Slicing Sets and Measures

Tuomas Orponen

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1 The Problem and Examples

Let (Ω, d) be a locally compact separable metric space, and let $B \subset \Omega$ be a compact set with $0 < \mathcal{H}^s(B) < \infty$ for some $s > 1$. We intersect (slice) B with a rectifiable set L of dimension less than s and then ask for $\dim[B \cap L]$. To gain intuition, think of Ω being $\mathbb{R}^2$ and L being a line. Even in this special case the problem is ill-posed for two reasons:

(i) Some lines are just too far away. Since B is a compact set, most lines L in $\mathbb{R}^2$ avoid B altogether, and then $\dim[B \cap L] = 0$. Not very interesting.

(ii) Let $B = [0,1] \times C \subset \mathbb{R}^2$, where $C \subset \mathbb{R}$ is a Cantor set. Then any line *not* parallel to the x-axis will intersect B in dimension $\dim C < 1$ (or zero). The lines parallel to the x-axis will, on the other hand, intersect B in dimension one (or zero). So there seems not to be a unique answer to the question on $\dim[B \cap L]$.

We must slightly lower our expectations to get past issues (i) and (ii). In the case of lines, we counter (i) by looking at all lines in a fixed direction simultaneously. Then some of them will certainly meet B, and we may ask if *positively many of them* will intersect B in some dimension. To get past issue (ii), we consider lines in all possible directions. In our example, we may state that *in almost all directions* positively many lines meet B in dimension $\dim C$. Following these ideas, we have the following formulation:

Definition 1 (. . . of the problem)**.** Let $B \subset \Omega$ be as above, and let $J \subset \mathbb{R}^n$ be an open set of 'parameters'. Suppose that we are given a collection of Lipschitz-mappings $\pi_\lambda \colon \Omega \to \mathbb{R}^m$, $\lambda \in J$, where $1 \leq m < s$. We call these mappings *projections*. Then we are to formulate sufficient conditions, as general as possible, for the projections π_λ so that the following statement holds: *for $\mathcal{L}^n$ almost every parameter $\lambda \in J$ positively many sets $\pi_\lambda^{-1}\{t\}$, $t \in \mathbb{R}^m$, intersect B in dimension $s - m$. In other words, for $\mathcal{L}^n$ almost every $\lambda \in J$ we have*

$$\mathcal{L}^m(\{t \in \mathbb{R}^m : \dim[B \cap \pi_\lambda^{-1}\{t\}] = s - m\}) > 0. \tag{2}$$

In case this result holds, we are also interested in the bounding from above the dimension of the set of *exceptional* parameters $\lambda \in J$ for which (2) fails.

Example 3. Let us get familiar with the formulation by looking at three examples of projection families:

(i) The directions of $\mathbb{R}^2$ can be parametrized by the interval $J = [0, 2\pi)$. Let $\pi_\lambda \colon \mathbb{R}^2 \xrightarrow{\sim} \mathbb{R}$ be the orthogonal projection onto the line passing through the origin in direction $e^{i\lambda}$, ie. $\pi_\lambda(x) = x \cdot (\cos\lambda, \sin\lambda)$. Then the sets $\pi_\lambda^{-1}\{t\}$ are all the lines of $\mathbb{R}^2$ orthogonal to $e^{i\lambda}$. In our example $B = [0,1] \times C$ above, we see that for almost every λ (almost every direction), positively many lines in this direction intersect B in dimension $\dim C = \dim B - 1$. Does this hold for all sets $B \subset \mathbb{R}^2$?

(ii) Let $J = \mathbb{R}^2$, and let $\pi_\lambda \colon \mathbb{R}^2 \to \mathbb{R}$ be the non-linear 'projection' $\pi_\lambda(x) = |x - \lambda|$. Then the sets $\pi_\lambda^{-1}\{r\}$, $r > 0$, are exactly the circles centered at λ. A positive answer to our problem for these projections would imply that for almost every $\lambda \in \mathbb{R}^2$ there are positively many radii $r > 0$ such that $\dim[B \cap S(\lambda, r)] = s - 1$. Also, if we manage to show that the *exceptional* parameters $\lambda \in J$ have dimension less than s, then we may actually find a point $\lambda \in B$ such that the previous conclusion holds. This is related to the Erdös problem: 'How many distinct distances are there between points in a set?'

