Homeomorphisms with lower bounds for moduli

D. Kovtonyuk and V. Ryazanov

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Abstract

We elucidate possibilities of lower estimates of moduli for families of surfaces of dimension $n - 1$ under mappings with finite distortion. In particular, it makes possible to investigate the boundary behavior of homeomorphisms of finite area distortion, especially, of finitely bi-Lipschitz homeomorphisms between quasi-extremal distance domains by Gehring–Martio.

1 Introduction

Many classes of the so-called mappings with finite distortion are intensively studied during the last years, see e.g. [AIKM], [FKZ], [GI], [HK1], [HK2], [HK3], [HM], [HP], [IKO1]–[IKO2], [IM], [IS], [Ka], [KKM1]–[KKM2], [KM], [KKMOZ], [KO], [KOR], [MV1]–[MV2], [On1]–[On3], [Pa], and [Ra1]–[Ra4]. So far the upper estimates of moduli have played the major role in the theory, see e.g. [MRSY1]–[MRSY6], [IR1]–[IR2], [RS] and our previous preprint [KR].

In this paper we consider the lower estimates of moduli. First recall the base concepts. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \to [1, \infty]$ be a measurable function. A homeomorphism $f : D \to \mathbb{R}^n$ is called a $Q$–homeomorphism if

\begin{equation}
M(f\Gamma) \leq \int_D Q(x) \cdot \rho^n(x) \, dm(x)
\end{equation}

for every family $\Gamma$ of paths in $D$ and every admissible function $\rho$ for $\Gamma$, see [MRSY3]–[MRSY6]. Here the notation $m$ refers to the Lebesgue measure in $\mathbb{R}^n$.

Recall that, given a family of paths $\Gamma$ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called admissible for $\Gamma$, abbr. $\rho \in \text{adm}\Gamma$, if

\begin{equation}
\int_\gamma \rho \, ds \geq 1
\end{equation}

for each $\gamma \in \Gamma$. The (conformal) modulus of $\Gamma$ is the quantity

\begin{equation}
M(\Gamma) = \inf_{\rho \in \text{adm}\Gamma} \int_D \rho^n(x) \, dm(x)
\end{equation}
In particular, the homeomorphisms \( f : D \to \mathbb{R}^n, n \geq 2 \), of the class \( W^{1,n}_{loc} \) with a locally integrable inner dilatation \( K_f(x, f) \) are \( Q \)-homeomorphisms with \( Q(x) = K_f(x, f) \).

The following localization and extension of the notion of \( Q \)-homeomorphisms was first introduced in \([RSY1]\) for \( n = 2 \) and then investigated in \([RS]\) for an arbitrary \( n \geq 2 \). It was motivated by Gehring’s ring definition of quasiconformality in \([Ge1]\).

Given a domain \( D \subseteq \mathbb{R}^n, n \geq 2 \), \( x_0 \in D, \varepsilon_0 < \text{dist}(x_0, \partial D) \), a measurable function \( Q : B(x_0, \varepsilon_0) \to [0, \infty] \), a homeomorphism \( f : D \to \mathbb{R}^n \) is called a ring \( Q \)-homeomorphism at \( x_0 \) if

\[
M(\Gamma(fS_1, fS_2)) \leq \int R Q(x) \cdot \eta^n(|x - x_0|) \, dm(x) \tag{1.4}
\]

for every ring

\[
R = R(x_0, r_1, r_2) = \{ x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2 \}, \quad 0 < r_1 < r_2 < \varepsilon_0 ,
\]

and every measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that

\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1 \tag{1.5}
\]

where

\[
S_i = S(x_0, r_i) = \{ x \in \mathbb{R}^n : |x - x_0| = r_i \}, \quad i = 1, 2 ,
\]

and \( \Gamma(C_1, C_2), C_i = fS_i \), denotes the family of all path \( \gamma : [a, b] \to \mathbb{R}^n \) which join \( C_1 \) and \( C_2 \).

We may assume in the above definition of the ring homeomorphism that \( Q \) is given in the whole domain \( D \) because every measurable function in \( B(x_0, \varepsilon_0) \) can be extended to a measurable function in \( D \), as in \([RS]\). There it was shown that (1.4) is equivalent to the inequality

\[
M(\Gamma(fS_1, fS_2)) \leq \frac{\omega_{n-1}}{I_{n-1}} \tag{1.6}
\]

where \( \omega_{n-1} \) is an area of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \),

\[
I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r q_{x_0}^{n-1}(r)} \tag{1.7}
\]

where \( q_{x_0}(r) \) is the mean of the function \( Q(x) \) over the sphere \( |x - x_0| = r \). Note that the infimum of the expression from the right in (1.4) is realized for the function

\[
\eta_0(r) = \frac{1}{I} : \frac{1}{r q_{x_0}^{n-1}(r)} .
\]

In the present paper, we study a similar notion in terms of modulus for surfaces of the dimension \( n - 1 \).
Below $H^k$, $k = 1, ..., n - 1$ denotes the $k$–dimensional Hausdorff measure in $\mathbb{R}^n$, $n \geq 2$. More precisely, if $A$ is a set in $\mathbb{R}^n$, then

$$H^k(A) = \sup_{\varepsilon > 0} H^k_\varepsilon(A),$$

where the supremum is taken over all countable collections of numbers $\delta_i \in (0, \varepsilon)$ such that some sets $A_i$ in $\mathbb{R}^n$ with diameters $\delta_i$ cover $A$. Here $V_k$ denotes the volume of the unit ball in $\mathbb{R}^k$.

Let $\omega$ be an open set in $\mathbb{R}^k$, $k = 1, ..., n - 1$. A (continuous) mapping $S : \omega \rightarrow \mathbb{R}^n$ is called a $k$–dimensional surface $S$ in $\mathbb{R}^n$. Sometimes we call the image $S(\omega) \subseteq \mathbb{R}^n$ by the surface $S$, too. The number of preimages

$$N(S, y) = N(S, y, \omega) = \text{card } S^{-1}(y) = \text{card } \{ x \in \omega : S(x) = y \}$$

is said to be a multiplicity function of the surface $S$ at a point $y \in \mathbb{R}^n$. In other words, $N(S, y)$ means the multiplicity of covering of the point $y$ by the surface $S$. It is known that multiplicity function is lower semi-continuous, i.e.,

$$N(S, y) \geq \liminf_{m \to \infty} N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, ...$ such that $y_m \rightarrow y \in \mathbb{R}^n$ as $m \rightarrow \infty$, see e.g. [RR], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure $H^k$, see e.g. [Sa], p. 52.

