that, if \( E \) is bounded, one needs to test only a single bounded open set \( \Omega \) containing \( E \) in showing that \( E \) has zero Besov \( B_p \)-capacity.

**Lemma 4.14.** Suppose that \( E \) is bounded and that there is a bounded neighborhood \( \Omega \) of \( E \) with \( \text{cap}_{B_p}(E, \Omega) = 0. \) Then \( \text{cap}_{B_p}(E) = 0. \)

**Proof.** The proof is similar to the proof of [Cos, Lemma 3.13] and omitted. \( \square \)

**Corollary 4.15.** Suppose \( X \) is a complete and unbounded Ahlfors \( Q \)-regular metric space as in Theorem 4.10. Let \( E \subset X \) be such that \( \text{cap}_{B_p}(E) = 0. \) Then \( \Lambda_h(E) = 0 \) for every measure function \( h : [0, \infty) \to [0, \infty) \) such that

\[
\int_0^1 h(t)t^{p'-1} \frac{dt}{t} < \infty.
\]

In particular, the Hausdorff dimension of \( E \) is zero and \( X \setminus E \) is connected.

Note that for every \( \varepsilon > 0 \) we can take \( h = h_\varepsilon : [0, \infty) \to [0, \infty) \) in Corollary 4.15, where \( h_\varepsilon(t) = (\ln t)^{1-p-\varepsilon} \) for every \( t \in (0, 1/2). \)
Proof. It is enough to assume, without loss of generality, that \( h : [0, \infty) \to [0, \infty) \) is a continuous measure function such that \( t \mapsto h(t)t^{-Q}, 0 < t < \infty \) is decreasing. (See Proposition 4.8.) If \( \text{cap}_{B_p}(E) = 0 \), then there exists a Borel set \( \tilde{E} \) such that \( E \subset \tilde{E} \) and \( \text{cap}_{B_p}(\tilde{E}) = 0 \), hence we can assume without loss of generality that \( E \) is itself Borel. Since \( \Lambda_h \) is a Borel regular measure and \( \Lambda_h(E) = 0 \) if and only if \( \Lambda^\infty_h(E) = 0 \), it is enough to assume that \( E \) is in fact compact. For \( E \) compact the claim follows obviously from Theorem 4.10.

The second claim is a consequence of the first claim because for every \( s \in (0, Q) \), the function \( h_s : [0, \infty) \to [0, \infty) \) defined by \( h_s(t) = t^s \) has the property (32). The third claim is an easy consequence of the second claim. \( \square \)

We also get upper bounds of the relative Besov \( p \)-capacity in terms of a certain Hausdorff measure.

**Proposition 4.16.** Let \( h : [0, \infty) \to [0, \infty) \) be an increasing homeomorphism such that \( h(t) = (\ln \frac{1}{t})^{1-p} \) for all \( t \in (0, \frac{1}{2}) \). Suppose \( (X, d, \mu) \) is a proper and unbounded Ahlfors \( Q \)-regular metric space. Let \( E \) be a compact subset of \( X \). There exists a constant \( C \) depending only on \( p \) and on the Ahlfors regularity of \( X \) such that \( \text{cap}_{B_p}(E, \Omega) \leq C \Lambda_h(E) \) for every bounded and open set \( \Omega \) containing \( E \).

**Proof.** The proof is similar to the proof of [Cos, Proposition 3.17] and omitted. \( \square \)

Proposition 4.16 gives another sufficient condition to obtain sets of Besov \( p \)-capacity zero.

**Theorem 4.17.** Let \( h : [0, \infty) \to [0, \infty) \) be an increasing homeomorphism such that \( h(t) = (\ln \frac{1}{t})^{1-p} \) for all \( t \in (0, \frac{1}{2}) \). Then \( \Lambda_h(E) < \infty \) implies \( \text{cap}_{B_p}(E) = 0 \) for every \( E \subset X \).

**Proof.** The proof is similar to the proof of [Cos, Theorem 3.16] and omitted. \( \square \)

5. **Besov capacity and quasicontinuous functions**

In this section we study a global Besov capacity and quasicontinuous functions in Besov spaces.

