Track Records

My research area is theoretical and mathematical physics. Generally, my research interests are the interplay between theoretical high energy physics and statistical physics. In a broad sense, my research interest is the connection between quantum field theory (QFT) and string theory on one hand, and statistical and stochastic systems such as lattice and continuous models on the other. I am mostly interested in the study of non-perturbative aspects of gauge theories, by using the rigorous mathematical methods from algebra, analysis, geometry, stochasticity and probability, and approximative methods from theoretical physics.

Since the beginning of my research career, I worked on various subjects in non-commutative, conformal and integrable QFTs, and also more generally in two-dimensional and higher-dimensional QFTs, statistical field theories and string theories. I have been studying about discrete lattice statistical models and their continuum limits as some QFT's. These studies are substantially facilitated by various mathematical methods from (a) representations of infinite-dimensional algebras, vertex operator algebras, symmetric polynomials (b) Schramm-Loewner evolution, (c) stochastic processes and combinatorics, (d) non-commutative geometry and algebraic (toric and tropical) geometry (e) discrete and continuous complex analysis and special functions (f) random matrix theory, random partitions, random fields such Gaussian free fields.

Past Research

In my M.Sc. thesis (2006-2008), I have studied the perturbative and non-perturbative approach to the problem of triviality in interacting scalar field theory in different dimensions, using computational lattice methods and renormalization group techniques. I have studied the scaling limit of (one-to-four dimensional) Ising model and lattice scalar field theory by using the theoretical and computational methods.

During the first two years of my Ph.D. (2009-2011) in theoretical physics division in department of Physics, I have studied perturbative and non-perturbative methods from field theory and string theory in computation of statistical properties of different interesting physical systems such as polymers and fluids etc. During the first two years, I have published three articles on the following subjects:

1) The aggregation phenomena of particle trajectories in turbulent flows by using the field theoretic path integral method and instanton contributions [8].
2) The non-perturbative modifications to Moyal star product, in noncommutative geometry, to obtain a finite noncommutative field theory and its interpretations in string theory [7].
3) The statistical hydrodynamical transport properties of the fictitious fluid on the horizon of the black holes, in the context of membrane paradigm, by using string theory. This work shed light on the microscopic origin of the membrane paradigm, from string theory point of view. We obtained the ratio of the shear viscosity to entropy density by applying the linear response theory to the highly excited strings which resemble the polymers. This ratio has become recently important and has been computed by different approaches such as AdS/CFT applied in strongly interacting systems like quark-gluon plasma [6].

Recent Research

In last two years of my Ph.D. (2011-2013) and two years of my postdoctoral research (2013-2015) in mathematical physics group in department of Mathematics, I am studying the analytic, algebraic and probabilistic methods in planar lattice models and their relations to (super) conformal and topological field theories. I have prepared five articles in the following topics:

1) **Lattice QFT of Ising model:** In a collaboration with C. Hongler and K. Kytölä, we studied the rigorous aspects of operator formalism, fermionic (1+1)-dimensional QFT, in 2D Ising model on domains with boundaries and its relation to recently introduced discrete analytic methods such as s-holomorphicity. We have constructed an algebraic framework which presents the geometric information of the domain in terms of an operator, called Poincare-Steklov operator. The scaling limit of this operator is well defined and therefore this can be considered as a starting point for an algebraic construction of rigorous scaling limit of the quantum states of the Ising model in s-holomorphic terms. The final goal would be a rigorous construction of well-defined quantum field theory for the Ising model [5].

2) **Boundary QFT of scaling limit of critical Ising model:** Future research on the foundations of QFT will examine the success of vivid ideas from probability and statistical physics such as the rigorous construction of QFT's by means of SLE! Roughly speaking, SLE may be used, instead of path integral approach, to construct a well-defined measure in the Fock space of QFT's. Rigorous construction of the quantum field theories which explain the scaling limit of the lattice models at criticality is an ambitious project. In that direction, we are developing the first steps, a fermionic conformal field theory which explains the scaling limit of the Ising model at criticality.
I have studied the algebraic and analytic construction of the 2D fermionic boundary conformal field theory (BCFT) of the scaling limit of the critical Ising model on bounded domains. Specially, the vertex operator algebra (VOA) and fermionic CFT in bounded domains and their relations to the scaling limit of the interfaces, Schramm Loewner evolution (SLE) curves, in a concrete example of the Ising model are studied. That leads to an explicit realization of the fermionic CFT/SLE_3 correspondence [4].