A $k$–dimensional Hausdorff area in $\mathbb{R}^n$ (or simply area) associated with a surface $S : \omega \rightarrow \mathbb{R}^n$ is given by

$$A_S(B) = A^k_S(B) := \int_B N(S, y) \, dH^k y$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set which is measurable with respect to $H^k$ in $\mathbb{R}^n$. The surface $S$ is rectifiable if $S(\mathbb{R}^n) < \infty$.

If $\rho : \mathbb{R}^n \rightarrow [0, \infty]$ is a Borel function, then its integral over $S$ is defined by the equality

$$\int_S \rho \, dA := \int_{\mathbb{R}^n} \rho(y) \, N(S, y) \, dH^k y.$$
We also set
\[ M(\Gamma) = M_n(\Gamma) \quad (1.15) \]
The modulus is itself an outer measure on the collection of all families \( \Gamma \) of \( k \)-dimensional surfaces.

Sometimes, under proofs, it is more convenient to use the following notion. A Lebesgue measurable function \( \rho : \mathbb{R}^n \to [0, \infty) \) is said to be \( p \)-extensively admissible for a family \( \Gamma \) of \( k \)-dimensional surfaces \( S \) in \( \mathbb{R}^n \), abbr. \( \rho \in \text{ext}_p \text{adm} \Gamma \), if
\[ \int_S \rho^k \, dA \geq 1 \quad (1.16) \]
for \( p \)-a.e. \( S \in \Gamma \). The \( p \)-extensive modulus \( \mathcal{M}_p(\Gamma) \) of \( \Gamma \) is the quantity
\[ \mathcal{M}_p(\Gamma) = \inf_{\rho \in \text{ext}_p \text{adm} \Gamma} \int_{\mathbb{R}^n} \rho^n(x) \, dm(x) \quad (1.17) \]
where the infimum is taken over all \( \rho \in \text{ext}_p \text{adm} \Gamma \). In the case \( p = n \), we use notations \( \mathcal{M}(\Gamma) \) and \( \rho \in \text{ext adm} \Gamma \), respectively. For every \( p \in (0, \infty) \), \( k = 1, \ldots, n - 1 \), and every family \( \Gamma \) of \( k \)-dimensional surfaces in \( \mathbb{R}^n \),
\[ \mathcal{M}_p(\Gamma) = M_p(\Gamma) \quad (1.18) \]
see Corollary 2.16 in [KR]. The same is also true for moduli with weights.

Given a domain \( D \subseteq \mathbb{R}^n \), \( n \geq 2 \), \( x_0 \in D \setminus \{ \infty \} \), a measurable function \( Q : D \to (0, \infty) \), we say that a homeomorphism \( f : D \to \mathbb{R}^n \) is a lower \( Q \)-homeomorphism at the point \( x_0 \) if
\[ M(f \Sigma_\varepsilon) \geq \inf_{\rho \in \text{adm} \Sigma_\varepsilon} \int_{D \cap R_\varepsilon} Q^{-1}(x) \, g^n(x) \, dm(x) \quad (1.19) \]
for every ring
\[ R_\varepsilon = \{ x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0 \} , \quad \varepsilon \in (0, \varepsilon_0) \]
where
\[ 0 < \varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0| = \sup_{x \in \partial D} |x - x_0| \quad (1.20) \]
and \( \Sigma_\varepsilon \) denotes the family of all intersections of the spheres
\[ S(r) = S(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \} , \quad r \in (\varepsilon, \varepsilon_0) , \]
with \( D \). Here \( \text{adm} \Sigma_\varepsilon \) consists of Borel functions \( g : \mathbb{R}^n \to [0, \infty] \) with
\[ \int_{D(r)} g^{n-1} \, dA \geq 1 , \quad \forall r \in (\varepsilon, \varepsilon_0) \quad (1.21) \]
where
\[ D(r) = D(x_0, r) = \{ x \in D : |x - x_0| = r \} = D \cap S(x_0, r) . \quad (1.22) \]
As usual, the notion can be extended to the case \( x_0 = \infty \in \overline{D} \) through applying the inversion \( T \) with respect to the unit sphere in \( \mathbb{R}^n \), \( T(x) = x / |x|^2 \), \( T(\infty) = 0 \), \( T(0) = \infty \).

We also say that a homeomorphism \( f : D \to \mathbb{R}^n \) is a **lower \( Q \)-homeomorphism** in \( D \) if \( f \) is a lower \( Q \)-homeomorphism at every point \( x_0 \in \overline{D} \).

We show here that the condition (1.19) is equivalent to the inequality:

\[
M(f \Sigma_\varepsilon) \geq \int_\varepsilon^{\varepsilon_0} \frac{dr}{||Q||_{n-1}(r)}
\]

where

\[
||Q||_{n-1}(r) = \left( \int_{B(r)} Q^{n-1} \, dA \right)^{\frac{1}{n-1}}.
\]

Note that the infimum in (1.19) is attained only for the function

\[
\varrho_0(x) = ||Q||_{n-1}(|x|) \cdot Q^{n-1}(x).
\]

Below we always assume that \( Q \equiv 0 \) outside of \( D \) and take the integrals in (1.24) over the whole spheres \( S_U = S(x_0, r) \).

Let \( \Sigma_\varepsilon^* \) be the family of all \( (n-1) \)-dimensional surfaces in \( D \) which separate the spheres \( |x - x_0| = \varepsilon \) and \( |x - x_0| = \varepsilon_0 \) in \( D \). Note that (1.23) implies the corresponding lower estimate for \( \Sigma_\varepsilon^* \) because \( \Sigma_\varepsilon \subseteq \Sigma_\varepsilon^* \) and hence \( \text{adm} \Sigma_\varepsilon^* \subseteq \text{adm} \Sigma_\varepsilon \). However, the inequality (1.23) for \( \Sigma_\varepsilon^* \) is not precise. The same is true for \( \Sigma_\varepsilon^{**} \) consisting of all closed sets \( C \) in \( D \) which separate the given spheres in \( D \). Indeed, \( \Sigma_\varepsilon \subseteq \Sigma_\varepsilon^{**} \) and hence \( \text{adm} \Sigma_\varepsilon^{**} \subseteq \text{adm} \Sigma_\varepsilon \), cf. [Z]. In the case of \( \Sigma_\varepsilon^{**} \), the definitions in the (1.11)–(1.15) are similar with \( N(C, y) \equiv 1 \). Thus, \( M(f \Sigma_\varepsilon) \) is majorized by \( M(f \Sigma_\varepsilon^*) \) as well as by \( M(f \Sigma_\varepsilon^{**}) \).

