VERTEX OPERATOR ALGEBRA, CONFORMAL FIELD THEORY AND STOCHASTIC LOEWNER EVOLUTION IN ISING MODEL

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ABSTRACT. We study the algebraic construction of the conformal field theory (CFT) and its relation to the Schramm Loewner evolution (SLE) in an example of the Ising model. We first obtain the rigorous scaling limit of the correlation functions of Ising free fermions on an arbitrary simply connected two-dimensional domain D in the explicit form of Pfaffians. Then, we study the algebraic and analytic aspects of the fermionic conformal field theory on D in terms of the Clifford vertex operator algebra (VOA). This construction leads to the fermionic Fock space of states and conformal field theory of the Fock space fields of the Ising free fermions. Furthermore, we investigate the conformal structure of the fermionic Fock space fields, namely their operator product expansions, correlation functions and differential equations. Finally, by using the Clifford VOA construction and the fermionic CFT, we investigate a rigorous realization of CFT/SLE correspondence in the case of the fermionic CFT/SLE_3 in two directions: operator formalism and correlation functions. By studying the relation between the operator formalism in VOA and SLE_3 martingale generators, we found an explicit Fock space for the SLE_3 martingale generators. Also we obtain a large collection of SLE_3 martingale observables in terms of the correlation functions of fermionic Fock space fields which are constructed from the Clifford VOA.

CONTENTS

1. INTRODUCTION

The Ising model was introduced in 1925 by W. Lenz as a model describing a ferromagnet on lattice. The Ising model consists of spins $\sigma_\alpha = \pm 1$, on the vertices α of the lattice, interacting by short range neighborhood self-interactions as well as interactions with the external magnetic field B. In the case of $B = 0$, in contrast to the one-dimensional model, two-dimensional Ising model in $\mathbb{Z}^2$ possesses a second order phase transition at critical inverse temperature ($\beta = \frac{1}{k_B T}$), $\beta_c = \frac{1}{2}\ln(\sqrt{2}-1)$, as shown in the pioneering works by Peierls, Kramers, Wannier and Onsager, [Pe36], [KrWa41], [Ons44] and [KaOn49]. At $\beta = \beta_c$ specific heat and magnetic susceptibility diverge to infinity and for $\beta > \beta_c$ spontaneous breaking of the symmetry leads to a nonzero magnetization. Moreover, due to infinite dimensional symmetry, many physical properties of the $2d$ Ising model such as free energy and spin correlations at $B = 0$ can be computed exactly.

The transfer matrix formalism is one of the approaches toward the exact results in Ising model, [Ba08]. Specially the free fermion operators play an essential role in this formalism, [Kau49]. Moreover, the Fock space representations of the transfer matrix formalism and free fermions have been studied extensively, for a review see [Pal07]. In this paper, we study the connections between the discrete fermionic formalism of the Ising model and the rigorous aspects of a fermionic conformal field theory that describes the scaling limit of the model.

There is a common belief that the scaling limit of the lattice models such as Ising model at criticality are described by a field theory with the conformal symmetry. However, it is known that the spin operator is not enough to describe the continuum limit of the theory. In fact, it is believed that the critical Ising model in the continuum limit is described by a conformal field theory, namely the theory of free fermionic fields. Roughly speaking, the fermionic field is identified with the scaling limit of the fermionic operator on the lattice. However, there were no exact proofs about the scaling limit of the Ising model free fermions and their correlation functions. Recently, the rigorous methods from discrete analysis and probability theory have provided exact proofs about the conformal invariance in the scaling limit of the Ising model at criticality, for a good review on general aspects of discrete holomorphicity see [Car09] and [DuSm11]. We have used these techniques to obtain the scaling limit of the correlation functions of Ising free fermions with specific boundary conditions, rigorously. The continuum correlation functions are obtained from the scaling limit of the lattice correlation functions obtained in [HKZ12]. By means of these methods, we find a proof of the Pfaffian formula for the correlation functions of free fermionic fields, in the scaling limit.

Moreover, a new rigorous formulation of the continuum Fock space of fields and their properties; CFTs on domains with boundaries in the case of Gaussian free fields is proposed in [KaMa11]. We have extended and adopted a similar formulation to obtain a formulation of the conformal field theory on bounded domains in the case of free fermion fields of the Ising model. In this approach, we have obtained the characteristic features of fermionic conformal field theory on a bounded domain such as transformation rules for fields and their correlation functions, the operator product expansion of fields, Virasoro algebra representation and the Ward identity.