3) **New Phase transitions in Chern-Simons matter theory:** Applying the machinery of random matrix theory and Toeplitz determinants we study the level k, U(N) Chern-Simons theory coupled with fundamental matter on S^2 at finite temperature T. This theory admits a discrete matrix integral representation, i.e. a unitary discrete matrix model of two-dimensional Yang-Mills theory. In this study, the partition function and phase structure of the Chern-Simons matter theory in an special case with Gross-Witten-Wadia potential are investigated.
We obtain an exact expression for the partition function of the Chern-Simons matter theory as a function of k,N,T, for finite values and in the asymptotic regime. In the Gross-Witten-Wadia case, we show that ratio of the Chern-Simons matter partition function and the continuous two-dimensional Yang-Mills partition function, in the asymptotic regime, is the Tracy-Widom distribution. Consequently, using the explicit results for free energy of the theory, new second-order and third-order phase transitions are observed. Depending on the phase, in the asymptotic regime, Chern-Simons matter theory is represented either by a continuous or discrete two-dimensional Yang-Mills theory, separated by a third-order domain wall [3].

4) **Microscopic entropy of crystal melting black holes:** In a collaboration with R. Szabo, we study the relations between random partitions and BPS spectrum of gauge theories. The microscopic counting of black hole microstates is one of the unsolved problems in quantum gravity. In this paper we study the thermodynamical properties of the BPS charged black holes given by branes wrapping the cycles in orbifolds in the limit of large parameters. These black holes are in one-to-one correspondence with crystal melting models and dimer model. These statistical models provide a systematic approach, via a statistical frame work, to understand the thermodynamics of the black holes. From this point of view, we derive explicit functional forms for the microscopic entropy and free energy of the BPS black holes and their values with exact numerical coefficients in the WKB approximation, by using techniques from crystal melting models and dimer model. Finally, we compare our results with the previously obtained macroscopic and microscopic results from different approaches such as supergravity and D-branes in IIA superstring theory. We found an agreement between these BPS black hole entropy and D4-$D0 entropy. However, our numerical results for some other BPS black holes seem to be new [2].

5) **Physical dimer model and quiver gauge theories:** In collaboration with M. Langvik, we are studying

the relation between superconformal index of N=1, four-dimensional superconformal gauge theories, and the partition function of isoradial dimer model. Our aim is to prove that the brane tilings and quiver gauge theories are completely described by dimer models. Furthermore, we plan to explore the RG space of gauge theories via the moduli space of spectral curve of the dimer model [1].

1) M. Langvik, A. Zahabi, Physical dimer model and quiver gauge theories, in preparation.

2) R. Szabo, A. Zahabi, Microscopic entropy of crystal melting black holes, in preparation.

3) A. Zahabi, New Phase transitions in Chern-Simons matter theory, arXiv:1505.00673

4) A. Zahabi, Vertex Operator Algebra, Conformal Field Theory and Schramm Loewner Evolution in Ising Model, arXiv: 1505.01405

5) C. Hongler, K. Kytölä, A. Zahabi, Discrete Holomorphicity and Ising Model Operator Formalism, Contemporary Mathematics 644, 79-115, (2015).

6) Y. Sasai, A. Zahabi, Shear Viscosity of a Highly Excited String and Black Hole Membrane Paradigm, Phys. Rev. D 83,026002 (2011).

7) M. Långvik, A. Zahabi, On Finite Noncommutativity in Quantum Field Theory, Int. J. Mod. Phys. A **25**, 2955 (2010) .

8) M. Chaichian, A. Tureanu, A. Zahabi, Solution of the Stochastic Langevin Equations for Clustering of Particles in Random Flows in Terms of Wiener Path Integral, Phys. Rev. E **81**, 066309 (2010).

Future Research

My general plan for future research is to investigate the connections and dualities between CFT's and string theory on the one hand, and different statistical continuous and lattice models and their continuum limits on the other hand by using non-perturbative and rigorous mathematical methods and techniques. This would basically lead to a better understanding of non-perturbative aspects of QFT and string theory, as well as a more transparent picture for continuous and lattice models and their continuum limit. Moreover, the relations between gauge theory-string theory and statistical physics have led to a new paradigm, the relations between gauge theories and integrability. One of the examples of this duality paradigm is the relation between crystal melting and dimer models, and string theory-gauge theories.

The topic of 2D statistical lattice models is a rich subject from physical and mathematical perspectives. The variety of exact mathematical methods, such as discrete holomorphicity and integrability, and useful physical information that these models provide, has made them as one of the main theme in studies toward understanding of different aspects of higher dimensional theories.

On the other hand, a natural extension of QFTs to supersymmetric field theories provides another approach to study more constraint field theories which might have a chance to become exactly solvable. Furthermore, these supersymmetric QFTs show some of the main features of realistic QFT such as chiral symmetry and confinement etc.

In this plan, the general goal is to study the relations between gauge theory/string theory and statistical physics which have been grown in two different but closely related directions. First, the holography principle (e.g. AdS/CFT) and its applications in condensed matter and statistical integrable systems. Second, the four-dimensional quiver gauge theories and their relations to lower dimensional discrete models and integrable systems such as dimer and crystal model.

There is a specific direction for my future research plan that is to further study of the mentioned relations between QFT, string theory and statistical systems. Beside the large scale structure of the research plan, which reflects my interests and desires in a broad and diverse sense from past works to present and future works, there is a concrete research plan in crystal and dimer models / string theory duality with concrete strategies and affordable steps.