This makes possible to find the corresponding estimates of distortion under lower \( Q \)-homeomorphisms and to investigate the removability of isolated singularities and other problems.

Moreover, here we state that homeomorphisms with finite area distortion studied in [KR] are lower \( Q \)-homeomorphisms with \( Q(x) = K_\varepsilon(x, f) \) where \( K_\varepsilon(x, f) \) is the outer dilatation of \( f \) at \( x \). In particular, this holds for the so-called finitely bi-Lipschitz homeomorphisms which are a natural extension of isometries as well as quasi-isometries, see [K].

Given a mapping \( \varphi : E \to \mathbb{R}^n \) and a point \( x \in E \subseteq \mathbb{R}^n \), let

\[
L(x, \varphi) = \limsup_{y \to x, y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y - x|},
\]

and

\[
l(x, \varphi) = \liminf_{y \to x, y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y - x|}.
\]
A mapping \( f : D \to \mathbb{R}^n \) is said to be of **finitely be–Lipschitz** if
\[
0 < l(x, f) \leq L(x, f) < \infty \quad \forall x \in D. 
\]

Recall that **outer dilatation** of \( f \) at \( x \) is defined by
\[
K_O(x, f) = \begin{cases} 
\frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x,f) \neq 0 \\
1, & \text{if } f'(x) = 0 
\end{cases}
\]
and otherwise we set \( K_O(x, f) = \infty \). Similarly, the **inner dilatation** of \( f \) at \( x \) is defined as
\[
K_I(x, f) = \begin{cases} 
\frac{|J(x,f)|}{|f'(x)|^n}, & \text{if } J(x,f) \neq 0 \\
1, & \text{if } f'(x) = 0 
\end{cases}
\]
and \( K_I(x, f) = \infty \) otherwise. Here \( f'(x) \) denotes the Jacobian matrix of \( f \), \( J(x,f) = \det f'(x) \) is its Jacobian, \( |f'(x)| \) is the operator norm of \( f'(x) \), i.e.
\[
|f'(x)| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}, \\
l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}.
\]

## 2 On mappings with finite area distortion.

Let \( \Omega \) be an open set in \( \mathbb{R}^n, n \geq 2 \). A mapping \( f : \Omega \to \mathbb{R}^n \) is said to be of **finite metric distortion**, abbr. \( f \in FMD \), if \( f \) has \((N)\)-property and
\[
0 < l(x, f) \leq L(x, f) < \infty \quad \text{a.e.}
\]
Note that a mapping \( f : \Omega \to \mathbb{R}^n \) is of \( FMD \) if and only if \( f \) is differentiable a.e. and has \((N)\)– and \((N^{-1})\)–properties, see Corollary 3.4 in [MRSY2]. Recall that a mapping \( f : X \to Y \) between measurable spaces \((X, \Sigma, \mu)\) and \((X', \Sigma', \mu')\) is said to have **\((N)\)-property** if \( \mu'(f(E)) = 0 \) whenever \( \mu(E) = 0 \). Similarly, \( f \) has the **\((N^{-1})\)-property** if \( \mu(E) = 0 \) whenever \( \mu'(f(E)) = 0 \).

We say that a mapping \( f : \Omega \to \mathbb{R}^n \) has **\((A_k)\)-property** if the two conditions hold:
\[
(A_k^{(1)}) : \text{for a.e. } k\text{–dimensional surface } S \text{ in } \Omega \text{ the restriction } f|_S \text{ has } (N)\text{–property}; \\
(A_k^{(2)}) : \text{for a.e. } k\text{–dimensional surface } S_\ast \text{ in } \Omega_\ast = f(\Omega) \text{ the restriction } f|_S \text{ has } (N^{-1})\text{–property for each lifting } S \text{ of } S_\ast.
\]

Here a surface \( S \) in \( \Omega \) is a **lifting** of a surface \( S_\ast \) in \( \mathbb{R}^n \) under a mapping \( f : \Omega \to \mathbb{R}^n \) if \( S_\ast = f \circ S \). We also say that a mapping \( f : \Omega \to \mathbb{R}^n \) is of **finite distortion of area in dimension** \( k = 1, \ldots, n-1 \), abbr. \( f \in FAD_k \), if \( f \in FMD \) and has the **\((A_k)\)-property**. Note that analogues of \((A_k)\)-properties and the classes \( FAD_k \) have been first formulated in the mentioned work [MRSY2] for \( k = 1 \) where it is additionally requested local rectifiability of \( S_\ast \) and \( S \) in \((A_1^{(1)})\)– and \((A_1^{(2)})\)-properties, respectively. Thus, the mapping class \( FLD \) (finite length...
distortion) in [MRSY_{2}] is a subclass of FAD_{1}. Finally, we say that a mapping \( f: \Omega \to \mathbb{R}^{n} \) is of \textit{finite area distortion}, abbr. \( f \in \text{FAD} \), if \( f \in \text{FAD}_{k} \) for every \( k = 1, \ldots, n - 1 \), see [KR].

2.2. Lemma. Let \( \Omega \) be an open set in \( \mathbb{R}^{n}, n \geq 2 \), and \( f: \Omega \to \mathbb{R}^{n} \) a FMD homeomorphism with \( (A_{k}^{(1)}) \)–property for some \( k = 1, \ldots, n - 1 \). Then

\[
M(f\Gamma) \geq \inf_{\varrho \in \text{adm} \Gamma} \int_{\Omega} K_{O}^{-1}(x,f) \varrho^{n}(x) \, dm(x)
\]

for every family \( \Gamma \) of \( k \)–dimensional surfaces \( S \) in \( \Omega \).

Proof. Let \( B \) be a (Borel) set of all points \( x \) in \( \Omega \) where \( f \) has a differential \( f'(x) \) and \( J(x,f) = \det f'(x) \neq 0 \). As known, \( B \) is the union of a countable collection of Borel sets \( B_{l}, l = 1, 2, \ldots \) such that \( f|_{B_{l}} \) is bi–Lipschitz, see e.g. 3.2.2 in [Fe]. Without loss of generality we may assume that \( B_{l} \) are mutually disjoint. Note that \( B_{0} = \Omega \setminus B \) and \( f(B_{0}) \) have the Lebesgue measure zero in \( \mathbb{R}^{n} \) for \( f \in \text{FMD} \). Thus, by Theorem 2.11 in [KR] \( A_{S}(B_{0}) = 0 \) for a.e. \( S \in \Gamma \) and hence by \( (A_{k}^{(1)}) \)–property \( A_{S_{*}}(f(B_{0})) = 0 \) for a.e. \( S \in \Gamma \) where \( S_{*} = f \circ S \).

