Taylor's conjecture on magnetic helicity conservation in magnetohydrodynamics

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Incompressible **magnetohydrodynamics (MHD)** is a continuum model for electrically conducting fluids.

In order to formulate the equations, we start with

$$\varrho(\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta u + \mathbf{J}\times\mathbf{B}, \tag{1}$$
$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$
$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}, \tag{3}$$
$$\partial_t\mathbf{B} + \nabla\times\mathbf{E} = \mathbf{0}, \tag{4}$$
$$\nabla\cdot\mathbf{B} = 0, \tag{5}$$

where $\nu > 0$. (We set $\rho = \mu_0 = 1$ for notational simplicity.)

By combining (3)–(4) with Ohm's law $\mathbf{E} = \mathbf{B}\times\mathbf{u} + \eta\mathbf{J}$ we get the induction equation

$$\partial_t\mathbf{B} + \nabla\times(\mathbf{B}\times\mathbf{u}) + \eta\nabla\times(\nabla\times\mathbf{B}) = \mathbf{0}.$$

We eliminate $\mathbf{E}$ and $\mathbf{J}$ altogether and formulate the MHD equations in $\mathbf{B}$, $\mathbf{u}$ and p:

MHD equations

The **(homogeneous,) incompressible, viscous, resistive MHD equations** consist of

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta u + (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{6}$$
$$\nabla \cdot \mathbf{u} = 0, \tag{7}$$
$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) + \eta \nabla \times (\nabla \times \mathbf{B}) = 0, \tag{8}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{9}$$

where $\nu, \eta > 0$.

When $\nu = \eta = 0$, (6)–(9) is called **ideal MHD**.

Given a bounded, simply connected domain $\mathcal{V} \subset \mathbb{R}^3$ with $\partial \mathcal{V} = \mathcal{S}$ we set the boundary conditions $\mathbf{u}|_{\mathcal{S}} = 0$; $\mathbf{B} \cdot \mathbf{n}|_{\mathcal{S}} = 0$; $\mathbf{E} \times \mathbf{n}|_{\mathcal{S}} = 0$, which leads to $(\nabla \times \mathbf{B}) \times \mathbf{n}|_{\mathcal{S}} = 0$.

In ideal MHD, we set $\mathbf{u} \cdot \mathbf{n}|_{\mathcal{S}} = \mathbf{B} \cdot \mathbf{n}|_{\mathcal{S}} = 0$.

Conserved integral quantities of ideal MHD

Continuously differentiable solutions **u**, **B** of *ideal* MHD conserve

$$\int_{\mathcal{V}} \frac{u^2 + B^2}{2}\, \mathrm{d}V := \int_{\mathcal{V}} \frac{|\mathbf{u}(x,t)|^2 + |B(\mathbf{x},t)|^2}{2}\, \mathrm{d}V \qquad \text{(\textbf{total energy})},$$

$$\int_{\mathcal{V}} \mathbf{u}\cdot\mathbf{B}\, \mathrm{d}V \qquad \text{(\textbf{cross helicity})},$$

$$\int_{\mathcal{V}} \mathbf{A}\cdot\mathbf{B}\, \mathrm{d}V \qquad \text{(\textbf{magnetic helicity})},$$

where $\nabla \times \mathbf{A} = \mathbf{B}$.

However, simulations point towards **anomalous dissipation** of total energy: when viscosity and resistivity tend to zero, the energy dissipation rate tends to a *positive* constant, see

- Mininni-Pouquet (Phys. Rev. Lett. 2009),
- Dallas-Alexakis (Astrophys. J. Lett. 2014),
- Linkmann-Berera-McComb-McKay (Phys. Rev. E 2015).

Taylor (Phys. Rev. Lett. 1974) conjectured that magnetic helicity is, by contrast, approximately conserved at very low resistivities.

Magnetic helicity

- Magnetic helicity was first studied by Woltjer (PNAS 1958).

- It is *gauge invariant* (independent of the choice of $\mathbf{A}$) in simply connected domains $\mathcal{V}$ (when $\mathbf{B} \cdot n|_{\partial \mathcal{V}} = 0$):

  If $\nabla \times \mathbf{A} = \nabla \times \mathbf{A}' = \mathbf{B}$, then $\mathbf{A} - \mathbf{A}' = \nabla \phi$ and

$$\int_{\mathcal{V}} (\mathbf{A} - \mathbf{A}') \cdot \mathbf{B} \, \mathrm{d}V = \int_{\mathcal{V}} \nabla \phi \cdot \mathbf{B} \, \mathrm{d}V = \int_{\mathcal{V}} \nabla \cdot (\phi \mathbf{B}) \, \mathrm{d}V = \int_{\mathcal{S}} \phi \underbrace{\mathbf{B} \cdot \mathbf{n}}_{=0} \mathrm{d}S = 0.$$

- It measures e.g. the linkage of magnetic field lines, see Moffatt (J. Fluid Mech. 1969).