Research Proposal

From crystal melting model/dimer model to toric geometry, quantum field theory and string theory

a. Present research related to research plan

<u>Introduction and general idea: Random partitions/quiver gauge theory/topological string theory</u>

Understanding the quantum nature of space-time is a long-standing problem in theoretical physics. Dualities between string theories and gauge theories give a deeper understanding of quantum space-time. In general, low-energy regime of branes in string theoey may explain the gauge field theory and hopefully the standard model of particle physics. In the duality context, when the target space of the string theory is a Calabi-Yau manifold, the corresponding field theory is a complicated quiver gauge theory which consists of a generalized diagrammatic approach to gauge theory, and is of special interest. There are many studies in string and brane theories, dual to these quiver gauge theories, for a review see [1]. Understanding a stack of D-brane on a tip of a Calabi-Yau manifold and its low energy limit which is a supersymmetric gauge theory is an especially important step towards deeper understanding of the string/gauge duality and quantum nature of space-time.

Exact nonperturbative solvability of quantum field theory is a closely related open problem. There has been considerable progress towards this problem especially in the case of supersymmetric gauge field theories, namely the “Seiberg-Witten solution” of $N = 2$ supersymmetric gauge theory [16] and Nekrasov rederivation of this result using instanton methods [17]. Moreover, the integrability and 2D CFT methods and techniques have been played important roles in these studies, namely the Bethe ansatz for supersymmetric $N = 4$ gauge theory by Minahan and Zarembo [18] and recent results by Alday, Gaiotto and Tachikawa on the relation between 4D $N = 2$ gauge theory and 2D Liouville field theory [19].

In order to better understand the string theory/gauge theory dualities in interesting cases with less susy ($N < 4$) and more complicated gauge groups, there have been considerable studies about the relation between the dimer model and crystal melting model on the one hand, and brane tiling physics and topological string theory and geometry of the Calabi-Yau spaces on the other [2]. In fact, crystal melting models provide a non-perturbative (statistical/stochastic and geometrical) approach towards this problem. These studies are mainly motivated by the seminal work of Kenyon, Okounkov and Sheffield on the limit shapes of the dimer model and amoeba [3] on the statistical/stochastic side, and also by the important papers by Okounkov, Reshetikhin and Vafa on quantum Calabi-Yau geometry and classical crystals [4] and the paper by Okounkov and Nekrasov on Seiberg-Witten theory and random partitions [7] on the string/gauge theory side. These connections between high energy and low energy physics can lead to a better understanding of non-perturbative aspects of QFT and string theory, as well as a more transparent picture for statistical systems.

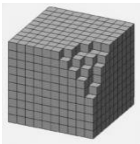
One particular aim of studying these relationships is to find a tool for counting the bound states of D-branes and BPS states in string theory by using crystal techniques. However, at high temperature limit, many atoms are removed from the crystal and the boundaries of the crystal form a specific shape which is called limit shape. Physically, these correspond to heavy bound states of many branes. The further connections of the scaling limit of the limit shape to the BPS counting and quantum Calabi-Yau manifold are discussed by Ooguri and Yamazaki [5], [6]. On the statistical physics side, there are probabilistic methods to obtain the limit shape of the melting crystal [3]. Form a high energy physics viewpoint, scaling limit of the thermodynamics limit of lattice models such as crystal melting and dimer models on toric topologies are closely related to brane tiling, supersymmetric gauge theories and topological string theories [1], [2], [4]. The geometric aspects of the dimer model, its relation to algebraic geometry and in special troic geometry is under active studies, for a good overall view on the subject see [20].

From another point of view, the relation between the certain statistical models of 2D and 3D random partitions, and gauge theories and string theory has been developed. The crystal melting model is closely related to the 3D partitions. The relation between the partitions and gauge theories is that the statistical properties of the random partitions are reflected in the dynamics of the gauge theories. As an example, the partition function of the four-dimensional $N = 2$ supersymmetric gauge theories is defined as the sum over 2D partitions weighted by a generalization of the Jack measure. At thermodynamic limit, the limit shape of the 2D partitions appears in this context precisely as the Seiberg-Witten curve [7]. Another example is the statistical model of random plane (3D) partitions and its interpretation as the 5D $N = 1$ supersymmetric SU(N) Yang-Mills theory on $R^4 \times S^1$ and as a Kähler gravity on local SU(N) geometry. The limit shape of the model is related to a hyperbolic curve as a 5D generalization of SW curve via the Ronkin function and the amoebas [8].

Moreover, the relation between plane partitions and quantum Kähler gravity has been studied, [9]. The origin of this topic is the connection between topological string theory and the statistical model of random plane partitions via the topological vertex. Furthermore, it has been shown by Ooguri and Yamazaki that the smooth geometry of the Calabi-Yau manifold emerges in the thermodynamic limit of the crystal melting model [6].