Let \( \varrho_{*} \in \text{ext adm} f\Gamma, \varrho_{*} \equiv 0 \) outside of \( f(D) \), and set \( \varrho \equiv 0 \) outside of \( D \) and

\[
\varrho(x) = \varrho_{*}(f(x)) \frac{|f'(x)|}{\|f'(x)\|}, \quad x \in D.
\]

Arguing piecewise on \( B_{l} \), we have by 3.2.20 and 1.7.6 in [Fe] that

\[
\int_{S} \varrho_{*}^{k} \, dA \geq \int_{S_{*}} \varrho_{*}^{k} \, dA \geq 1
\]

for a.e. \( S \in \Gamma \) and, thus, \( \varrho \in \text{ext adm} \Gamma \).

By the change of variables for the class FMD, see Proposition 3.7 in [MRSY_{2}],

\[
\int_{\Omega} K_{O}^{-1}(x,f) \varrho^{n}(x) \, dm(x) = \int_{f(\Omega)} \varrho_{*}^{n}(y) \, dm(y)
\]

and (2.3) follows.

2.4. Remark. It is easy to see by the well–known Lusin theorem that

\[
\inf_{\varrho \in \text{ext adm} \Gamma} \int_{\Omega} K_{O}^{-1}(x,f) \varrho^{n}(x) \, dm(x) = \inf_{\varrho \in \text{adm} \Gamma} \int_{\Omega} K_{O}^{-1}(x,f) \varrho^{n}(x) \, dm(x),
\]

see similar arguments to (2.17) in [MRSY_{2}]. The expressions in (2.5) are particular cases of moduli with weights.

Combining Lemma 3.10 in [KR] with Lemma 2.2 we have the following statement.
2.6. Theorem. Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 2$, and let a homeomorphism $f: \Omega \to \mathbb{R}^n$ belong to $FAD_k$ for some $k = 1, \ldots, n - 1$. Then, for every family $\Gamma$ of $k$–dimensional surfaces $S$ in $\Omega$, $f$ satisfies the double inequality

$$\inf_{\Omega} \int K^{-1}_O(x, f) \cdot g^n(x) dm(x) \leq M(f\Gamma) \leq \inf_{\Omega} \int K_1(x, f) \cdot g^n(x) dm(x)$$

(2.7)

where the infimums are taken over all $g \in \text{adm} \Gamma$.

2.8. Corollary. Every homeomorphism $f: D \to \mathbb{R}^n$ of finite area distortion in the dimension $n - 1$ is a lower $Q$–homeomorphism with $Q(x) = K_O(x, f)$.

3 The main lemma on lower $Q$–homeomorphisms

We start first from the following general statement.

3.1. Lemma. Let $(X, \mu)$ be a measure space, $p \in (1, \infty)$ and let $\varphi: X \to (0, \infty)$ be a measurable function. Set

$$I(\varphi, p) = \inf_{\alpha} \int_X \varphi \alpha^p \, d\mu$$

(3.2)

where the infimum is taken over all measurable functions $\alpha: X \to [0, \infty]$ such that

$$\int_X \alpha \, d\mu = 1.$$

(3.3)

Then

$$I(\varphi, p) = \left[ \int_X \varphi^{-\lambda} \, d\mu \right]^{-\frac{1}{\lambda}}$$

(3.4)

where

$$\lambda = \frac{q}{p}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

(3.5)

i.e. $\lambda = 1/(p - 1) \in (0, \infty)$. Moreover, the infimum in (3.2) is attained only under the function

$$\alpha_0 = C \cdot \varphi^{-\lambda}$$

(3.6)

where

$$C = \left( \int_X \varphi^{-\lambda} \, d\mu \right)^{-1}.$$

(3.7)

Proof. Indeed, by the Hölder inequality

$$1 = \int_X \alpha \, d\mu = \int_X \left( \varphi^{-\frac{q}{p}} \right)^{\frac{1}{q}} [\varphi \alpha^p]^{\frac{1}{p}} \, d\mu \leq \left[ \int_X \varphi^{-\frac{q}{p}} \, d\mu \right]^{\frac{1}{q}} \cdot \left[ \int_X \varphi \alpha^p \, d\mu \right]^{\frac{1}{p}}.$$
and the equality holds if and only if
\[ c \cdot \varphi^{-\frac{2}{p}} = \varphi \cdot \alpha^p \text{ a.e.}, \]
see e.g. [HLP] or [Ru].

\[ C = c^2 \text{ in (3.7), i.e.} \]

\[ C = \left( \int_X \varphi^{-\frac{1}{p-1}} d\mu \right)^{-1} \]

and

\[ \alpha_0(x) = \left( \int_X \varphi^{-\frac{1}{p-1}} d\mu \right)^{-1} \cdot \varphi^{-\frac{1}{p-1}}(x). \]

3.8. Theorem. Let \( D \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \), \( x_0 \in \overline{D} \), and let \( Q : D \to (0, \infty) \) be a measurable function. A homeomorphism \( f : D \to \mathbb{R}^n \) is a lower \( Q \)-homeomorphism at \( x_0 \) if and only if

\[ (3.9) \quad M(f\Sigma_\varepsilon) \geq \int_\varepsilon^{\varepsilon_0} \frac{dr}{\| Q \|_{n-1}(r)} \quad \forall \varepsilon \in (0, \varepsilon_0) \]

where

\[ (3.10) \quad 0 < \varepsilon_0 < d_0 = \sup_{x \in \overline{D}} |x - x_0| = \sup_{x \in \partial D} |x - x_0|, \]

\( \Sigma_\varepsilon \) denotes the family of all the intersections of \( D \) with the spheres \( S(r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \} \), \( r \in (\varepsilon, \varepsilon_0) \) and

\[ (3.11) \quad \| Q \|_{n-1}(r) = \left( \int_{D(r)} Q^{n-1} \, dA \right)^{\frac{1}{n-1}} \]

is the \( L_{n-1} \)-norm of \( Q \) over \( D(r) = \{ x \in D : |x - x_0| = r \} = D \cap S(r) \). The infimum of the expression from the right in (1.19) is attained only for the function

\[ \varrho_0(x) = \| Q \|_{n-1}(|x|) \cdot Q^{n-1}(x). \]