(iii) As a slightly more exotic possible application, consider $B = \Omega = \{-1, 1\}^{\mathbb{N}}$, $J = (1/2, 1)$ and let $\pi_\lambda \colon \Omega \to \mathbb{R}$ be defined by

$$\pi_\lambda(x_1, x_2, \ldots) := \sum_{j=1}^{\infty} x_j \lambda^j.$$

If Ω is equipped with a suitable metric, we have $\dim \Omega > 1$. In this case $\pi_\lambda^{-1}\{t\}$ is the collection of all sequences $(x_1, x_2, \ldots) \in \Omega$ such that the previous series has value $t \in \mathbb{R}$. By elementary means it is not easy to construct even two distinct sequences with the same π_λ-value. A positive answer to our problem for these projections would show that, for almost every $\lambda \in J$, there are uncountably many such sequences for positively many values of $t \in \mathbb{R}$.

2 The Upper Bound

The following integralgeometric inequality holds for all Lipschitz-maps $\pi \colon \Omega \to \mathbb{R}^m$: if $m \le s$, then

$$\int^* \mathcal{H}^{s-m}(B \cap \pi^{-1}\{t\})\, d\mathcal{L}^m y \le \alpha(m) Lip(f)^m \mathcal{H}^s(B).$$

For the (easy) proof, see [Ma1], Theorem 7.7. It follows immediately that if $\mathcal{H}^s(B) < \infty$, then for any parameter $\lambda \in J$, almost all the sets $B \cap \pi_\lambda^{-1}\{t\}$, $t \in \mathbb{R}^m$, have finite $\mathcal{H}^{s-m}$ measure. In particular, $\dim[B \cap \pi_\lambda^{-t}\{t\}] \le s - m$ for every $\lambda \in J$ and $\mathcal{L}^m$ almost every $t \in \mathbb{R}^m$. Thus the dimension estimate in (2) is the best possible we can hope for any families of Lipschitz-projections.

3 The Lower Bound

3.1 Slicing with Hyperplanes

In the case of slicing subsets of $\mathbb{R}^n$ with lower dimensional hyperplanes the statement formulated as part of Definition 1 is true. This is due to Marstrand (lines in a plane) and (for general dimensions) Mattila, see [Ma2]. For example, in the lines-in-a-plane case we have

Theorem 4. *Let $B \subset \mathbb{R}^2$ be a Borel set with $\dim B = s > 1$. Write $\pi_\lambda(x) := x \cdot (\cos \lambda, \sin \lambda)$ for $\lambda \in (0, 2\pi)$. Then for $\mathcal{L}^1$ almost every $\lambda \in (0, 2\pi)$ we have*

$$\mathcal{L}^1(\{t \in \mathbb{R} : \dim[B \cap \pi_\lambda^{-1}\{t\}] = s - 1\}) > 0.$$

In this situation we also have an estimate for the dimension of exceptional parameters, due to yours sincerely. If $E \subset (0, 2\pi)$ is the set of parameters λ such that the conclusion of the previous theorem fails, we have $\dim E \leq 2 - \dim B$. This bound cannot be improved.

3.2 The Generalisation

What is the key feature of orthogonal projections (or, equivalently, m-dimensional hyperplanes in $\mathbb{R}^n$) that enables the preceding result? A hint is given by the following observation: whenever $\lambda \in J$ is such that (2) holds, then clearly $\mathcal{L}^m(\pi_\lambda(B)) > 0$. This means that any family of projections satisfying the statement in Definition 1 cannot decrease dimension arbitrarily. About ten years ago, around the year 2000, there was an extensive article in Duke Mathematical Journal by Yuval Peres and Wilhelm Schlag, [PS], where sufficient conditions were given for a family of projections $\pi_\lambda \colon \Omega \to \mathbb{R}^m$, $\lambda \in J$, to be 'dimension preserving'. If the projections are $\mathbb{R}$-valued and the parameter set J is an open interval, the most restrictive and central condition, called *transversality*, reads as follows: there exists $\delta > 0$ such that