On the other hand, vertex operator algebra (VOA) provides a concrete mathematical language for CFT, [Ka98]. The vertex operator algebra is an algebraic construction for conformal field theory in terms of formal power series. The general VOA has been adopted in different cases for different purposes such as VOA for bosonic and fermionic fields. In this paper we have used the Clifford VOA for fermionic fields which has the Clifford algebra symmetry in addition to Virasoro algebra symmetry. The Clifford VOA, as an equivalent algebraic formalism to fermionic conformal field theory, turns out to be useful in study of scaling limit of the Ising model at criticality. Specially, we have obtained the Fock space of fermionic states in terms of VOA vector space.

From a different perspective, Schramm Loewner evolution plays a crucial role in this picture. The SLE is a stochastic process that is defined by a stochastic differential equation, the Loewner equation with the Brownian motion as a driving force. In general, SLE curves explain the scaling limit of the interfaces of the statistical lattice models on domains with boundary, at critical temperature. Specially, it has been proved in

[CDHKS12], that the scaling limit of the interfaces in $2d$ critical Ising model is described by a Schramm Loewner evolution, SLE_3. In SLE, the probability measures of the interface curves satisfy the conformal symmetry and the Markov property. These are physically expected conditions that the scaling limit of the interfaces should satisfy.

In this paper we combine the approach of Clifford VOA and the Fock space of conformal fermionic fields in order to obtain a unified picture of a conformal field theory describing the scaling limit of the critical Ising model. To have a unified picture, we need a mapping between the Clifford VOA and correlation functions of the Fock space fields which satisfies the axioms that are reflecting the analytic and algebraic aspects of the underlying conformal symmetry in the scaling limit of Ising model at criticality. This has been done through the main theorem of this paper.

Another aspect of this study refers to the well-known CFT/SLE correspondence, [BaBe06]. We employ the aforementioned framework of the VOA and Fock space fields approach to CFT, and their inter-relation in order to concretely investigate an example of the CFT/SLE correspondence in the case of Ising model. We have obtained the results indicating a rigorous realization of the fermionic CFT/SLE_3 in terms of Clifford VOA and Fock space of fermionic conformal fields. The Clifford VOA provides a fermionic Fock space for the SLE_3 martingale generators and furthermore, a large collection of SLE_3 martingale observables are explicitly written in terms of the correlation functions of the Fock space fields that are corresponding fields to the states of the Clifford VOA.

2. Fermionic theory of Ising model

In this section we explain first the transfer matrix formalism which is an approach towards an exact solution of two-dimensional Ising model on a rectangle with specific boundary conditions. Specifically, we express the correlation functions of any operators in this formalism. Then, we introduce the notion of the free fermions in Ising model and their correlation functions on the lattice. Eventually, by using methods from discrete complex analysis the scaling limit of the fermionic correlation functions on the half plane and other conformally equivalent domains are obtained. These are CFT correlation functions in the Fock space construction of the free fermionic fields.

2.1. **Transfer matrix formalism in Ising model.** The Ising model on the domain $\Lambda_{M,N} = \{(j,i) \in \mathbb{Z}^2 \mid |j| \leq M, |i| \leq N\}$, consists of spins $\sigma_\alpha = \pm 1$, on the vertices α of the lattice in the domain $\Lambda_{M,N}$. The model is parameterized by the inverse temperature β and nearest neighbor interaction coupling J between $< \alpha, \alpha' >$, the pairs of sites that are nearest neighbors. The Ising model is defined by its partition function which contains all the physical and geometrical information of the model via the Hamiltonian and the domain geometry and its boundary conditions,

$$Z_\Lambda(\beta) = \sum_{\sigma \in \mathcal{C}_\Lambda} \exp\left(\beta \sum_{<\alpha,\alpha'>\subset\Lambda} J\sigma_\alpha\sigma_{\alpha'}\right), \tag{1}$$

where the sum is over spin configurations σ in $\mathcal{C}_\Lambda = \{\pm 1\}^{\Lambda_{M,N}}$ which satisfy the boundary conditions. For simplicity we can set $J = 1$.

As we mentioned, the transfer matrix formalism can be used to calculate the partition function and correlation functions of operators such as spin, energy etc. in planar Ising model on the rectangle with the specific boundary conditions. In order to calculate the partition function and correlation functions in transfer matrix formalism, the sums over all configurations in partition function and correlation functions are divided into the multiple sums over the configurations of the rows, $\mathcal{C}_\Lambda(\text{row}) = \{\pm 1\}^{2M+1}$.

