Stochastic equilibrium in financial markets

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Abstract

In this paper we will study the price-forming of securities in purely financial markets when the agents have quadratic utility functions for final wealth. We will emphasize a model where the utility parameters are sampled and agents’ acts are somewhat random even in a homogeneous environment. In the scale of the whole economy some behavior is still expected and we study the deviations from this behavior.

1 Security demand and equilibrium

Consider a set of agents $i = 1, \ldots, n$ acting on a two-period financial markets with securities $j = 1, \ldots, \ell$ bearing risk and a safe security $j = \ell + 1$ with a fixed payoff. At the next period there are states $s = 1, \ldots, S$ one of which will reveal. The securities have state-dependent payoffs tomorrow in money, $\psi^j(s)$.

\[
\Psi = \begin{pmatrix}
\psi^1(1) & \psi^2(1) & \cdots & \psi^{\ell+1}(1) \\
\psi^1(2) & \psi^2(2) & \cdots & \psi^{\ell+1}(2) \\
\vdots & \vdots & \ddots & \vdots \\
\psi^1(S) & \psi^2(S) & \cdots & \psi^{\ell+1}(S)
\end{pmatrix}
\]

Especially for the $\ell + 1$st commodity $\psi^{\ell+1}(s) \equiv 1 \ \forall \ s = 1, \ldots, S$ for which the price $p^{\ell+1} = 1$. Hence it can be considered as the numeraire. For the different states agents assign probabilities $q_i(s)$, $i = 1, \ldots, n$. Furthermore, agents have initial endowments in assets $e^1_i, \ldots, e^{\ell+1}_i$ and a utility function $u_i : \mathbb{R} \to \mathbb{R}$ for final wealth with a special quadratic form:

\[
u_i(x) = x - \frac{x^2}{2a_i}
\] (1.1)
The parameter $a_i^{-1}$ has a risk-aversion interpretation – the bigger it is, the further we are from risk-neutrality. Argument $x$ refers to the terminal wealth of a feasible and optimal consumption allocation, or portfolio, as used more often.

We assume that the portfolio-holders or agents have unique beliefs $q$ of future and agreement on $\Psi$. We define the portfolios and future beliefs by

$$x_i = \begin{pmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^{\ell+1} \end{pmatrix}, \quad q = \begin{pmatrix} q(1) \\ q(2) \\ \vdots \\ q(S) \end{pmatrix}.$$

We will first discuss the selection of an optimal and feasible portfolio. For this, choose one agent and suppress the agent index $i$ everywhere.

### 1.1 Individual security demand

Recall that the instantaneous utility of a terminal wealth was $u : \mathbb{R} \rightarrow \mathbb{R}$. If we see this from today, the utility will be $u : \mathbb{R}^S \rightarrow \mathbb{R}^S$, as there are $S$ states tomorrow. The utility of a whole portfolio $x$ will then be $u : \mathbb{R}^{(\ell+1)\times S} \rightarrow \mathbb{R}^S$

$$u(\Psi x) = (u(\Psi(1)x), \ldots, u(\Psi(S)x))^\top. \quad (1.2)$$

To define the optimal portfolio, an agent wants to maximize a utility function $U : \mathbb{R}^{\ell+1} \rightarrow \mathbb{R}$. A natural choice is the expected utility $^1$

$$U(x) = q^\top u(\Psi x). \quad (1.3)$$

Besides optimal, the portfolio must also be feasible and thus we have a convex programming problem

$$\max\{U(x) = q^\top u(\Psi x) \mid p^\top x = p^\top e\}. \quad (1.4)$$

The Lagrangean is

$$L(x; \lambda) = q^\top u(\Psi x) - \lambda p^\top (x - e)$$

and the first-order condition is

$$\nabla L(x; \lambda) = \nabla(\Psi x) u'(\Psi x) q - \lambda \nabla(x - e) p = \Psi^\top u'(\Psi x) q - \lambda p = 0,$$

where $u'(\Psi x) = \text{diag}[u'(\Psi(s)x)] \in \mathbb{R}^{S\times S}$. This produces the system,