Backgrounds:

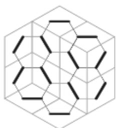
The *crystal melting model* (CM) is a statistical mechanics model that describes a melting corner of a semi-infinite crystal. Cubic crystal model is represented by 3D Young diagrams, "plane partitions", and therefore, CM is referred to as a model of "*random plane partitions*". The partition function is



$$Z = \sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = M(q),$$

where $|\pi|$ represent the number of cubes in the configuration π in the plane partition. This model is exactly solvable since its partition function can be obtained as "MacMahon function" $M(q)$.

The *dimer model* is a model of diatomic molecules on the surface of a crystal. It is a planar graph $G = (V, E)$, with vertices V and edges E. A *dimer configuration* of G is a *perfect matching* of G , which is a subset of edges C such that every vertex of G is incident to a unique edge of C. The partition function is given by the energy of a dimer covering $\varepsilon(C)$ that is defined as the sum of the energies of those bonds covered with dimers:



$$Z = \sum_{C\in\mathcal{M}(G)} e^{-\varepsilon(C)/kT}.$$

Dimer model is also exactly solvable since its partition function can be written as a "Pfaffian of the Kasteleyn matrix K, the associated signed, weighted, adjacency matrix of the graph"

$$Z = |Pf(K)| = \sqrt{|detK|}.$$

Characteristic polynomial of the dimer model in periodic planar graph on the torus $G_n = G/n\mathbb{Z}^2$ is defined by

$$P(z,w) = \sum_{M\in\mathcal{M}(G)} e^{-\varepsilon(M)} z^{h_x} w^{h_y} (-1)^{h_x h_y},$$

where h_x, h_y to be the horizontal and vertical height change of periodic graph M around the torus.

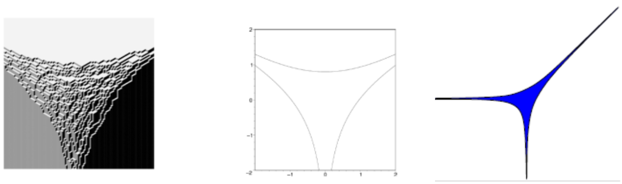
Another closely related subject is toric geometry. There is correspondence between the dimer model and toric Calabi-Yau geometry. This correspondence is based on the idea that the D3-branes are probes on the Calabi-Yau manifold and therefore they can be used to study the cone geometry. Furthermore, the moduli space of D-branes provides the tools for D-branes to probe the neighbourhood of the singularity. This moduli space can be parameterized naturally by the dimer model. The Newton polynomial of the toric Calabi-Yau manifold X is defined by

$$P(z,w) = \sum_i c_i(t) z^{n_i} w^{m_i},$$

where the exponents (n_i, m_i) correspond to lattice points of the toric diagram and $c_i(t)$'s are functions of the Kähler moduli t of the toric Calabi-Yau 3-fold X.

Another interesting concept is the amoeba, [12]. A toric manifold X is specified by its Newton polynomial or equivalently the characteristic polynomial $P(z,w)$. Consider the equation $P(z,w) = 0$ which defines the spectral curve. The amoeba is obtained by a logarithmic projection of the spectral curve. In fact, the amoeba is defined as the set of solutions to the spectral curve, with a logarithmic mapping:

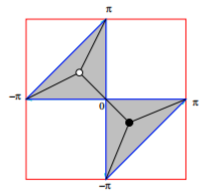
$Amoeba(P) = Log(P^{-1}(0))$ with $Log(z,w) = (\log|z|, \log|w|)$.



Left: limit shape of cubic CM, middle: boundary of the projection of limit shape, right: amoeba

From the statistical point of view, as it is shown in the theorem 3 part c, amoeba is the phase diagram of the limit shape. Roughly, this means that atoms inside the amoeba are removable, with the ones outside cannot be removed without removing other atoms first. For a complicated X, solving the boundary of the amoeba is also a difficult task, requiring Monte-Carlo methods.

In addition to amoeba, it is natural to define the alga as the phase projection mapping of the spectral curve: $Alga(P) = Arg(P^{-1}(0))$ with $Arg(z,w) = (\arg(z), \arg(w))$.



Alga of the cubic CM and its corresponding dimer

b. Purpose of proposed research:

<u>Questions driving the proposed research and hypothesis</u>

The general purpose of this research plan is to deeply understand the three-fold connections between the geometrical, statistical (probabilistic-stochastic) and stringy aspects of the limit shape and its projections; amoeba and alga in the crystal melting model, quiver gauge theory, brane tiling and topological string theory in a unified picture. In general, first we want to systematically study the limit shape and its amoeba and alga projections in different crystal models from the probabilistic and geometric points of views. Second, we wish to find the interpretations and implications of these results in gauge theory and string theory.

The combinatorial and geometrical aspects of the relation between dimer model, brane tiling and quiver gauge theories are understood in a good level but the statistical and probabilistics aspects of this relation and its implications and importance are not completely investigated and understood. Especially, clarifications on the relation between statistical setting of dimer model, and geometrical setting of brane tiling and quiver gauge theories are beneficial for both sides. One approach to this problem is via the study of the limit shape.