Proof. Note that, in view of the Lusin theorem, in (1.19)

\[ \inf_{g \in \text{adm} \Sigma_\varepsilon} \int_{R_\varepsilon} Q^{-1}(x) \varrho^n(x) \, dm(x) = \inf_{g \in \text{ext adm} \Sigma_\varepsilon} \int_{R_\varepsilon} Q^{-1}(x) \varrho^n(x) \, dm(x), \]

see (1.16) for the definition of \( \text{ext adm} \Sigma_\varepsilon \). Moreover, for every \( g \in \text{ext adm} \Sigma_\varepsilon \),

\[ A(r) = \int_{D(r)} \varrho^{n-1} \, dA \neq 0 \text{ a.e.} \]

is a measurable function in the parameter \( r \), say by the Fubini theorem. Thus, we may request the equality \( A(r) \equiv 1 \text{ a.e.} \) instead of (1.16) and

\[ \inf_{g \in \text{ext adm} \Sigma_\varepsilon} \int_{R_\varepsilon} Q^{-1}(x) \varrho^n(x) \, dm(x) = \int_\varepsilon^{\varepsilon_0} \left( \inf_{\alpha \in \Lambda(\varepsilon)} \int_{D(r)} Q^{-1}(x) \alpha^p(x) \, dA \right) dr \]
where $p = n/(n-1) > 1$ and $I(r) \text{ denotes the set of all measurable function } \alpha \text{ on the surface } D(r) = S(r) \cap D \text{ such that}$

$$\int_{D(r)} \alpha \, dA = 1.$$ 

Hence Theorem 3.8 follows by Lemma 3.1 with $X = D(r)$, the $(n-1)$-dimensional area as a measure $\mu$ on $X$, $\varphi = \frac{1}{Q}|D(r)|$ and $p = n/(n-1) > 1$.

3.12. **Corollary.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, $Q : D \to (0, \infty)$ a measurable function and let $f : D \to \mathbb{R}^n$ be a lower $Q$-homeomorphism at $x_0$. Then

$$M(f \Sigma \varepsilon) \geq \omega_{n-1} \frac{\varepsilon_0}{\varepsilon} \int_{\varepsilon_0}^{\varepsilon} \frac{dr}{r \cdot q_{n-1}(r)} \quad \forall \varepsilon \in (0, \varepsilon_0)$$

(3.13)

where

$$q_{n-1}(r) = \left( \int_{S(r)} q^{n-1} \, dA \right)^{1/(n-1)}$$

(3.14)

where

$$q(x) = \begin{cases} Q(x), & x \in D, \\ 0, & x \in \mathbb{R}^n \setminus D. \end{cases}$$

(3.15)

4 **Estimates of distortion under hyper $Q$-homeomorphisms**

In what follows, we use the spherical (chordal) metric $h(x, y) = |\pi(x) - \pi(y)|$ in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ where $\pi$ is the stereographic projection of $\mathbb{R}^n$ onto the sphere $S^n(\frac{1}{2} e_{n+1}, \frac{1}{2})$ in $\mathbb{R}^{n+1}$:

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \quad h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y.$$ 

(4.1)

Thus, by definition $h(x, y) \leq 1$ for all $x$ and $y \in \overline{\mathbb{R}^n}$. The spherical (chordal) diameter of a set $E \subset \overline{\mathbb{R}^n}$ is

$$h(E) = \sup_{x, y \in E} h(x, y).$$ 

(4.2)

Note that

$$h(x, y) \leq |x - y|$$

(4.3)

for all $x, y \in \mathbb{R}^n$ and

$$h(x, y) \geq \frac{1}{2} |x - y|$$

(4.4)

for all $x$ and $y$ in the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ with the equality in (4.4) on $\partial \mathbb{B}^n$. 


**4.5. Lemma.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $f : D \to \mathbb{R}^n$ be a lower $Q$–homeomorphism at $x_0 \in D$ and let $0 < \varepsilon < \varepsilon_0 < \text{dist}(x_0, \partial D)$. Then

$$
(4.6) \quad h(fS_\varepsilon) \leq \frac{\alpha_n}{h(fS_{\varepsilon_0})} \cdot \exp \left( -\int_\varepsilon^{\varepsilon_0} \frac{dr}{r q_{n-1}(r)} \right)
$$

where $\alpha_n = 2\lambda_n^2$ with $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$ and $\lambda_n^{\frac{1}{n}} \to e$ as $n \to \infty$.

$$
(4.7) \quad q_{n-1}(r) = \left( \int_{|x-x_0|=r} Q^{n-1}(x) \, dA \right)^{\frac{1}{n-1}},
$$

$S_\varepsilon$ and $S_{\varepsilon_0}$ denote the spheres in $\mathbb{R}^n$ centered at $x_0$ with radii $\varepsilon$ and $\varepsilon_0$, correspondingly.

**Proof.** Set $E = fS_\varepsilon$ and $F = fS_{\varepsilon_0}$. By the known Gehring lemma

$$
(4.8) \quad \text{cap } R(E, F) \geq \text{cap } R_T \left( \frac{1}{h(E) h(F)} \right)
$$

where $h(E)$ and $h(F)$ denote the spherical diameters of $E$ and $F$, correspondingly, and $R_T(s)$ is the Teichmüller ring

$$
(4.9) \quad R_T(s) = \mathbb{R}^n \setminus ([-1, 0] \cup [s, \infty]), \quad s > 1,
$$

see e.g. 7.37 in [Vu_1] or [Ge_2]. It is also known that

$$
(4.10) \quad \text{cap } R_T(s) = \frac{\omega_{n-1}}{(\log \Psi(s))^{n-1}}
$$

where the function $\Psi$ admits the good estimates:

$$
(4.11) \quad s + 1 \leq \Psi(s) \leq \lambda_n^2 \cdot (s + 1) < 2\lambda_n^2 \cdot s, \quad s > 1,
$$

see e.g. [Ge_2], p. 225–226, and (7.19) and (7.22) in [Vu_1]. Hence the inequality (4.8) implies that

$$
(4.12) \quad \text{cap } R(E, F) \geq \frac{\omega_{n-1}}{(\log 2\lambda_n^2 h(E) h(F))^{n-1}}.
$$

By Theorem 3.13 in [Z] and (3.13) we have

$$
(4.13) \quad \text{cap } R(E, F) \leq \frac{1}{M^{n-1}(f\Sigma_\varepsilon)} \leq \frac{\omega_{n-1}}{(\int_{\varepsilon}^{\varepsilon_0} \frac{dr}{r q_{n-1}(r)})^{n-1}}
$$

because $f\Sigma_\varepsilon \subset \Sigma(fS_\varepsilon, fS_{\varepsilon_0})$ where $\Sigma(fS_\varepsilon, fS_{\varepsilon_0})$ consists of all $(n-1)$–dimensional surfaces which separate $fS_\varepsilon$ and $fS_{\varepsilon_0}$.