5.1. **Besov Capacity.**

**Definition 5.1.** For a set \( E \subset X \) define

\[
\text{Cap}_{B_p}(E) = \inf \{ ||u||_{L^p(X)}^p + [u]_{B_p(X)}^p : u \in S(E) \},
\]

where \( u \) runs through the set

\[
S(E) = \{ u \in B_p(X) : u = 1 \text{ in a neighborhood of } E \}.
\]

Since \( B_p(X) \) is closed under truncations and the norms do not increase, we may restrict ourselves to those functions \( u \in S(E) \) for which \( 0 \leq u \leq 1 \). We get the same capacity if we consider the apparently larger set of admissible functions, namely

\[
\tilde{S}(E) = \{ u \in B_p(X) : u \geq 1 \text{ \( \mu \)-a.e. in a neighborhood of } E \}.
\]

Moreover, we have the following lemma:
Lemma 5.2. If $K$ is compact, then
\[ \text{Cap}_{B_p}(K) = \inf\{||u||_{L^p(X)}^p + [u]_{B_p(X)}^p : u \in S_0(K)\} \]
where $S_0(K) = S(K) \cap Lip_0(X)$.

Proof. Let $u \in S(K)$. Since $B_p(X) = B^0_p(X)$, we may choose a sequence of functions $\varphi_j \in Lip_0(X)$ converging to $u$ in $B_p(X)$. Let $U$ be a bounded and open neighborhood of $K$ such that $u = 1$ in $U$. Let $\psi \in Lip(X)$, $0 \leq \psi \leq 1$ be such that $\psi = 1$ in $X \setminus U$ and $\psi = 0$ in $\tilde{U} \subset U$, an open neighborhood of $K$. From Lemma 3.7 we see that the functions $\psi_j = 1 - (1 - \varphi_j)\psi$ converge to $1 - (1 - u)\psi$ in $B_p(X)$. This establishes the assertion since $1 - (1 - u)\psi = u$. \hfill $\Box$

We have a result similar to Theorem 4.2, namely:

Theorem 5.3. The set function $E \mapsto \text{Cap}_{B_p}(E)$, $E \subset X$ is a Choquet capacity. In particular

(i) If $E_1 \subset E_2$, then $\text{Cap}_{B_p}(E_1) \leq \text{Cap}_{B_p}(E_2)$.

(ii) If $E = \bigcup_i E_i$, then
\[ \text{Cap}_{B_p}(E) \leq \sum_i \text{Cap}_{B_p}(E_i). \]

We have introduced two different capacities, and it is next shown that they have the same zero sets.

Let $\Omega, \tilde{\Omega}$ be bounded and open subsets of $X$ such that $\Omega \subset \subset \tilde{\Omega}$. Let $\eta \in Lip_0(\tilde{\Omega})$ be a cut-off function as in Remark 3.8. Suppose $K$ is a compact subset of $\Omega$. Then, if $u \in S_0(K)$, we have that $u\eta$ is admissible for the condenser $(K, \tilde{\Omega})$. Therefore
\[ \text{cap}_{B_p}(K, \tilde{\Omega}) \leq ||u\eta||_{B_p(\tilde{\Omega})}^p \leq ||u\eta||_{B_p(\tilde{\Omega})}^p \leq C ||u||_{B_p(X)}^p \]
where $C$ depends only on $Q$, $p$, $\mu$, $\text{diam} \tilde{\Omega}$ and $\text{dist}(\Omega, X \setminus \tilde{\Omega})$. (See Remark 3.8.) Since $||u||_{B_p(X)} = ||u||_{L^p(X)} + [u]_{B_p(X)}$, we have
\[ ||u||_{B_p(X)}^p \leq 2^p(||u||_{L^p(X)}^p + [u]_{B_p(X)}^p). \]
From (33) and (34) we get, by taking the infimum over all $u \in S_0(K)$, that
\[ \text{cap}_{B_p}(K, \tilde{\Omega}) \leq 2^p C \text{Cap}_{B_p}(K), \]
where $C$ is the constant from (33).

Since both $\text{cap}_{B_p}(\cdot, \tilde{\Omega})$ and $\text{Cap}_{B_p}(\cdot)$ are Choquet capacities, we obtain:

Theorem 5.4. There exists $C > 0$ depending only on $Q$, $p$, $\mu$, $\text{dist}(\Omega, X \setminus \tilde{\Omega})$, and $\text{diam} \tilde{\Omega}$ such that
\[ \text{cap}_{B_p}(E, \tilde{\Omega}) \leq C \text{Cap}_{B_p}(E) \]
for every $E \subset \Omega$.