\[ U = f \circ g : \mathbb{R}^{\ell+1} \rightarrow \mathbb{R}^S \rightarrow \mathbb{R}, \text{ where } g(x) = \Psi x \text{ and } f(y) = q^\top u(y). \]
\[
\begin{align*}
&\sum_{s=1}^{S} q(s) u'\left[\sum_{j=1}^{\ell+1} \psi^j(s) x^j\right] \psi^1(s) = \lambda p^1 \\
&\sum_{s=1}^{S} q(s) u'\left[\sum_{j=1}^{\ell+1} \psi^j(s) x^j\right] \psi^2(s) = \lambda p^2 \\
&\vdots \\
&\sum_{s=1}^{S} q(s) u'\left[\sum_{j=1}^{\ell+1} \psi^j(s) x^j\right] \psi^{\ell+1}(s) = \lambda p^{\ell+1}.
\end{align*}
\]

Now put \(u'(x) = 1 - \frac{a}{x}\). The system of equations (*) can be written shortly as

\[
\mu_{\psi} - \frac{1}{a} \Sigma_{\psi} x = \lambda p,
\]

where \(\mu_{\psi} = \Psi^\top q\) and

\[
[\Sigma_{\psi}]_{j,k} = \sum_{s=1}^{S} \psi^j(s) \psi^k(s) q(s).
\]

For \(\psi^{\ell+1}(\cdot) = 1 \in \mathbb{R}^S\) and \(\lambda^{\ell+1} = 1\), hence \(\lambda = 1 - \frac{1}{a} \mu_{\psi}^\top x\) and

\[
\mu_{\psi} - \frac{1}{a} \Sigma_{\psi} x = p - \frac{1}{a} \mu_{\psi}^\top x = 0 = p - \frac{1}{a} p_{\mu_{\psi}}^\top x.
\]

The demand i.e. optimal and feasible portfolio is then

\[
x(p) = a[p \otimes \mu_{\psi} - \Sigma_{\psi}]^{-1}(p - \mu_{\psi}),
\]

where \(p \otimes \mu_{\psi}\) denotes the tensor (Kronecker-) product \(p_{\mu_{\psi}}^\top \in \mathbb{R}^{(\ell+1) \times (\ell+1)}\).

1.2 Equilibrium

We now add the subindex \(i\) in \(a\), \(x(p)\) and \(e\) to indicate the agent. Denote the individual excess demand by

\[
\zeta_i(p) = x_i(p) - e_i = a_i[p \otimes \mu_{\psi} - \Sigma_{\psi}]^{-1}(p - \mu_{\psi}) - e_i,
\]

a vector in \(\mathbb{R}^\ell\), like \(e_i\) and \(p - \mu_{\psi}\), while \([p \otimes \mu_{\psi} - \Sigma_{\psi}]^{-1}\) is in \(\ell \times \ell\). For an economy with \(n\) agents we use the following notation:

\[
\zeta(p) = \bar{a} \otimes [[p \otimes \mu_{\psi} - \Sigma_{\psi}]^{-1}(p - \mu_{\psi})] - \bar{e},
\]

where \(\bar{e}, \zeta(p) \in \mathbb{R}^{n \times \ell}\), \(\bar{a} \in \mathbb{R}^n\) and hence the rest is in \(\mathbb{R}^{n \times \ell}\) as ought to be. The total excess demand \(\bar{Z}(p)\) is the sum of the \(n\) individual excess demands

\[
\bar{Z}(p) = \zeta(p)^\top 1.
\]
We get the market clearing condition of equilibrium:
\[
[\tilde{a} \otimes [(p \otimes \mu_\psi - \Sigma_\psi)^{-1}(p - \mu_\psi)]]^\top 1 - e^\top 1 = 0.
\]

Let us look at the equilibrium prices of the securities. They satisfy
\[
(p \otimes \mu_\psi - \Sigma_\psi)^{-1}(p - \mu_\psi)\tilde{a}^\top 1 - \bar{e}^\top 1 = 0.
\]