The specific purpose of this research plan is to further systematic study the statistics and dynamics such as energy dependence and fluctuations of the limit shape, amoebas and algas of 2D and 3D random partitions with given measure, in certain dimer and crystal models by using the probability methods. Then, by using the algebraic geometry methods we want to study and indicate the implications and relevance of the above results, such as moduli dependence of the amoeba for brane tiling, topological string theory and supersymmetric gauge theories, and their dualities.

Toric geometry information fixes some features of the limit shape and its amoeba and alga e.g. the spine of the amoeba are given by the (p, q) web of the toric diagram. However, other features of the amoeba such as the interior parts are controlled by the statistical parameters such as energy of dimer or equivalently by the algebraic geometry parameters such as Kähler moduli. Despite few studies on the problem, there is no complete understanding of the properties and characteristics of mentioned aspects. We are aiming for a complete classification and physical meaning and applications of the limit shape, amoeba and alga of the mentioned model.

Moreover, further studies on the symmetries and characteristics, and statistical properties such as dynamics and fluctuations of the amoeba are beneficial from geometrical and physical points of view for other applications. We plan to pursue this goal via several simple but nontrivial and concrete examples and calculations. The more ambitious goal for further study is to fully understand and clarify the implications and importance of limit shapes for string theory and gauge theory. However, for the present research, there are mainly three research questions:

Specific Objectives:

1) Suppose that the statistical crystal/dimer models, measure of 3D random partitions, and their corresponding supersymmetric gauge theories and topological string theory are given. Then, how the morphology (geometry and topology) of the limit shape and amoeba, and the 5D SW curves depends on:

   a) The parameters of the characteristic polynomial of the model, such as the statistical measure, statistical ensemble and energy distribution of crystals/dimers ε.
   b) The algebraic geometry parameters, coefficients of the Newton polynomial of the amoeba, such as the Kähler moduli t of the Calabi-Yau manifold. A clear understanding of the relation between these two different, but related, parameter spaces of the problem is of special interest.

   The similar questions for the 2D partitions, its limit shape and SW curve can be proposed. For example, different choices of the measures lead to different gauge theories, e.g. a generalization of the Jack measure on random partitions gives the $N = 2$ 4D susy gauge theory. This is an example of a more general probabilistic approach towards the understanding and classification of gauge theories and their embedding in string theories, which has not studied completely.
   In general, there are no such studies about changing the measure in plane partitions for 3D CM and its implications in high energy physics. There are few studies about 2D crystal models and there are fewer studies about the changing of the energy for the random 2D partitions.
   The more ambitious question is the classification of the amoebas according to different statistical/geometrical models or different parameters of the same statistical/geometrical model.

   Another closely related problem as introduced in the background part is the alga. The alga is not well studied from mathematical point of view. From physical point of view, it has been observed that the alga gives the dimer model [21]. The purpose is to find the lattice statistical model origins of the alga from the crystal model in a similar way that amoeba is related to limit shape. Furthermore, it would be interesting to investigate the statistics for the alga similar to what have been done for amoeba. Studying the relation of the alga and the 5d gauge theories (5d SW) might be interesting.

2) The probabilistic questions about central limit theorem corrections to the limit shape and local statistics of the surface in various regions of the limit shape are the next natural questions. The initial studies about these questions are performed (theorem 3 part c). In this direction, first we want to further study the probabilistic methods and stochastic processes that explain the fluctuations of the limit shape of the crystal melting in a special case, with the volume measure. The goal is to specify the Schur processes for this special measure and its natural generalizations. Then, we want to answer the following question: what are the implications of the dynamics and fluctuation of the limit shape for gauge theories and string theory? It would be fruitful to further study the Schur processes in the dynamics of the supersymmetric gauge theories. For example, statistical fluctuations lead to the corrections to the limit shape (in 2D and 3D) and that could be related to the corrections to SW curves (in 4D and 5D). Furthermore, the chamber dependence of the BPS invariants can be studied from the fluctuations and quantization of the limit shape which has been initiated in [31].

   It has been observed that the thermodynamic limit of the partition function of the crystal melting model gives the genus-0 topological string partition function of the toric Calabi-Yau manifold [5]. Understanding the role of measure and energy, and the statistics and fluctuations of the limit shape and the amoeba in the scattering amplitudes of topological string theory is one of the goals of this plan. On the other hand, since the genus-0 topological string partition function is given by the limit shape of the crystal model, this proposal can be used to calculate the higher genus partition function of topological string theory by using the stochastic processes and correlation functions of the crystal model which reflect the statistics and fluctuations of the limit shape [10], [11].

Moreover, the different phases e.g. gaseous, liquid and solid, of the amoeba in the dimer model á la Kenyon is known (theorem 2 in part c). However, understanding the behavior of the limit shape and the amoeba when the parameters of the model such as energy change (this is partially understood in theorem 1, 2 part c), provides an approach towards this question: how the phases of the dimer model evolve as the parameters of the model change. It would be interesting to apply this result to specific models with specific energies and geometries and finding their phase diagrams and then study the implications of these results for gauge theories and string theories dual to these models such as brane tiling.