Corollary 5.5. If $\text{Cap}_{B_p}(E) = 0$, then $\text{cap}_{B_p}(E) = 0$.

We also have a converse result, namely:

Theorem 5.6. If $\text{cap}_{B_p}(E) = 0$, then $\text{Cap}_{B_p}(E) = 0$.

Proof. The proof is similar to the proof of [Cos, Theorem 4.6] and omitted. \hfill $\Box$
Remark 5.7. For $E \subset X$ compact we see from the proof of Lemma 4.14 and Theorem 5.6 that it is enough to have $\text{cap}_{B_p}(E, \Omega) = 0$ for one bounded open set $\Omega \subset X$ with $E \subset \Omega$ in order to have $\text{Cap}_{B_p}(E) = 0$.

It is desirable to know when a set is negligible for a Besov space. If there is an isometric isomorphism between two normed spaces $X$ and $Y$ we write $X = Y$. In particular, if $E$ is relatively closed subset of $\Omega$, then by
\[ B^0_p(\Omega \setminus E) = B^0_p(\Omega) \]
we mean that each function $u \in B^0_p(\Omega)$ can be approximated in $B_p$-norm by functions from $\text{Lip}_0(\Omega \setminus E)$.

**Theorem 5.8.** Suppose that $E$ is a relatively closed subset of $\Omega$. Then
\[ B^0_p(\Omega \setminus E) = B^0_p(\Omega) \]
if and only if $\text{Cap}_{B_p}(E) = 0$.

**Proof.** Suppose that $\text{cap}_{B_p}(E) = 0$. Let $\varphi \in \text{Lip}_0(\Omega)$ and choose a sequence $u_j$ of functions in $B_p(X)$ such that $0 \leq u_j \leq 1$, $u_j = 1$ in a neighborhood of $E$ and $u_j \to 0$ in $B_p(X)$. For every $j \geq 1$ we define $w_j = (1 - u_j)\varphi$. Then from Remark 3.9 and the properties of the functions $\varphi$ and $u_j$, it follows that $w_j$ is a bounded sequence of functions in $B_p(X)$, compactly supported in $\Omega \setminus E$. Lemma 3.13 implies that $w_j$ is a sequence in $B^0_p(\Omega \setminus E)$. Moreover, Lemma 3.7 implies, since $\varphi - w_j = u_j\varphi$ for every $j \geq 1$ and since $||u_j||_{B_p(X)} \to 0$, that $w_j$ converges to $\varphi$ in $B_p(X)$. Since $w_j$ is a sequence in $B^0_p(\Omega \setminus E)$, it follows that $\varphi \in B^0_p(\Omega \setminus E)$. Hence
\[ B^0_p(\Omega) \subset B^0_p(\Omega \setminus E) \]
and since the reverse inclusion is trivial, the sufficiency is established.

For the only if part, let $K \subset E$ be compact. It suffices to show that $\text{Cap}_{B_p}(K) = 0$. Choose $\varphi \in \text{Lip}_0(\Omega)$ with $\varphi = 1$ in a neighborhood of $K$. Since $B^0_p(\Omega \setminus E) = B^0_p(\Omega)$, we may choose a sequence of functions $\varphi_j \in \text{Lip}_0(\Omega \setminus K)$ such that $\varphi_j \to \varphi$ in $B_p(\Omega)$. Consequently
\[ \text{Cap}_{B_p}(K) \leq \left( \lim_{j \to \infty} ||\varphi_j - \varphi||_{L^p(X)}^p + [\varphi_j - \varphi]_{B_p(X)}^p \right) = 0, \]
and the theorem follows. \qed

5.2. Quasicontinuous functions. We show that for each $u \in B_p(X)$ there is a function $v$ such that $u = v \mu$-a.e. and that $v$ is $B_p$-quasicontinuous, i.e. $v$ is continuous when restricted to a set whose complement has arbitrarily small Besov $B_p$-capacity. Moreover, this quasicontinuous representative is unique up to a set of Besov $B_p$-capacity zero.

**Definition 5.9.** A function $u : X \to \mathbb{R}$ is $B_p$-quasicontinuous if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\text{Cap}_{B_p}(G) < \varepsilon$ and the restriction of $u$ to $X \setminus G$ is continuous.