- For a review see Berger (Plasma Phys. Control. Fusion 1999).

Force-free magnetic fields

- In both astrophysical and laboratory experiments, plasmas tend to evolve towards a (Beltrami) state

$$\nabla \times \mathbf{B}(x) = \lambda \mathbf{B}(x) \qquad (\lambda \text{ constant}), \tag{10}$$

  see e.g. Qin-Liu-Li-Squire (Phys. Rev. Lett. 2012).

- First, Lüst and Schlüter (Z. Astrophys. 1954) and Woltjer (B.A.N. 1958) suggested that various cosmic magnetic fields are **force-free**, i.e., the Lorentz force $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$; then $\nabla \times \mathbf{B} = \lambda \mathbf{B}$, but λ could be non-constant.

- Woltjer (PNAS 1958) showed that magnetic helicity $\int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V$ is conserved in time by (smooth) solutions of ideal MHD.

- Woltjer noted that by minimizing the magnetic energy $\int_{\mathcal{V}} B^2/2$ under the constraint $\int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V = K$ one obtains (10).

- Woltjer suggested that the plasma relaxes towards a state that minimises magnetic energy subject to the (fixed) magnetic helicity.

(Too) many conserved quantities of ideal MHD

- In ideal MHD, magnetic helicity is conserved for all volumes $\mathcal{V}_i(t) \subset \mathcal{V}$ moving with the fluid that are magnetically closed at $t = 0$ (i.e. $\mathbf{B}(\mathbf{x}, 0) \cdot \mathbf{n}(\mathbf{x})|_{\partial \mathcal{V}_i(0)} = 0$).
- This can lead to numerous conserved quantities $K_i := \int_{\mathcal{V}_i(t)} \mathbf{A}(x,t) \cdot \mathbf{B}(x,t) \, \mathrm{d}V$, sometimes called *subhelicities*.
- Formally, if one minimises magnetic energy subject to all the constraints $\int_{\mathcal{V}_i} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V = K_i$, one gets $\nabla \times \mathbf{B}(x,t) = \lambda_i(t) \mathbf{B}(x,t)$ in each $\mathcal{V}_i(t)$. Here $\lambda_i(t)$ is directly related to the initial conditions.
- However, a relaxed state is observed to be essentially independent of initial conditions! (See Ortolani-Schnack: Magnetohydrodynamics of Plasma Relaxation, 1993)

Taylor's relaxation theory

- Taylor (Phys. Rev. Lett. 1974) argued that with non-zero resistivity η, the subhelicities $K_i = \int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V$ would no longer conserved but $K_0 = \int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V$ itself would remain a good invariant.

- Thus, when $\eta > 0$, one should minimise the magnetic energy subject to the constraint that $K_0 = \int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}V$ remain constant.

- It is then hypothesised that during an initial turbulent state, magnetic fields in various laboratory settings shed extra magnetic energy until they reach a quiescent 'relaxed state' where $\nabla \times \mathbf{B} = \lambda \mathbf{B}$.

- The resulting **Taylor relaxation theory** is discussed in Ortolani-Schnack: Magnetohydrodynamics of Plasma Relaxation, 1993.

- However, Taylor's theory does not address the dynamics that lead to the relaxed state $\nabla \times \mathbf{B} = \lambda \mathbf{B}$. A dynamical explanation was proposed in Qin-Liu-Li-Squire (Phys. Rev. Lett. 2012).

Taylor's conjecture

- **Taylor's conjecture (1974)**: (when $\mathcal{V}$ is simply connected and $\mathbf{B} \cdot \mathbf{n}|_{\mathcal{S}} = 0$), magnetic helicity is approximately conserved in $\mathcal{V}$ for very small resistivities $\eta > 0$.

- When $\nu, \eta > 0$,

$$\partial_t \int_{\mathcal{V}} \mathbf{A}(x,t) \cdot \mathbf{B}(x,t) \, \mathrm{d}V = -2\eta \int_{\mathcal{V}} \mathbf{B}(x,t) \cdot \nabla \times \mathbf{B}(x,t) \, \mathrm{d}V,$$
$$\partial_t \int_{\mathcal{V}} (B^2 + u^2)/2 \, \mathrm{d}V = -\nu \int_{\mathcal{V}} |\nabla \times \mathbf{u}|^2 \, \mathrm{d}V - \eta \int_{\mathcal{V}} |\nabla \times \mathbf{B}|^2 \, \mathrm{d}V.$$

- For small resistivities $\eta > 0$, Berger (Geophys. Astrophys. Fluid Dynamics 1984) showed (under certain extra physical hypotheses) that magnetic helicity dissipates much slower than magnetic energy.