$$\left| \frac{\pi_\lambda(x) - \pi_\lambda(y)}{d(x, y)} \right| \leq \delta \quad \Longrightarrow \quad \left| \frac{\partial_\lambda \pi_\lambda(x) - \partial_\lambda \pi_\lambda(y)}{d(x, y)} \right| \geq \delta$$

for all $x, y \in \Omega$. Intuitively, if some π_{λ_o}, $\lambda_o \in J$, maps two points x and y too close to each other, then 'the next π_λ' will already map x and y fairly far apart.[1] Under the transversality hypothesis and the other (weaker) conditions defined in the article, the authors proved results showing that, for example, if $\dim B \leq m$, then $\dim \pi_\lambda(B) = \dim B$ for almost all $\lambda \in J$. From the results in [PS] it also follows that $\dim B > m$ implies $\mathcal{L}^m(\pi_\lambda(B)) > 0$ for almost every $\lambda \in J$. For orthogonal projections these facts have been well-known for decades, but the projection formalism in [PS] also covers our examples (ii) and (iii) from the first chapter.

Then it seemed like a natural conjecture that a family of projections having good dimension preserving properties should also behave well with respect to the problem stated in Definition 1. This idea had also occurred to the authors of [JJN] in 2004, and, indeed, Theorem 4 admits the following generalisation:

[1] This is clear in the case of orthogonal projections: if x and y are projected to the same point in one direction, then the projection rotated by α degrees will already send x and y to distance $c\alpha$ from each other.

Theorem 5. *Let $\pi_\lambda \colon \Omega \to \mathbb{R}^m$, $\lambda \in J$, be a family of Lipschitz-projections satisfying the conditions in [PS]. Suppose that $B \subset \Omega$ is a Borel set of dimension $s > m$. Then, for $\mathcal{L}^n$ almost every $\lambda \in J$ it holds that*

$$\mathcal{L}^m(\{t \in \mathbb{R}^m : \dim[B \cap \pi_\lambda^{-1}\{t\}] = s - m\}) > 0.$$

With the results of [PS] at our disposal, the proof of Theorem 5 is practically the same as that of Theorem 4. However, in [JJN] there was no attempt to prove bounds for the set E of the exceptional parameters $\lambda \in J$. The Fourier-analytic method used to obtain $\dim E \leq 2 - \dim B$ in connection with Theorem 4 no longer worked in this case. Finding and proving the optimal estimates is an ongoing project, but here is one result:

Theorem 6. *Let $\pi_\lambda \colon \Omega \to \mathbb{R}$, $\lambda \in J \subset \mathbb{R}$, be a family of Lipschitz-projections satisfying the conditions in [PS]. We also have to assume that $x \mapsto \partial_\lambda \pi_\lambda(x)$ is Lipschitz. Then, given $1 < r \leq s$ and a Borel set $B \subset \Omega$ with $\dim B = s$, we have*

$$\mathcal{L}^1(\{t \in \mathbb{R} : \dim[B \cap \pi_\lambda^{-1}\{t\}] \geq r - 1\}) > 0$$

for every $\lambda \in J$ except for a set $E \subset J$ satisfying $\dim E \leq 1 + r - s$.

The assumptions are satisfied by the projections in example (iii). Our methods also extend to cover the projections in example (ii) even though the parameter set is multidimensional: there we have the same result, except that one has to replace $\dim E \leq 1 + r - s$ by $\dim E \leq 2 + r - s$. We hope to sharpen these estimates in the very near future.