A sequence of functions $\psi_j : X \to \mathbb{R}$ converges $B_p$-quasiformly in $X$ to a function $\psi$ if for every $\varepsilon > 0$ there is an open set $G$ such that $\text{Cap}_{B_p}(G) < \varepsilon$ and $\psi_j \to \psi$ uniformly in $X \setminus G$.

We say that a property holds $B_p$-quasieverywhere, or simply q.e., if it holds except on a set of Besov $B_p$-capacity zero.
**Theorem 5.10.** Let \( \varphi_j \in C(X) \cap B_p(X) \) be a Cauchy sequence in \( B_p(X) \). Then there is a subsequence \( \varphi_k \) which converges \( B_p \)-quasiformly in \( X \) to a function \( u \in B_p(X) \). In particular, \( u \) is \( B_p \)-quasicontinuous and \( \varphi_k \to u \) \( B_p \)-quasieverywhere in \( X \).

**Proof.** The proof is similar to the proof of [HKM93, Theorem 4.3] and omitted. \( \square \)

Theorem 5.10 implies the following corollary.

**Corollary 5.11.** Suppose that \( \varphi \in L_{\text{lip}} \). Since \( \text{Proof.} \)

Theorem 5.11 implies the following corollary.

**Theorem 5.12.** Let \( u \in B_p(X) \). Then \( u \in B_p^0(\Omega) \) if and only if there exists a \( B_p \)-quasicontinuous function \( v \in B_p(X) \) such that \( u = v \) \( \mu \)-a.e. in \( \Omega \) and \( v = 0 \) q.e. in \( X \setminus \Omega \).

**Proof.** Fix \( u \in B_p^0(\Omega) \) and let \( \varphi_j \in L_{\text{lip}}(\Omega) \) be a sequence converging to \( u \) in \( B_p(\Omega) \). By Theorem 5.10 there is a subsequence of \( \varphi_j \) which converges \( B_p \)-quasieverywhere in \( X \) to a \( B_p \)-quasicontinuous function \( v \) in \( X \) such that \( u = v \) \( \mu \)-a.e. in \( \Omega \) and \( v = 0 \) q.e. in \( X \setminus \Omega \). Hence \( v \) is the desired function.

To prove the converse, we assume first that \( \Omega \) is bounded. Because the truncations of \( v \) converge to \( v \) in \( B_p(\Omega) \), we can assume that \( v \) is bounded. Without loss of generality, since \( v \) is \( B_p \)-quasicontinuous and \( v = 0 \) q.e. outside \( \Omega \) we can assume that in fact \( v = 0 \) everywhere in \( X \setminus \Omega \). Choose open sets \( G_j \) such that \( v \) is continuous on \( X \setminus G_j \) and \( \text{Cap}_{B_p}(G_j) \to 0 \). By passing to a subsequence, we may pick a sequence \( \varphi_j \in B_p(X) \) such that \( 0 \leq \varphi_j \leq 1 \), \( \varphi_j = 1 \) everywhere in \( G_j \), \( \varphi_j \to 0 \) \( \mu \)-a.e. in \( X \), and

\[
||\varphi_j||_{B_p(X)}^p + [\varphi_j]_{B_p(X)}^p \to 0.
\]

Then from Remark 3.9 we have that \( w_j = (1 - \varphi_j)v \) is a bounded sequence in \( B_p(\Omega) \). Moreover, for every \( j \geq 1 \), we have \( \lim_{x \to y, x \in \Omega} w_j(x) = 0 \) for all \( y \in \partial \Omega \). Thus, from Lemma 3.14, we have that \( w_j \) is a sequence in \( B_p^0(\Omega) \). Clearly \( w_j \to v \) in \( L_p(X) \) and pointwise \( \mu \)-a.e. in \( X \). This, together with the boundedness of the sequence \( w_j \) in \( B_p^0(\Omega) \), implies via Mazur’s lemma that \( v \in B_p^0(\Omega) \). The proof is complete in case \( \Omega \) is bounded.