Denote
\[
S(p) \triangleq (p \otimes \mu_\psi - \Sigma_\psi)^{-1}(p - \mu_\psi) = \frac{\bar{e}^\top 1}{\tilde{a}^\top 1}\
\]
\[
\Leftrightarrow p - \mu_\psi = \frac{1}{\tilde{a}^\top 1}(p \otimes \mu_\psi e^\top 1 - \Sigma_\psi e^\top 1)\
\]
\[
\Leftrightarrow p - \frac{1}{\tilde{a}^\top 1}p\mu_\psi e^\top 1 = \mu_\psi - \frac{1}{\tilde{a}^\top 1}\Sigma_\psi e^\top 1.
\]

We get the formula for the equilibrium prices,
\[
p = \frac{\tilde{a}^\top 1\mu_\psi - \Sigma_\psi e^\top 1}{\tilde{a}^\top 1 - \mu_\psi e^\top 1} = \bar{p}_n. \tag{1.7}
\]

**Remark 1.1.** Write \(a \overset{\Delta}{=} \theta^1, e^1 \overset{\Delta}{=} \theta^2, \ldots, e^\ell \overset{\Delta}{=} \theta^{\ell+1}\) and the total excess demand \(\bar{Z}(\theta; p)\) can be defined more precisely
\[
\bar{Z}(\theta; p) = \zeta(\theta_1; p) + \ldots + \zeta(\theta_n; p) \overset{\Delta}{=} A(p)\bar{S}(\theta), \tag{1.8}
\]
where \(A(p)\) is a \(\mathbb{R}^{\ell \times (\ell+1)}\)-matrix,
\[
A(p) = \begin{pmatrix}
S(p) & -1 & 0 & \ldots & 0 \\
S(p) & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S(p) & 0 & 0 & \ldots & -1
\end{pmatrix}
\]
and \(\bar{S}(\theta)\) is the sum of the individual characteristics. Using vector \(\zeta(p)\),
\[
\bar{Z}(\theta; p) = \zeta(p)^\top 1 = A(p)\Theta^\top 1\text{ so that }\zeta(p) = \Theta A(p)^\top, \Theta \in \mathbb{R}^{n \times (\ell+1)}.
\]

**Remark 1.2 (Capital asset prices).** Put \(m \overset{\Delta}{=} (\mu_\psi - p), W \overset{\Delta}{=} p^\top e\) and \(C_\psi = [C^{i,j}_\psi] \overset{\Delta}{=} [\cov(\psi^j, \psi^k)] = \Sigma_\psi - \mu_\psi \otimes \mu_\psi\). We can write (1.5) as
\[
x(p) = (a - W)[C_\psi + m \otimes m]^{-1}m. \tag{1.9}
\]
Write
\[
[C_\psi + m \otimes m]x = (a - W)m \Leftrightarrow \[\Sigma_\psi - p \otimes \mu_\psi - \mu_\psi \otimes p + p \otimes p]x = am - (m \otimes p)e.
\]
The last two terms cancel from both sides by the equilibrium condition \(x = e\), which results in (1.5). The security demand of (1.9) is proportional to \(C_\psi^{-1}m\), the solution of the *mean-variance* formulation of the CAPM. See [4].
2 Random economy

Take, not only $s$, but also $a$ and $e$ as random variables with a joint-distribution $f(a, e)$. We define

$$
\mu(p) = \mathbb{E} \zeta_i(p) = \int \int \zeta_i(a, e; p) f(a, e) da de.
$$

Each $\zeta_i(p)$ is a realization via $(a, e)$. When $\mu(p^*) = 0$ we call $p^*$ an expected equilibrium price. Let us solve the expected equilibrium prices:

$$
\mu(p) = \mathbb{E} a[p \otimes \mu - \Sigma \psi]^{-1}(p - \mu) - \mathbb{E} e = 0 \iff \\
\frac{1}{\mathbb{E} a} p\mu^\top \mathbb{E} e - p = \frac{1}{\mathbb{E} a} \Sigma \psi \mathbb{E} e - \mu \psi \iff \\
p = \frac{\Sigma \psi \mathbb{E} e - \mathbb{E} a \mu \psi}{\mu^\top \psi \mathbb{E} e - \mathbb{E} a} = p^*.
$$

(2.1)

Recall that (1.7) equals

$$
\bar{p}_n = \frac{\Sigma \psi \frac{1}{n} e^\top 1 - \frac{1}{n} a^\top 1 \mu \psi}{\mu^\top \frac{1}{n} e^\top 1 - \frac{1}{n} a^\top 1}.
$$

Now we see that w.p. 1 as $n \to \infty$, $\bar{p}_n \to p^*$. This is the law of large numbers.