Finally, combining (4.12) and (4.13) we obtain (4.6).
5 On removability of isolated singularities

By Theorem 3.8 similarly to the proof of Lemma 4.5 we obtain the following statement.

5.1. Theorem. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, $Q : D \to (0, \infty)$ be a measurable function and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be a lower $Q$–homeomorphism. Suppose that

$$\int_0^{\varepsilon_0} \frac{dr}{r \cdot q_{n-1}(r)} = \infty$$

(5.2)

where $\varepsilon_0 < \text{dist}(x_0, \partial D)$ and

$$q_{n-1}(r) = \left( \int_{|x-x_0|=r} Q^{n-1}(x) \, dA \right)^{\frac{1}{n-1}},$$

(5.3)

then $f$ has a homeomorphic extension to $D$.

5.4. Corollary. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be a lower $Q$–homeomorphism. If

$$\int_{|x-x_0|=r} Q^{n-1}(x) \, dA = O \left( \log^{n-1} \frac{1}{r} \right)$$

(5.5)

as $r \to 0$ then $f$ has a homeomorphic extension to $D$.

5.6. Corollary. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and let $f : D \setminus \{x_0\} \to \mathbb{R}^n$ be a lower $Q$–homeomorphism. If

$$\int_{|x-x_0|=r} Q^{n-1}(x) \, dA = O \left( \log \log \log \cdots \log \frac{1}{r} \right)^{n-1},$$

(5.7)

as $r \to 0$ then $f$ has a homeomorphic extension to $D$.

5.8. Corollary. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and $f : D \setminus \{x_0\} \to \mathbb{R}^n$ a homeomorphism of the class $FAD_{n-1}$. If

$$\int_{|x-x_0|=r} K^{n-1}_O(x, f) \, dA = O \left( \log^{n-1} \frac{1}{r} \right)$$

(5.9)

as $r \to 0$ then $f$ has a homeomorphic extension to $D$.

5.10. Remark. In particular, (5.9) holds if

$$K_O(x, f) = O \left( \log \frac{1}{|x-x_0|} \right)$$

(5.11)

as $x \to x_0$. 
6 On continuous extension to boundary points

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a domain. $\partial D$ is said to be **strongly accessible** if, for nondegenerate continua $E$ and $F$ in $\overline{D}$,

$\quad M(\Delta(E, F; D)) > 0$ (6.1)

and **weakly flat** if, for nondegenerate continua $E$ and $F$ in $\overline{D}$ with $E \cap F \neq \emptyset$,

$\quad M(\Delta(E, F; D)) = \infty$ (6.2)

where $\Delta(E, F; D)$ is the family of all paths joining $E$ and $F$ in $D$. It is known that every weakly flat boundary is strongly accessible, see Lemma 5.6 in [MRSY6].

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is called **locally connected at** $x_0 \in \partial D$ if $x_0$ has an arbitrarily small neighborhood $U$ such that $U \cap D$ is connected. Every Jordan domain $D$ in $\mathbb{R}^n$ is locally connected at every point of $\partial D$, see [Wi], p. 66.

6.3. **Lemma.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in \partial D$, $Q : D \to (0, \infty)$ be a measurable function and let $f : D \to \mathbb{R}^n$ be a lower $Q$–homeomorphism at $x_0$. Suppose that the domain $D$ be locally connected at $x_0$ and the domain $D' = f(D)$ has a strongly accessible boundary. If

$\quad \int_0^{\varepsilon_0} \frac{dr}{\|Q\|_{n-1}(r)} = \infty$ (6.4)

where

$\quad 0 < \varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0| = \sup_{x \in \partial D} |x - x_0|$ (6.5)

and

$\quad \|Q\|_{n-1}(r) = \left( \int_{D \cap S(x_0, r)} Q^{n-1} \, dA \right)^{\frac{1}{n-1}}$ (6.6)

then $f$ extends by continuity to $x_0$.

**Proof.** We must show that the cluster set $E = C(x_0, f) = \{y \in \mathbb{R}^n : y = \lim_{k \to \infty} f(x_k), x_k \to x_0, x_k \in D\}$ is a singleton. Note that $E$ is a continuum because $D$ is locally connected at $x_0$. Let us assume that the continuum $E$ is not degenerate.

Let $\Gamma_\varepsilon$ be a family of all paths joining the spheres $S_\varepsilon = \{x \in \mathbb{R}^n : |x - x_0| = \varepsilon\}$ and $S_0 = \{x \in \mathbb{R}^n : |x - x_0| = \varepsilon_0\}$.

Arguing similarly to the Section 4 and 5 on the base of Theorem 3.8 we have that $M(f\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ in view of (6.4).

On the other hand, $M(f\Gamma_\varepsilon) \geq M_0 = M(\Delta(fS_0, E; D'))$ and by the strong accessibility of $\partial D'$ we have that $M_0 > 0$. The contradiction disproves the above assumption.
7 On quasiextremal distance domains

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is called a \textbf{quasiextremal distance domain}, abbr. a \textbf{QED domain}, if

\begin{equation}
M(\Delta(E, F; \mathbb{R}^n)) \leq K \cdot M(\Delta(E, F; D))
\end{equation}

for some $K \geq 1$ and for all pairs of disjoint continua $E$ and $F$ in $D$, see [GM]. It is known that the inequality (7.1) also holds in a QED domain for every pair of disjoint continua $E$ and $F$ in $D$, see Theorem 2.8 in [HK₂], p. 173, cf. Lemma 6.11 in [MV], p. 35. The latter implies (7.1) for nondegenerate intersecting continua $E$ and $F$ in $D$, too. Hence QED domains have weakly flat boundaries, see (6.2), cf. Lemma 3.1 in [HK₂], p. 196. Every QED domain is \textbf{quasiconvex}, i.e., each pair of points $x_1$ and $x_2$ in $D$ can be joined by a rectifiable arc $\gamma$ in $D$ such that

\begin{equation}
s(\gamma) \leq a \cdot |x_1 - x_2|,
\end{equation}

see Lemma 2.7 in [GM], p. 184. Hence $D$ is locally connected at $\partial D$, cf. also Lemma 2.11 in [GM], p. 187, and [HK₂], p. 190.