Let us define the transfer matrix of the $2d$ Ising model. The transfer matrix $V_M : \mathcal{H} \to \mathcal{H}$ is a linear transformation on the Hilbert space $\mathcal{H} = \bigotimes_{j=-M}^{M} \mathbb{C}_j^2$. The transfer matrix is defined as $V_M = V_1^{\frac{1}{2}} V_2 V_1^{\frac{1}{2}}$

where the action of V_1 on the basis of the Hilbert space $e_\sigma = \bigotimes_{j=-M}^{M} \begin{bmatrix} \frac{1+\sigma_j}{2} \\ \frac{1-\sigma_j}{2} \end{bmatrix}$, is defined by

$$V_1 e_\sigma = \exp\left(\beta \sum_{j=-M}^{M-1} \hat{\sigma}_j \hat{\sigma}_{j+1}\right) e_\sigma, \tag{2}$$

where $\hat{\sigma}_j = 1 \otimes ... \otimes 1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes 1 \otimes ... \otimes 1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_j$ is the spin operator and also suppose that $b \in \mathbb{R}$ and σ and $\rho \in \mathcal{C}_\Lambda(\text{row})$, then we define the matrix elements of V_2 as

$$(V_2)_{\rho\sigma} = e^{-2b} \exp\left(\sum_{j=-M}^{M} \beta(j) \rho_j \sigma_j\right), \tag{3}$$

where ρ_j and σ_j are the row configurations evaluated at $j-th$ column and $\beta(j)$ is

$$\beta(j) = \begin{cases} b & \text{for } |j| = M \\ \beta & \text{for } |j| \neq M \end{cases}. \tag{4}$$

Moreover the action of V_2 on the Hilbert space is computed by $V_2 e_\sigma = \sum_\rho V_2 e_\rho$. We are interested in the action of V_2 in the limit $b \to \infty$; $\lim_{b\to\infty} V_2 e_\sigma = \sum_{\rho, \rho_{\pm M} = \sigma_{\pm M}} V_2 e_\rho$.

It has been discussed in proposition (1.1.1) in [Pal07], that the correlation function of $\mathcal{O}_A = \prod_{i\in A} \mathcal{O}_i$ as a product of linear operators $\mathcal{O}_i$ such as spin etc. in a subset A of the domain Λ, a finite collection of sites, with specific boundary conditions, in the limit $b \to \infty$, is given by

$$< \mathcal{O}_A >_\Lambda = \frac{< e_\sigma^N | V_1^{\frac{1}{2}} V_M \mathcal{O}_{A_{N-1}} V_M \mathcal{O}_{A_{N-2}} ... \mathcal{O}_{A_{-N+1}} V_M V_1^{\frac{1}{2}} | e_\sigma^{-N} >}{Z_\Lambda}, \tag{5}$$

where $\mathcal{O}_{A_i}$ denotes the restriction of $\mathcal{O}_A$ to the $i-$th row, $Z_\Lambda =< e_\sigma^N | V_1^{\frac{1}{2}} V_M^{2N} V_1^{\frac{1}{2}} | e_\sigma^{-N} >$ is the partition function and $e_\sigma^{\pm N}$ is the Hilbert space representation of the $\pm N - th$ row configuration. We take eq. (5) as a definition of the correlation functions.

2.2. **Ising free fermions.** The method of free fermions was introduced in 1949 by Kaufman in order to compute the free energy of the Ising model. This is one of the powerful methods besides other methods such as combinatorial methods, that have led to the integrability paradigm in the Ising model, [Ba08].

In order to discuss the free fermions in Ising model we introduce a representation of the Clifford algebra in Ising model. Suppose that W is a finite-dimensional complex vector space with a nondegenerate complex bilinear form denoted by $(\cdot, \cdot)$. A Clifford algebra $Cliff(W)$ on the vector space W is defined as an associative algebra with unit e and set of generators in W satisfying $ab + ba = (a, b)e$.