Assume that \( \Omega \) is unbounded. We can assume again, without loss of generality, that \( v \) is bounded and that \( v = 0 \) everywhere in \( X \setminus \Omega \). We fix \( x_0 \in X \). For every \( k \geq 2 \) let \( \varphi_k \in L_{\text{lip}}(B(x_0,k^2)) \) be such that \( 0 \leq \varphi_k \leq 1 \), \( \varphi_k = 1 \) on \( B(x_0,k) \) and \( [\varphi_k]_{B_p(X)} \leq C(\ln k)^{1-p} \). (See (24).) Then \( v_k = v\varphi_k \in B_p^0(\Omega \cap B(x_0,k^2)) \subset B_p^0(\Omega) \) for every \( k \geq 2 \) and like in Theorem 3.12, we get

\[
||v - v_k||_{B_p(X)} \to 0,
\]

which implies that \( v \in B_p^0(\Omega) \). This finishes the proof. \( \square \)
We denote by

\[ Q^B_p = Q^B_p(X) \]

the set of all functions \( u \in B_p(X) \) such that there exists a sequence \( \varphi_j \in C(X) \cap B_p(X) \) converging to \( u \) both in \( B_p(X) \) and \( B_p \)-quasiuniformly. It follows immediately from Theorem 5.10 that the functions in \( Q^B_p \) are \( B_p \)-quasicontinuous and for each \( v \in B_p(X) \) there is \( u \in Q^B_p \) such that \( u = v \) \( \mu \)-a.e. We soon show that, conversely, each \( B_p \)-quasicontinuous function \( v \) of \( B_p(X) \) belongs to \( Q^B_p \).

**Theorem 5.13.** Let \( u \in Q^B_p \). If \( u \geq 1 \) \( B_p \)-quasieverywhere on \( E \), then

\[ \text{Cap}_{B_p}(E) \leq ||u||_{L^p(X)}^p + [u]_{B_p(X)}^p. \]

**Proof.** The proof is similar to the proof of [HKM93, Lemma 4.7] and omitted. \( \square \)

This result has the following corollary.

**Corollary 5.14.** Suppose that \( \Omega \) is open and bounded and let \( E \subset \subset \Omega \). Let \( u \in Q^B_p \). Suppose that \( u \geq 1 \) quasieverywhere on \( E \) and that \( u \) has compact support in \( \Omega \). Then

\[ \text{cap}_{B_p}(E, \Omega) \leq [u]_{B_p(\Omega)}^p. \]

We know that \( \text{Cap}_{B_p} \) is an outer capacity. It satisfies the following compatibility condition (see [Kil98]):

**Theorem 5.15.** Suppose that \( G \) is open and \( \mu(E) = 0 \). Then

\[ \text{Cap}_{B_p}(G) = \text{Cap}_{B_p}(G \setminus E). \]

(37)

**Proof.** The proof is very similar to the proof of [Cos, Theorem 4.15] and omitted. \( \square \)

We state now the uniqueness of a \( B_p \)-quasicontinuous representative.

**Theorem 5.16.** Let \( f \) and \( g \) be \( B_p \)-quasicontinuous functions on \( X \) such that

\[ \mu(\{x : f(x) \neq g(x)\}) = 0. \]

Then \( f = g \) \( B_p \)-quasieverywhere on \( X \).

**Proof.** The proof is verbatim the proof from [Kil98, p. 262]. \( \square \)

Combining Theorem 5.13 and Theorem 5.16 we obtain the following corollary.

**Corollary 5.17.** Suppose that \( E \subset X \). Then

\[ \text{Cap}_{B_p}(E) = \inf \{|||u||_{L^p(X)}^p + [u]_{B_p(X)}^p| \}, \]

where the infimum is taken over all \( B_p \)-quasicontinuous \( u \in B_p(X) \) such that \( u = 1 \) \( B_p \)-quasieverywhere on \( E \).

Corollary 5.11 and Theorem 5.16 imply that each \( u \in B_p(X) \) has a "unique" quasicontinuous version.

**Corollary 5.18.** Suppose that \( u \in B_p(X) \). Then there exists a \( B_p \)-quasicontinuous function \( v \) such that \( u = v \) \( \mu \)-a.e. Moreover, if \( \tilde{v} \) is another \( B_p \)-quasicontinuous function such that \( u = \tilde{v} \) \( \mu \)-a.e., then \( v = \tilde{v} \) \( B_p \)-quasieverywhere.