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to be \textbf{uniform} if the inequalities (7.2) and

\begin{equation}
\min_{i=1,2} s(\gamma(x_i, x)) \leq b \cdot d(x, \partial D)
\end{equation}

hold for some $\gamma$ and for all $x \in \gamma$ where $\gamma(x_i, x)$ is the part of $\gamma$ between $x_i$ and $x$, see [MS]. Every uniform domain is a QED domain but there exist QED domains which are not uniform, see [GM]. Bounded convex domains provide simple examples of uniform domains.

7.4. \textbf{Theorem.} Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in \partial D$, $Q : D \rightarrow (0, \infty)$ be a measurable function and let $f : D \rightarrow \mathbb{R}^n$ be a lower $Q$–homeomorphism at $x_0$. Suppose that $D$ and $D' = f(D)$ are QED domains. If

\begin{equation}
\int_0^{\varepsilon_0} \frac{dr}{\|Q\|_{n-1}(r)} = \infty
\end{equation}

where

\begin{equation}
0 < \varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0| = \sup_{x \in \partial D} |x - x_0|
\end{equation}

and

\begin{equation}
\|Q\|_{n-1}(r) = \left( \int_{D \cap S(x_0, r)} Q^{n-1} dA \right)^{\frac{1}{n-1}},
\end{equation}

then $f$ extends by continuity to $x_0$. 
8 On singular null-sets for extremal distances

A closed set \( X \subset \mathbb{R}^n \), \( n \geq 2 \), is called a null-set for extremal distances, abbr. a \textit{NED set}, if
\[
M(\Delta(E, F; \mathbb{R}^n)) = M(\Delta(E, F; \mathbb{R}^n \setminus X)) \tag{8.1}
\]
for every pair of disjoint continua \( E \) and \( F \subset \mathbb{R}^n \setminus X \).

8.2. Remark. It is known that, if \( X \subset \mathbb{R}^n \) is a NED set, then
\[
|X| = 0 \tag{8.3}
\]
and \( X \) does not locally disconnect \( \mathbb{R}^n \), i.e.,
\[
dim X \leq n - 2 \tag{8.4}
\]
Conversely, if \( X \subset \mathbb{R}^n \) is closed and
\[
H^{n-1}(X) = 0, \tag{8.5}
\]
then \( X \) is a NED set, see [Va2].

Here \( H^{n-1}(X) \) denotes the \((n-1)\)-dimensional Hausdorff measure of a subset \( X \) in \( \mathbb{R}^n \). We also denote by \( C(X, f) \) the \textit{cluster set} of a mapping \( f : D \to \mathbb{R}^n \) in a set \( X \subset D \),
\[
C(X, f) := \{ y \in \mathbb{R}^n : y = \lim_{k \to \infty} f(x_k), \ x_k \to x_0 \in X, \ x_k \in D \}. \tag{8.6}
\]

Note that the complements of NED sets in \( \mathbb{R}^n \) are a very particular case of QED domains considered in the previous section. Thus, arguing locally, we obtain by Theorem 7.4 the following statement.

8.7. Theorem. Let \( D \) be a domain in \( \mathbb{R}^n \) and let \( f : D \setminus X \to \mathbb{R}^n \), \( n \geq 2 \), be lower \( Q \)-homeomorphism at \( x_0 \in X \) where \( X \subset D \). Suppose that \( X \) and \( C(X, f) \) are NED sets. If
\[
\int_0^{\varepsilon_0} \frac{dr}{\| Q \|_{n-1}(r)} = \infty \tag{8.8}
\]
where \( \varepsilon_0 < \text{dist}(x_0, \partial D) \) and
\[
\| Q \|_{n-1}(r) = \left( \int_{|x-x_0|=r} Q^{n-1}(x) \, dA \right)^{\frac{1}{n-1}}, \tag{8.9}
\]
then \( f \) extends by continuity to \( x_0 \).
9 Lemma on cluster sets under lower \( Q \)-homeomorphisms

9.1. Lemma. Let \( D \) and \( D' \) be domains in \( \mathbb{R}^n \), \( n \geq 2 \), \( z_1 \) and \( z_2 \) distinct points in \( \partial D \) and \( f \) a lower \( Q \)-homeomorphism of \( D \) onto \( D' \) with \( Q \in L^{n-1}(D) \). If \( D \) is locally connected at \( z_1 \) and \( z_2 \) and \( \partial D' \) is weakly flat, then

\[
C(z_1, f) \cap C(z_2, f) = \emptyset. \tag{9.2}
\]

9.3. Remark. In fact, it is sufficient for (9.2) to request in Lemma 9.1 instead of the condition \( Q \in L^{n-1}(D) \) that \( Q \in L^{n-1}(D \cap U) \) for some neighborhood \( U \) of one of the points \( z_i, i = 1, 2 \).

Furthermore, it follows from our proof below it is sufficient for (9.2) even that \( Q \) is integrable on

\[
D(r) = \{ x \in D : |x - z_1| = r \} = D \cap S(z_1, r)
\]

for some set of \( r < |z_1 - z_2| \) of a positive linear measure.

Proof. Without loss of generality, we may assume that the domain \( D \) is bounded. Let \( d = |z_1 - z_2| \). By the Fubini theorem the set

\[
E = \{ r \in (0, d) : Q|_{D(r)} \in L^{n-1}(D(r)) \}
\]

has a positive linear measure because \( Q \in L^{n-1}(D) \). Choose \( \varepsilon \) and \( \varepsilon_0 \in (0, d) \) such that

\[
E_0 = \{ r \in E : r \in (\varepsilon, \varepsilon_0) \}
\]

has a positive measure. The choice is possible because of a countable subadditivity of the linear measure and because of the exhaustion

\[
E = \bigcup_{m=1}^{\infty} E_m
\]

where

\[
E_m = \{ r \in E : r \in (1/m, d - 1/m) \}.
\]

Note that each of the spheres \( S(z_1, r), r \in E_0, \) separates the points \( z_1 \) and \( z_2 \) in \( \mathbb{R}^n \) and \( D(r), r \in E_0, \) in \( D \). Thus, by Theorem 3.8 we have that

\[
M(f \Sigma_\varepsilon) > 0 \tag{9.4}
\]

where \( \Sigma_\varepsilon \) denotes the family of all intersections of the spheres

\[
S(r) = S(z_1, r) = \{ x \in \mathbb{R}^n : |x - z_1| = r \}, \ r \in (\varepsilon, \varepsilon_0),
\]

with \( D \).