We define a finite-dimensional, irreducible spin representation of Clifford algebra $Cliff(W'_M)$, so called Brauer-Weyl representation, acting on $\bigotimes_{j=-M}^{M} \mathbb{C}_j^2$ space, with generators

$$p_k = \left\{ \prod_{j=-M}^{k-\frac{1}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_j \right\} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{k+\frac{1}{2}}, \quad q_k = \left\{ \prod_{j=-M}^{k-\frac{3}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_j \right\} \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}_{k-\frac{1}{2}}, \tag{6}$$

where for p_k, $k \in I_M - \frac{1}{2}$, and for q_k, $k \in I_M + \frac{1}{2}$ and $I_M = \{-M, -M+1, ..., M\}$. The Clifford algebra generators of this representation with some normalization factors, $\{\frac{p_k}{\sqrt{2}}, \frac{q_k}{\sqrt{2}}\}$, are orthonormal basis of a complex vector space W'_M,

$$W'_M = Span(\{p_k | k \in I_M - \frac{1}{2}\} \cup \{q_k | k \in I_M + \frac{1}{2}\}) = W_M \oplus (\mathbb{C}_{p_{-M-\frac{1}{2}}} + \mathbb{C}_{q_{M+\frac{1}{2}}}). \tag{7}$$

It can be easily checked that p_k and q_k satisfy the anti-commutation Clifford relations

(8) $$p_k p_l + p_l p_k = 2\delta_{kl}, \quad q_k q_l + q_l q_k = 2\delta_{kl}, \quad p_k q_l + q_l p_k = 0.$$

Then, the lattice fermion ψ_k and anti-fermion $\bar{\psi}_k$ operators are defined on the mid-points of horizontal edges of two-dimensional rectangular lattice as

(9) $$\psi_k = A_\psi(q_k + p_k), \quad \bar{\psi}_k = A_{\bar{\psi}}(-q_k + p_k),$$

where A_ψ and $A_{\bar{\psi}}$ are normalization factors.

In order to simplify the notation we define time-dependent fermion operators by using the transfer matrix V_M in the domain $\Lambda_{M,N}$ as

(10) $$\psi(k + \mathrm{i}m) = V_M^{-m} \psi_k V_M^m.$$

Furthermore, the transfer matrix can be written in terms of Clifford algebra generators p_k and q_k and thus the time evolution of the free fermions can be explicitly calculated via the conjugation by the transfer matrix, [HKZ12]. This conjugation is called induced rotation and it is denoted by $T(V_M)$. The induced rotation is a linear transformation, $T(V_M) : W'_M \to W'_M$ such that for all $v \in W'_M$

(11) $$T(V_M)v = V_M^{-1} v V_M.$$

Note that the induced rotation preserves the bilinear form, $(T(V_M)a, T(V_M)b) = (a, b)$ for $a, b \in W$.

Moreover, the normalized lattice correlation functions of fermion operators in the domain Λ (for simplicity we write the domain $\Lambda_{M,N}$ as Λ) with plus boundary conditions, $< \prod_{i=1}^{2n} \psi(z_i) >_\Lambda^{(+)}$, in the transfer matrix formalism can be defined from eq. (5) as

(12) $$< \prod_i \psi(z_i) >_\Lambda^{(+)} = \frac{1}{Z} < + + | \prod_i \psi(z_i) | + + >,$$

with $z_i = k_i + \mathrm{i}m_i$, $Z =< + + | + + >$ is the partition function and

$$| + + >= V_M^N V_1^{\frac{1}{2}} | e_{(+)}^{-N} >, \ \ < + + | =< e_{(+)}^N | V_1^{\frac{1}{2}} V_M^N,$$

where $e_{(+)}$ corresponds to a row configuration in which all the spins are plus.

It has been discussed in section (1.3) and (4.2) in [Pal07], that the naive scaling limit of the transfer matrix formalism for the free fermions leads to the Dirac equation and at critical temperature, one can observe that the free fermions (anti-fermions) are holomorphic (anti-holomorphic) functions.

In next section, we explain a rigorous approach to derive the scaling limit of the lattice fermion correlation functions.

2.3. **Discrete holomorphicity and scaling limit of the correlation functions.** The methods of discrete complex analysis and discrete holomorphic functions [Smi06], [Smi10a], [Smi10b], [ChSm09] [ChSm11], [IkCa09] and [RaCa07] provide the possibility to perform the rigorous scaling limit of the s-holomorphic functions and observables as well as other advantages. Thus, by using the relations between s-holomorphic functions and fermion correlation functions we can obtain the scaling limit of the correlation functions, rigorously. The obtained results from this method coincide with the vacuum correlation functions of free fermions in conformal field theory which is believed for a long time that it describes the continuum limit of the free fermions of the Ising model, [McWu73] and [DMS96].