- Mathematical version, see Caflisch-Klapper-Steele (Comm. Math. Phys. 1997):

Conjecture (Taylor 1974)

Magnetic helicity does not dissipate in the inviscid, non-resistive limit.

Leray-Hopf solutions

- Given smooth initial datas $\mathbf{u}_0$ and $\mathbf{B}_0$, it is wide open whether the Cauchy problem for MHD has a smooth solution.
- However, a weak 'Leray-Hopf solution' exists, see Sermange-Temam (Commun. Pure Appl. Math. 1984).
- Denote

$$L^2_\sigma(\mathcal{V}, \mathbb{R}^3) := \{\mathbf{v} \colon \int_\mathcal{V} v^2 \,\mathrm{d}V < \infty,\ \nabla \cdot \mathbf{v} = 0,\ \mathbf{v} \cdot n|_{\partial \mathcal{V}} = 0\}.$$

Definition

Let $\mathbf{u}_0, \mathbf{B}_0 \in L^2_\sigma(\mathcal{V}, \mathbb{R}^3)$. Then $(\mathbf{u}, \mathbf{B})$ is called a **Leray-Hopf solution** if

- $\mathbf{u}$ and $\mathbf{B}$ solve viscous, resistive MHD with initial datas $\mathbf{u}_0, \mathbf{B}_0$,
- $\mathbf{u}, \mathbf{B} \in L^\infty(0, T; L^2) \cap L^2(0, T; W^{1,2})$,
- $\mathbf{u}$ and $\mathbf{B}$ satisfy, at every $t \in (0, T)$, the *energy inequality*

$$\begin{aligned} & \frac{1}{2} \int_\mathcal{V} (u^2 + B^2) \,\mathrm{d}V \\ \leq \quad & \frac{1}{2} \int_\mathcal{V} (u_0^2 + b_0^2) \,\mathrm{d}V - \int_0^t \int_\mathcal{V} (\nu |\nabla \times \mathbf{u}|^2 + \eta |\nabla \times \mathbf{B}|^2) \,\mathrm{d}V \,\mathrm{d}\tau. \end{aligned}$$

Solution of (the mathematical version of) Taylor's conjecture

Theorem (Faraco-L. (to appear in Comm. Math. Phys.))

Suppose

- $\nu_j, \eta_j \searrow 0$ *when* $j \to \infty$.
- *At each* j, $(\mathbf{u}_j, \mathbf{B}_j)$ *is a Leray-Hopf solution with initial data* $(\mathbf{u}_{0,j}, \mathbf{B}_{0,j})$,
- $\mathbf{u}_j \rightharpoonup \mathbf{u}$, $\mathbf{B}_j \rightharpoonup \mathbf{B}$, $\mathbf{u}_{0,j} \rightharpoonup \mathbf{u}_0$, $\mathbf{B}_{0,j} \rightharpoonup \mathbf{B}_0$.

Then $\mathbf{B}$ *conserves magnetic helicity (a.e. in time).*

Similar ideas had been used elsewhere:

- Cheskidov-Lopes Filho-Nussenzveig Lopes-Shvydkoy (Comm. Math. Phys. 2016): on $\mathbb{T}^2$, if a solution $\mathbf{u}$ of Euler equations is a weak inviscid limit of Leray-Hopf solutions, then (under reasonably mild assumptions on initial datas,) $\mathbf{u}$ conserves kinetic energy.
- Constantin-Ignatova-Nguyen 2018: in SQG (in bounded, smooth domains), weak inviscid limits of Leray-Hopf solutions conserve the Hamiltonian.

Results on magnetic helicity conservation in ideal MHD

Mathematical results on magnetic helicity conservation in ideal MHD:

- $\mathbf{u} \in C([0,T]; B^{\alpha_1}_{3,\infty}(\mathbb{T}^3;\mathbb{R}^3))$ and $\mathbf{B} \in C([0,T]; B^{\alpha_2}_{3,\infty}(\mathbb{T}^3;\mathbb{R}^3))$, $\alpha_1 + 2\alpha_2 > 0$: Caflisch-Klapper-Steele (Comm. Math. Phys. 1997)
- $\mathbf{u}, \mathbf{B} \in L^3([0,T); L^3(\mathbb{T}^3,\mathbb{R}^3))$: Kang-Lee (Nonlinearity 2007) and Aluie (Ph.D. dissertation 2009) with Eyink
- Not in general conserved by $\mathbf{u}, \mathbf{B} \in L^\infty(0,T; L^2(\mathbb{T}^3,\mathbb{R}^3))$: Beekie-Buckmaster-Vicol (arXiv 2019)
- Weak L^2-limits of solutions (and subsolutions) $\mathbf{u}, \mathbf{B} \in L^3([0,T); L^3(\mathbb{T}^3,\mathbb{R}^3))$: Faraco-L.-Székelyhidi (arXiv 2019)

Thank you for your attention!