2.1 The Gärtner-Ellis theorem

The total characteristic is denoted

$$
S(\theta) = \theta_1 + \ldots + \theta_n \doteq (a_1, e_1^\top) + \ldots + (a_n, e_n^\top),
$$

which has the (limiting) free energy function

$$
c_\theta(u) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left\{ \exp[u^\top \bar{S}(\theta)] \right\}.
$$

The convex conjugate (or the Legendre–Fenchel transform) of it is

$$
I_\theta(x) \doteq \sup_u [u^\top x - c_\theta(u)].
$$

(2.2)

According to the Gärtner-Ellis theorem, for an open set $G$ and a closed set $F$, the LDP holds for $n^{-1}S$:

$$
\lim_{n \to \infty} \sup \frac{1}{n} \log \mathbb{P}\{n^{-1}S(\theta) \in F\} \leq - \inf_{x \in F} I_\theta(x)
$$

and

$$
\lim_{n \to \infty} \inf \frac{1}{n} \log \mathbb{P}\{n^{-1}S(\theta) \in G\} \geq - \inf_{x \in G} I_\theta(x).
$$

For instance, with $\theta_i$ iid, $\mathbb{P}(n^{-1}S(\theta) \approx x) \approx e^{-n I_\theta(x)}$, where $x \not= \mathbb{E} \theta_1$. In this special case the Gärtner-Ellis theorem is called the Cramér’s theorem.
2.2 Deviations from the expected behavior

We are interested in the asymptotics of \( P(n^{-1} \bar{Z}(\theta; p) \approx 0) \) while \( \mu(p) \neq 0 \). Equally one may think of the event \( p \neq p^* \) while the prices \( p \) seem to be in equilibrium i.e. with zero total excess demand \( \bar{Z}(\theta; p) \), which was defined as \( \bar{Z}(\theta; p) = \zeta_1(\theta; p) + \ldots + \zeta_n(\theta; p) \equiv A(p)S(\theta) \), where \( A(p) \) was defined in remark (1.1).

With this linear form, we see that the function \( \bar{Z} \) is continuous and satisfies the requirements of the contraction principle, see e.g. [3], Theorem 4.2.1. By the contraction principle, the LDP holds for \( n^{-1} \bar{Z}(\theta; p) \) with an excess demand-rate

\[
I(z; p) = \inf_{y: A(p)y = x} I_\theta(y). \tag{2.3}
\]

For the random equilibrium prices take \( z = 0 \) representing the equation \( \bar{Z}(\theta; p) = 0 \). Our equilibrium-rate is then

\[
I(0; p) = \sup_{u \in \mathbb{R}^{\ell+1}} [0 - c(u; p)] = - \inf_{u \in \mathbb{R}^{\ell+1}} c(u; p).
\]

Note that \( c(u; p) \) is not \( c_\theta(u) \) but a different function. However \( u^\top \bar{Z}(\theta; p) = (A(p)^\top u)^\top S(\theta) \) which implies

\[
c(u; p) = c_\theta(A(p)^\top u) \quad \text{and} \quad I(p) = - \inf_{u \in \mathbb{R}^{\ell+1}} c_\theta(A(p)^\top u) = -c_\theta(A(p)^\top u(p)),
\]

where \( u(p) \) is a unique minimum as the function \( c_\theta(\cdot) \) is convex. In this point

\[
\nabla_u c_\theta(A(p)^\top u) = 0.
\]

Using the convex duality: \( \nabla_x I_\theta(x) = u(p) \), s.t. \( \nabla_u c_\theta(A(p)^\top u) = x \), we get

\[
I(p) = -c_\theta(A(p)^\top \nabla_x I_\theta(x)).
\]