4 Slicing Measures

In geometric measure theory, all questions related to the Hausdorff dimension of a particular Borel set B can be translated to (and characterised by) questions concerning the Radon measures supported on B. The main tool is Frostman's lemma and its corollaries. This is also true for the problems stated above. Instead of slicing sets directly, we slice the (Frostman) measures supported on those sets and study the dimensional properties of the these *sliced measures*. In this presentation there is no time for proving any actual results, but we may finish the talk by sketching the idea of slicing a Radon measure on a compact separable metric space (Ω, d) with respect to a continuous projection. So, let μ be a Radon measure on Ω, and let $\pi \colon \Omega \to \mathbb{R}^m$ be a continuous mapping (projection). Fix any continuous function $\varphi \colon \Omega \to \mathbb{R}$, and define the signed measure μ_φ on Ω by

$$\mu_\varphi(B) = \int_B \varphi \, d\mu, \qquad B \in \operatorname{Bor} \Omega.$$

The measure μ_φ can be projected to a signed Radon measure $\pi_\sharp \mu_\varphi$ on $\mathbb{R}^n$ by the formula $\pi_\sharp \mu_\varphi(B) = \mu_\varphi(\pi^{-1}(B))$. Then it follows from a standard result in geometric measure theory that the 'derivative'

$$\Lambda_t \varphi := D(\pi_\sharp \mu_\varphi, \mathcal{L}^n, t) = \lim_{\delta \to 0} \frac{\pi_\sharp \mu_\varphi(B(t,\delta))}{(2\delta)^n} = \lim_{\delta \to 0} (2\delta)^{-n} \int_{\pi^{-1}(B(t,\delta))} \varphi \, d\mu$$

exists for $\mathcal{L}^n$ almost every $t \in \mathbb{R}^n$. Now, it follows from the fact that Ω is separable that also $C(\Omega)$ is separable. Hence we may find a countable dense set of functions $\{\varphi_1, \varphi_2, \ldots\} \subset C(\Omega)$. Since *for every* $j \in \mathbb{N}$ the limit $\Lambda_t \varphi_j$ exists for $\mathcal{L}^n$ almost every $t \in \mathbb{R}^n$, we conclude (using subadditivity) that for $\mathcal{L}^n$ *almost every* $t \in \mathbb{R}^n$ the limit $\Lambda_t \varphi_j$ exists for every $j \in \mathbb{N}$. Then it is an easy exercise to verify that, for these almost every $t \in \mathbb{R}^n$ the limit $\Lambda_t \varphi$ actually exists *for every* $\varphi \in C(\Omega)$. One also checks that $\varphi \mapsto \Lambda_t \varphi$ is a positive linear functional on $C(X)$ for these $t \in \mathbb{R}^n$, whence the Riesz representation theorem immediately yields a Radon measure μ_t on Ω such that

$$\lim_{\delta \to 0} (2\delta)^{-n} \int_{\pi^{-1}(B(t,\delta))} \varphi \, d\mu = \Lambda_t \varphi = \int \varphi \, d\mu_t, \qquad \varphi \in C(X).$$

Then the support of μ_t is contained in $\pi^{-1}\{t\}$, so that it may justly be called the slice of μ on $\pi^{-1}\{t\}$. Now the first serious problem arises from the question: *can μ_t be a trivial measure for every $t \in \mathbb{R}^n$?* In general, the answer is 'yes'. On the other hand, if $\pi_\sharp \mu \ll \mathcal{L}^n$, then the answer is 'no'. This follows from the Fubini-type formula

$$\int_B \int \varphi \, d\mu_t \, d\mathcal{L}^n(t) = \int_{\pi^{-1}(B)} \varphi \, d\mu, \qquad B \in \operatorname{Bor} \mathbb{R}^n,$$

which holds in case $\pi_\sharp \mu \ll \mathcal{L}^n$. Namely, if we choose $B = \mathbb{R}^n$ and $\varphi \equiv 1$, the previous formula turns into

$$\int \mu_t(\Omega) \, d\mathcal{L}^n(t) = \mu(\Omega),$$

which shows that $\mathcal{L}^n$-positively many of the measures μ_t must be non-trivial. The next step would be to investigate the dimensions of the measures μ_t given information on the dimension of μ, but this couldn't possibly fit into a one-hour presentation.

References

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