3) Finding a geometrical approach to obtain the limit shape and the amoeba of crystal model based on geometrical and topological information of the model is an interesting problem. On the statistical physics side, there are probabilistic, variational methods to obtain the limit shape of the crystal melting model in 2D and 3D [3]. Our aim is to study a simpler way of finding the limit shape, by means of the deterministic methods inspired by geometric picture of the model, formulized in algebraic geometry language. With Klaus Larjo we have proposed a way for finding the limit shape and amoeba of the crystal model on arbitrary geometry in 2D and 3D. The idea is to use just the geometrical data of the model and finding the curves of equal number of removed atoms from the crystal, combinatorialy. We guess that this simple geometrical/combinatorial algorithm gives the limit shape in the “tropical limit”. The purpose is to prove/disprove this guess.

Once we obtain a thorough understanding of the answers to the above three research problems in the simplest, nontrivial examples, then the generalizations to crystal melting with other measures, or with an external potential, or on the more complicated Calabi-Yau manifolds might be possible and possibly will have interesting implications for string theory and gauge theory. Even though there are some desperate hints, but these questions are not yet fully addressed and understood in the existing literature.

c. Proposed plan:

Methodology:

In general, the analytic methods and techniques employed to study the proposed research questions are in two different but closely related categories:

1) Toric algebraic geometry and its relation to brane tiling and quiver gauge theory, [1], [2]: Consider a stack of D3-branes in the space, $M^{1,3} \times X$ where X can be a Calabi-Yau threefold or a troic Calabi-Yau cone. Consider the case where the stack is located on orbifold point in Kähler moduli space of the cone. Then the D-branes break into fractional branes, and low energy gauge theory on the branes involves products of several gauge groups, SU(N) factors, coupled to matter in bifundamnetal representations. This theory can pictorially be represented as the quiver diagram, where the gauge group is the vertex and the scalar fields x_i are the arrows. The geometry of the internal space X determines the supersymmetry and gauge symmetry as well as matter content of the low energy field theory. This setup has been understood very well for special examples of the internal space X such as conifold etc.

The quiver can be placed on a torus T^2 and then opened up into a planar quiver that encodes all the information of the theory e.g. the superpotential terms W which are given by the faces in the new diagram. The dual of this graph is a bipartite graph called the brane tiling or dimer model.

Gauge theory	Planar quiver	Brane tiling
Gauge group	Vertex	Face
Scalar field	Arrow	Edge
Superpotential term	Face	Vertex

These graphs carry the path algebra; the elements are paths on the planar quiver, and the product of two paths is defined to be the concatenation of path. By introducing an equivalence relation among the set of paths in the algebra we can define the factor algebra which automatically satisfy the F-term relations $\frac{dW}{dx_i} = 0$. The elements of the factor algebra correspond to atoms of the crystal. Any path starting at i_0 can be given in the form $v_{i_0 j}\omega^n$, where $v_{i_0 j}$ is the shortest path between vertices i_0 and j, and ω is the loop around a vertex in the brane tiling or the superpotential term. $v_{i_0 j}\omega^n$ defines a crystal; $v_{i_0 j}$ correspond to atoms on the face of the crystal, while paths $v_{i_0 j}\omega^n$, with $n > 0$ are atoms inside the crystal.

It was shown by Douglas and Aspinwall [23, 24] that BPS D-branes can be understood as elements in the derived category of coherent sheaves on the Calabi-Yau manifold: $D(Coh\ X)$.

It is known that for toric X the derived category of coherent sheaves is isomorphic to the derived category of θ-stable A-modules, $D(Coh\ X) \cong D(A_\theta - Mod)$; where A is the factor algebra defined earlier. The derived categories describe topological branes all over the moduli space of the Calabi-Yau manifold. However, only some of these branes are stable, and it is these ones that θ-stability picks out. Physically, θ-stability will ensure that the configurations satisfy the D-term conditions. However, we do not care about the whole moduli space; we only care about the region in moduli space where X is a toric cone. Moving away from this point in moduli space would correspond to (partly) resolving the conical singularity; as happens in the story of the resolved conifold. But we want to keep the singularity, at least for now. It turns out that in this region of moduli space, the `derived' part drops off, and we are left with

$$\{BPS\ branes\ on\ X\} \cong D(Coh\ X)|_{toric}^{\theta-\text{stab.}} \cong\ A - Mod \cong Ideals\ of\ A\ .$$

Thus, we are interested in modules of the factor algebra A. But for a toric X satisfying θ-stability, modules of A are in one to one correspondence with ideals of A. We thus conclude that molten crystal configurations are in one-to-one correspondence with bound states of D-branes on X. All atoms carry D-brane charges (D0 and D2), and removing atoms is equivalent to adding branes to the bound state. So the original unmelted crystal corresponds to simply the D6-brane with no other branes, and removing atoms from the full crystal adds D0's and D6's into the state.