Especially for the equilibrium prices \( x = 0 \) and the rate will be

\[
I(p) = -c_\theta(A(p)^\top \nabla_x I_\theta(x)|_{x=0}). \tag{2.4}
\]

To make things more clear we will next present an example where the characteristic parameters are independently sampled from the multinormal distribution.
Example 2.1. Preferences $\theta_i$ i.i.d. $\sim mn(\theta, Q)$ with mean $\theta = E\theta_1$ and covariance matrix $Q = E[(\theta_1 - \theta)(\theta_1 - \theta)^\top]$. Assume $Q$ invertible.

Now $S_n(\theta) = \sum_{i=1}^n \theta_i \sim mn(n\theta, nQ)$ i.e. the density is

$$f(\theta) = [(2\pi)^{-1/2}|Q|^{1/2}\exp[-1/2(\theta - \hat{\theta})^\top Q^{-1}(\theta - \hat{\theta})].$$

The Laplace transform of $\theta$ is well-known,

$$E[e^{u^\top\theta}] = e^{u^\top\theta + \frac{1}{2}u^\top Qu}$$

and correspondingly for $S_n(\theta)$

$$E[e^{u^\top S_n}] = e^{nu^\top\theta + \frac{1}{2}u^\top Qu}. \quad (2.5)$$

Log of this is $c_\theta(u)$ and the convex conjugate of it is $I_\theta(x) = \sup_{u \in \mathbb{R}^{r+1}} [u^\top x - c_\theta(u)]$

$$= \sup_{u \in \mathbb{R}^{r+1}} [u^\top x - nu^\top\theta - \frac{n}{2}u^\top Qu]. \quad (2.6)$$

$\nabla_u I_\theta(x) = 0 \Rightarrow$ optimum $\check{u} = Q^{-1}(\frac{x}{n} - \check{\theta})$. Substitute to (2.6).

$$I_\theta(x) = \left[Q^{-1}(\frac{x}{n} - \check{\theta})\right]^\top x - n\left[Q^{-1}(\frac{x}{n} - \check{\theta})\right]^\top \check{\theta} - \frac{n}{2}\left[Q^{-1}(\frac{x}{n} - \check{\theta})\right]^\top Q\left[Q^{-1}(\frac{x}{n} - \check{\theta})\right]$$

$$= \frac{1}{2}\left[Q^{-1}(\frac{x}{n} - \check{\theta})\right]^\top (x - n\check{\theta})$$

$$= \frac{n}{2}(\frac{x}{n} - \check{\theta})^\top Q^{-1}(\frac{x}{n} - \check{\theta}) \quad (2.7)$$

The LDP holds with rate $I_\theta(x)$. Put $\zeta(\theta) = aS(p) - e$ where $\theta_i, i = 1, \ldots, n$. In matrix form $Z_n(\theta; p) = A(p)S_n(\theta)$, which is a continuous transformation. Thus due to the contraction principle we have that for $Z_n(\theta; p) = S_n(\zeta(\theta)) = A(p)S_n(\theta)$ and the LDP holds for $n^{-1}Z_n(\theta; p)$ with rate $I(z; p) = \inf_{\gamma: A(p)\gamma = z} I_\theta(y)$.

The rate at which the probability of seeing a random equilibrium price at a large economy, with pricesystem $p$ s.t. $p \neq p^*$ was of the form $I(p) = -\inf_{u \in \mathbb{R}^{r+1}} c_\theta(A(p)^\top u)$, equivalent to that of

$$I(p) = -c_\theta(A(p)^\top u(p))$$

$$= -c_\theta(A(p)^\top \nabla_x I_\theta(x)|_{x=0})$$

$$= -c_\theta(-A(p)^\top Q^{-1}\theta)$$

$$= n[A(p)^\top Q^{-1}\theta]^\top$$

$$= \frac{n}{2}[A(p)^\top Q^{-1}\theta]^\top Q[A(p)^\top Q^{-1}\theta].$$
References