For \( i = 1, 2 \), let \( C_i \) be the cluster set \( C(z_i, f) \) and suppose that \( C_1 \cap C_2 \neq \emptyset \). Since \( D \) is locally connected at \( z_1 \) and \( z_2 \), there exist neighborhoods \( U_i \) of \( z_i \) such that \( W_i = D \cap U_i \) is connected and \( U_1 \subseteq B^n(z_1, \varepsilon) \) and \( U_2 \subseteq \mathbb{R}^n \setminus B^n(z_1, \varepsilon_0) \).
Set $\Gamma = \Gamma(W_1, W_2; D)$. By (9.4)

$$(9.5) \quad M(f\Gamma) \leq \frac{1}{M^{n-1}(f\Sigma\varepsilon)} < \infty,$$

see Theorem 3.13 in [Z] and Theorem 5.13 in [Ma], cf. also [Ca], [He], [HK2] and [Sh].

However, $\partial D'$ is weakly flat and $W_i, i = 1, 2$ are non-degenerate continua in $\overline{D}$ with a non-empty intersection contradicting (9.5). Thus, the assumption $C_1 \cap C_2 \neq \emptyset$ was not true.

As an immediate consequence of Lemma 9.1 we have the following statement.

9.6. Theorem. Let $D$ and $D'$ be domains in $\mathbb{R}^n$, $n \geq 2$, $D$ be locally connected on $\partial D$ and $\partial D'$ be weakly flat. If $f$ is a lower $Q$–homeomorphism of $D$ onto $D'$ with $Q \in L^{n-1}(D)$, then $f^{-1}$ has a continuous extension to $\overline{D'}$.

9.7. Remark. In view of Remark 9.3, really it is sufficient to request in Theorem 9.6 that $Q$ is integrable in a neighborhood of $\partial D$ only.

10 On homeomorphic extension to boundaries

Combining results of Sections 6–9 we obtain the following statements.

10.1. Theorem. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to (0, \infty)$ belong to $L^{n-1}(D)$ and let $f : D \to \mathbb{R}^n$ be a lower $Q$–homeomorphism in $D$. Suppose that the domain $D$ be locally connected on $\partial D$ and the domain $D' = f(D)$ have a strongly accessible boundary. If at every point $x_0 \in \partial D$

$$(10.2) \quad \int_{0}^{\varepsilon_0} \frac{dr}{\|Q\|_{n-1}(r)} = \infty$$

where

$$(10.3) \quad 0 < \varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0| = \sup_{x \in \partial D} |x - x_0|$$

and

$$(10.4) \quad \|Q\|_{n-1}(r) = \left( \int_{\partial \cap S(x_0, r)} Q^{n-1}(x) dA \right)^{\frac{1}{n-1}},$$

then $f$ has a homeomorphic extension to $\overline{D}$.

10.5. Theorem. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to (0, \infty)$ belong to $L^{n-1}(D)$ and let $f : D \to \mathbb{R}^n$ be a lower $Q$–homeomorphism in $D$. Suppose that $D$ and $D' = f(D)$ are $Q\mathbb{E}\mathbb{D}$ domains. If the condition (10.2) holds at every point $x_0 \in \partial D$, then $f$ has a homeomorphic extension to $\overline{D}$. 
10.6. **Theorem.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to (0, \infty)$ belong to $L^{n-1}(D)$ and let $f : D\setminus X \to \mathbb{R}^n$, $n \geq 2$, $X \subset D$, be lower $Q$–homeomorphism. Suppose that $X$ and $C(X, f)$ are NED sets. If the condition (10.2) holds at every point $x_0 \in X$ for $\varepsilon_0 < \text{dist}(x_0, \partial D)$ where

\begin{equation}
||Q||_{n-1}(r) = \left( \int_{|x-x_0|=r} Q^{-1}(x) \, dA \right)^{\frac{1}{n-1}},
\end{equation}

then $f$ has homeomorphic extension to $D$.

10.8. **Remark.** The results of the section are valid if, instead of the condition $Q \in L^{n-1}(D)$, either $Q \in L^{n-1}(D \cap U)$ where $U$ is a neighborhood of $\partial D$ or $Q \in L^{n-1}(U)$ where $U$ is a neighborhood of $X$. By Corollary 5.7 in [IR$_1$], the condition $Q \in L^{n-1}(U)$ in Theorem 10.6 can be omitted at all if $\text{dim} X = 0$, i.e., if the set $X$ is totally disconnected.

10.9. **Corollary.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$ and let $f : D \to \mathbb{R}^n$ be a homeomorphism of the class $\text{FAD}_{n-1}$. Suppose that the domain $D$ be locally connected on $\partial D$ and the domain $D' = f(D)$ have a strongly accessible boundary. If at every point $x_0 \in \partial D$

\begin{equation}
K_O(x, f) = O \left( \log \frac{1}{|x-x_0|} \right)
\end{equation}

as $x \to x_0$, then $f$ has a homeomorphic extension to $\overline{D}$.

10.11. **Corollary.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $f : D \to \mathbb{R}^n$ be a homeomorphism of the class $\text{FAD}_{n-1}$. Suppose that $D$ and $D' = f(D)$ are QED domains. If the condition (10.10) holds at every point $x_0 \in \partial D$, then $f$ has a homeomorphic extension to $\overline{D}$.

10.12. **Corollary.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $f : D\setminus X \to \mathbb{R}^n$ be a homeomorphism of the class $\text{FAD}_{n-1}$. Suppose that $X$ and $C(X, f)$ are NED sets. If the condition (10.10) holds at every point $x_0 \in X$, then $f$ has a homeomorphic extension to $D$ which belongs to the class $\text{FAD}_{n-1}$.

10.13. **Remark.** In particular, the conclusion of Theorem 10.6 and Corollary 10.12 is valid if $X$ is closed set with

\begin{equation}
H^{n-1}(X) = 0 = H^{n-1}(C(X, f)).
\end{equation}

Thus, the results of the paper extend the well–known Gehring–Martio–Vuorinen theorems for quasiconformal mappings to lower $Q$–homeomorphisms and, in particular, to homeomorphisms with finite area distortion and, especially, to finitely $\text{be–Lipschitz}$ homeomorphisms , see [GM], p. 196, and [MV], p. 36, cf. [Na], [Va$_1$], [Vu$_2$] and [Vu$_3$], and also the corresponding results for $Q$–homeomorphisms in [MRSY$_6$] and [IR$_1$]–[IR$_2$].
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**References**


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