In summary, a full understanding of BPS branes is difficult, but for a toric X the derived category drastically simplifies, and all the information about bound states of branes on X is contained in a crystalline structure. Moreover, we have the correspondence between the perfect matching of the dimer model on two-torus and the bi-fundamental fields of gauged linear sigma model in the Kähler quotient construction of the toric Calabi-Yau manifold [1]. The Kähler moduli of Calabi-Yau are parameters of the quiver quantum mechanics.

2) Probability theory and Combinatorics of 2D and 3D random partitions such as dimer / crystal models [3], [10], [11]: The limit shape of the 2D and 3D random partitions is a hypersurface which minimizes the free energy and surface tension. The problem of finding the limit shape of 2D partitions has been studied extensively, and recently in the 3D case it is solved by Kenyon, Okounkov and Sheffield, [3]. Moreover, they have formulated a probabilistic framework to study the fluctuations of the height function.

In another direction, stochastic processes are used to study the dynamics of the models. The fluctuations of the limit shape can also be studied by these stochastic processes especially with determinantal random processes, [10, 26]. Conjecturally, the limit shape controls the answers to all probabilistic questions such as "central limit theorem" corrections to limit shape. For example, the Gaussian correction to the limit shape is given by the massless free field in the corresponding conformal structure [25]. Furthermore, the correlation functions of Schur processes in 3D random partitions can be studied by using free fermions [11].

We expect that the following theorems can be used to further study the moduli space and dynamics of $N = 1,2$ 5D gauge theories [8] and their embedding in string theory.

> **Theorem 1** [13] "By varying the edge energies all Harnack curves can be obtained as the characteristic polynomial of a planar dimer model".
>
> **Theorem 2** [3] "The measure in dimer model is respectively frozen, rough, or smooth according to whether the inverse of the gradient of the Ronkin function (Legendre dual of the free energy) is in the closure of an unbounded complementary component of the amoeba, in the interior of the amoeba, or in the closure of a bounded component of the amoeba".
>
> **Theorem 3** [25] "In the thermodynamic limit, the scaling limit of the scaled height function converges to a mean value whose surface is the limit shape. The fluctuation of the unrescaled height function will converge in law to a random process in that domain. In simple case, for honeycomb dimers on unbounded domain with $\varepsilon = 0$, the height fluctuations converge to Gaussian process."

These results make our study towards: 1) the relation between the algebraic moduli space of amoebas and the statistical parameter space of limit shapes and 2) dynamics of the limit shape much more tractable.

<u>Programme of research:</u>

The proposed plan to answer the research questions consists of few initial affordable steps of which the starting points have been studied:

A) Research question 1: The formation of the limit shape, its relation to amoebas, and their algebraic and probabilistic properties for 2D and 3D random partitions is understood. Furthermore, we have found the bijections between the crystal melting model with volume measure in the periodic and non-periodic cases, and dimer model on the same region. These results are the starting point for further studies toward understanding the proposed research question 1.

   As we mentioned, the limit shape and amoeba is closely related to toric Calabi-Yau manifold. In fact, the limit shape of the crystal in the thermodynamic limit coincides with a projection of the mirror Calabi-Yau manifold [5, 21]. An essential result is the equivalency between Newton polynomial for the mirror of a toric Calabi-Yau manifold and the characteristic polynomial of its corresponding dimer/crystal model on the torus which has been proved in [21] and in section 4.2 of [6].

   On example which shows the response of amoeba shape with respect to the parameters of the Newton polynomial is given in fig. 17 of [1]. In addition, it has been shown that the algebraic Harnack curves are related to the limit shapes and amoebas of the dimer model with different dimer energies, theorem 1 [13]. The full descriptive understanding of this theorem (in various examples and also general case) and its physical and geometrical meaning and implications especially its relation to the above example is not available now. To our knowledge there are few cases in the literature that can be considered as the physical (quiver gauge theory and brane tiling) and geometrical (Gromov-Witten theory) examples and consequences of the above theorem [21], [27]. However, the relation between these examples that are extremely similar has not pointed out.

The very first step of this study would be the investigation of the tropical limit ($t \rightarrow \infty$) in different examples and its statistical meaning and realization. For example, by changing the energy or equivalently the Kähler moduli one can move around the moduli space of the troic manifolds and in the tropical limit obtain the pyramid partition (resolved conifold) from the plane partition [28]. One of the purposes of this study is to understand such a transition and the implications of changing the parameters of characteristic polynomial or Newton polynomial e.g. for resolution of singularities etc., in full generality.

The relation between the alga and the dimer model is proposed in [21]. However, its relation to crystal model is not well understood. It is not clear how it emerges in the thermodynamic limit of the crystal model and what are the statistical implications of the alga, if any. In addition to the immediate guess from [21] about the relation between alga and dimer model and the relation between dimer and crystal model, there are other concrete ideas inspired by the relation between crystal model and the complex Burgers equation [22]. In fact, the Burgers equation is obtained from crystal model via a map which is similar to the alga map.