We have a result similar to Corollary 5.18 for locally integrable functions with finite \( B_p \)-seminorm.
Corollary 5.19. Suppose that \( u \in L^1_{\text{loc}}(X) \) such that \([u]_{B_p(X)} < \infty\). Then there exists a \( B_p \)-quasiconstant Borel function \( v \) such that \( u = v \mu\text{-a.e.} \). Moreover, if \( \tilde{v} \) is another \( B_p \)-quasiconstant Borel function such that \( u = \tilde{v} \mu\text{-a.e.} \), then \( v = \tilde{v} \) \( B_p \)-quasieverywhere.

Proof. We prove the ”uniqueness” first. Suppose \( v, \tilde{v} \) are two \( B_p \)-quasiconstant Borel functions such that \( v = u \mu\text{-a.e.} \) and \( \tilde{v} = u \mu\text{-a.e.} \). Let \( w = v - \tilde{v} \). We notice that \( w \) is \( B_p \)-quasiconstant and belongs to \( B_p(X) \) because \( w = 0 \) \( \mu\text{-a.e.} \) in \( X \). Hence from Corollary 5.18 we have that \( w = 0 \) \( B_p \)-quasieverywhere. The ”uniqueness” is proved.

We prove now the existence. Fix \( x_0 \in X \). For every integer \( k \geq 1 \) we choose a \( 2^{1-k} \)-Lipschitz function \( \eta_k \) supported in \( B(x_0,2^{k+1}) \) such that \( \eta_k = 1 \) on \( B(x_0,2^k) \). We have

\[
\eta_{k+1} \eta_k = \eta_k
\]

for every integer \( k \geq 1 \). For a fixed integer \( k \geq 1 \), we define \( u_k = \eta_k u \). Then \( u_k \in L^p(X) \) because \( u \in L^p_{\text{loc}}(X) \) and \( \eta_k \in L^{p_0}(B(x_0,2^{k+1})) \). Moreover, from Lemma 3.10, it follows that \[ \eta_k u - \eta_k u_{B(x_0,2^k)} \in B_p(X) < \infty. \] From this and the fact that \( \eta_k \in B_p(X) \), imply that \( u_k \in B_p(X) \). Therefore, from Corollary 5.11 it follows that there exists \( u_k \in B_p(X) \) a \( B_p \)-quasiconstant Borel function such that \( \tilde{u}_k = u_k \mu\text{-a.e.} \) in \( X \). In particular, since \( \eta_k = 1 \) in \( B(x_0,2^k) \), this implies that \( u_k = u \mu\text{-a.e.} \) in \( B(x_0,2^k) \). So, for every integer \( k \geq 1 \) we have that \( \tilde{u}_{k+1} \) is a \( B_p \)-quasiconstant Borel representative of \( \eta_{k+1} u \), hence \( \eta_k \tilde{u}_{k+1} \) is a \( B_p \)-quasiconstant Borel representative of \( \eta_k \eta_{k+1} u = u_k \), where the equality follows from the definition of \( u_k \) and (38). This implies that both \( \eta_k \tilde{u}_{k+1} \) and \( \tilde{u}_k \) are two \( B_p \)-quasiconstant Borel representatives of \( u_k \in B_p(X) \), hence from Corollary 5.18 we can assume that \( \tilde{u}_k = \eta_k \tilde{u}_{k+1} \) in \( B(x_0,2^k) \). Since \( \eta_k = 1 \) on \( B(x_0,2^k) \), this means in particular that we can assume that \( \tilde{u}_k(x) = \tilde{u}_{k+1}(x) \) for every \( x \) in \( B(x_0,2^k) \).

So, we constructed a sequence of \( B_p \)-quasiconstant Borel functions \( \tilde{u}_k \) in \( B_p(X) \) satisfying the following properties:

\[
\tilde{u}_k(x) = u(x) \quad \text{for } \mu\text{-a.e. } x \in B(x_0,2^k) \\
\tilde{u}_l(x) = \tilde{u}_k(x) \quad \text{for every } x \in B(x_0,2^k) \text{ and } l \geq k \geq 1.
\]

We define \( \tilde{u} : X \rightarrow \overline{\mathbb{R}} \) by

\[
\tilde{u}(x) = \lim_{k \to \infty} \tilde{u}_k(x).
\]

Thus, \( \tilde{u} \) is a \( B_p \)-quasiconstant Borel function and \( u = \tilde{u} \mu\text{-a.e.} \). This proves the existence of a \( B_p \)-quasiconstant Borel representative of \( u \). The claim follows.

\[ \square \]

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References


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