B) Research question 2: The fluctuation of the limit shapes in the dimer model can be studied using different techniques and methods. The correlation functions of the dimer model, as an exactly solvable model, is known [3, 11] and moreover, many studies about the stochastic growth process for 2D and 3D random partitions have been performed in the existing literature [10, 26]. Assuming the relation between the dimer/crystal model and topological string theory, these techniques and methods can be used to calculate the higher genus partition function of the topological string theory. The expected results of this study can be compared with the recently obtained results about the higher genus partition functions of the topological string theory from the matrix model [14, 15].

C) Research question 3: Our deterministic approach to the limit shape problem is based on the toric geometry data such as zig-zag paths. We use the fact that ridges of the crystal are along the zig-zag paths (which turn maximally left and right) or tentacles of the amoeba. By a geometrical method, we obtained the curves of equal number of removed atoms *n* in the thermodynamics limit. That is, if we consider all the configurations with *n* atoms removed, then atoms outside of this curve will not have been touched, while the atoms inside the curve have been removed in some of the configurations. This result is expected to match with the limit shape of the melting crystal and its projection, amoeba, in the tropical limit, in which the measure and energy are erased from the model. Therefore, by using the toric geometry data, we provide a simple method for studying the complicated problem of finding the limit shapes of the melting crystals on arbitrary geometries. Finally, we expect that the answer to the research question (1) facilitate the study and pave the road for the answering of research question (3).

d. Expected results and impacts:

The proposed study and research connect two quite distinct areas of mathematics and mathematical physics: Dimer/crystal models and geometrical string theory. The project shed light on the relation between the dimer and crystal melting statistical models, and toric geometry, quiver gauge theory and topological string theory by reformulating the existing results in one side to the other and providing a deeper insights and extending the techniques in the both sides.

In a broad sense, it will provide a new statistical engineering of QFTs. Making clear what is the physical meanings and implications of the energy of the dimer model and its variation in the brane tiling, toric geometry, quiver gauge theory and topological string theory is one of the very first expectations of this study. This study will reveal many special properties of crystal melting partition functions. Specifically, dynamics and statistics of the limit shapes will be studied from algebraic geometry and probabilistic point of views, and this study would lead to new insights and results for the dynamics and statistics of supersymmetric gauge theories and topological string theory such as the chamber dependence of the BPS invariants and higher genus partition function of the topological string theory, among other things. We expect to provide a clear understanding of the moduli space of the amoebas in toric tropical algebraic geometry based on statistical information of the model. We expect a quantitative understanding of this concrete question: how does varying the energy of dimers in the Kenyon setting corresponds to the moving around in the moduli space of the toric manifold, i.e. to resolving the toric singularity. In fact, we expect to develop a statistical approach to partially or completely resolve the singularities of the toric Calabi-Yau threefold. Finally, we expect a new simple way to find the limit shapes of the crystal models on arbitrary geometries.

e. Future perspective studies:

Quantization of classical dimer and crystal models by stochastic quantization methods [29] and non-commutative geometry methods [31], and its relation to topological M-theory [30] need further speculations and investigations.

Another possible project would be much more ambitious and challenging. It starts with the observation that the bound states that the crystals count form a classical configuration space that needs to be quantized. A suitable method of quantization is given by stochastic quantization [29]. Stochastic quantization works by introducing an additional `time' direction. This seems very reminiscent of lifting to M-theory. Is there a connection to topological M-theory?

Topological M-theory was discussed and formulated in [30], and it essentially comes about by introducing a new time direction to some form-theories of gravity. Topological M-theory reduces to type-A/B models with suitable choices of forms. This is very reminiscent of geometric quantization, where one chooses a polarization, which corresponds to choosing to work with the position or momentum basis in the quantum system. Assuming the connection to quantum crystals, there should be a corresponding choice to decide whether one works with A/B model. The crystals are firmly in the A-model; can we find the B-model starting from the quantum crystals?

Relevance to academic beneficiaries:

This project will be of large interest for both mathematical statistical physics and high energy physics communities, and hopefully, bridge the gap between them. For researchers in dimer model and related probabilistic descriptions of scaling limits, it will provide an understanding of the powerful algebraic/geometric tools from string theory such as mirror symmetry. For researchers in QFT and string theory, where the accent is usually put on physical and geometric aspects of QFT and string theory, the probabilistic and stochastic methods in the dimer/crystal models would lead to a completely new direction of research. The project is likely to give rise to a larger visibility of all the mentioned topics in general in the mathematical and physical communities, accelerating the developments in all these areas. It will also foster the general ideas of tropical algebraic geometry, which may find applications in completely new domains of studies in theoretical physics.

Certainly, all experts and collaborators of the project have displayed, and will continue to display interest in this research. Hence, discussions not only will help me in performing the research, but will help disseminating the results and ideas faster.

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