In a naive sense, the scaling limit of the fermion correlation functions in critical Ising model on a strip with lattice mesh size δ is defined by taking first the semi-infinite volume limit $N \to \infty$ and then taking the continuum limit, $M \to \infty$, $\delta \to 0$, while the width of the strip $M\delta$ is kept fixed. For example, the scaling limit of boundary state $| + + >$ is expected to behave like $| + + > \stackrel{N\to\infty}{\longrightarrow} |0>_M \stackrel{M\to\infty,\delta\to 0}{\longrightarrow} |0>$, in which $|0>$ is a CFT vacuum.

The first step towards the rigorous scaling limit of the Ising free fermions is to find the scaling limit of the lattice fermion correlation functions. In order to that, we have obtained the relations between s-holomorphic functions and fermions correlation functions in [HKZ12], then we need to obtain the scaling limit of s-holomorphic functions $F^{\uparrow}_{z'}(z)$ and $F^{\downarrow}_{z'}(z)$ which will be defined in the next sentence. These functions are parafermionic observables in the Ising model. They are solutions of the Riemann boundary value problems (the explicit form of the Riemann boundary conditions in the case of rectangle for the scaling limit of the functions can be found in eq. (14). For a general definition of Riemann boundary value problems see section (2) of [HKZ12]) with a discrete singularity at $z = z'$. In other words, they are s-holomorphic functions and they satisfy the Riemann boundary conditions,

$$F^{\uparrow}_{z'}(z) = \frac{1}{Z} \sum_{\sigma \in \mathcal{C}_{z'\uparrow}} \alpha_c^{L(\sigma)} e^{-\frac{\mathrm{i}}{2} w(z' \rightsquigarrow z)}, \quad F^{\downarrow}_{z'}(z) = \frac{1}{Z} \sum_{\sigma \in \mathcal{C}_{z'\downarrow}} \alpha_c^{L(\sigma)} e^{-\frac{\mathrm{i}}{2} w(z' \rightsquigarrow z)}, \tag{13}$$

where $Z = \sum_{\sigma \in \mathcal{C}^+} \alpha_c^{L(\sigma)}$ is a partition function with plus boundary conditions, and the sum in $F^{\uparrow}_{z'}(z)$ ($F^{\downarrow}_{z'}(z)$) is over collections of dual edges in all graphical expansions $\sigma \in \mathcal{C}_{z'\uparrow}$ ($\sigma \in \mathcal{C}_{z'\downarrow}$) consisting of loops and a path stars at z' upward (downward) and ends at z either from above or below, $\alpha_c = e^{-2\beta_c}$, $L(\sigma)$ is the total number of edges in the configuration σ and w is the winding number of directed path starting at z' and ending at z. The points z, z' are midpoints of horizontal edges of the lattice.

So far we have discussed only the rectangular domain but the parafermionic observables can be defined similarly in any square lattice domain [HoSm10b]. However, we want to study the scaling limit of the parafermionic observables on the rectangle at the critical point, $\beta = \beta_c$. In general, a continuous domain Λ can be approximated with the discrete domain Λ_δ, as a subgraph of the square lattice $\delta\mathbb{Z}^2$, when the small lattice mesh size δ tends to zero, $\delta \to 0$.

It has been shown that scaling limit of the s-holomorphic functions which satisfy the Riemann boundary conditions exists and the convergence of the parafermionic observables as $\delta \to 0$ can be controlled by the methods of discrete complex and harmonic analysis, [HoSm10b, Hon10a]. The result of these studies can be summarized as follow: the functions $\frac{F^{\uparrow}_{z'}(z)}{\delta}$, $\frac{F^{\downarrow}_{z'}(z)}{\delta}$ converge uniformly on compact subsets of $\Lambda \setminus \{z'\}$ to the unique holomorphic functions with Riemann boundary values and the appropriate residue, $\lim_{\delta \to 0} \frac{F^{\uparrow}_{z'}(z)}{\delta} = f^{\uparrow}_{z'}(z)$ and $\lim_{\delta \to 0} \frac{F^{\downarrow}_{z'}(z)}{\delta} = f^{\downarrow}_{z'}(z)$.

Then, similar to the discrete case, we have the Riemann boundary value problem for the scaling limit of the Ising parafermionic observables. The residue calculations on the lattice are performed by considering couple of combinatorial cases and using the fact that the contour integral of s-holomorphic function is zero. Then, with the help of lattice residue calculations and Riemann boundary value problem we obtain the following statement in the scaling limit,

$$\begin{cases} f^{\uparrow}_{z'}(z) \; and \; f^{\downarrow}_{z'}(z) \; are \; holomorphic \; on \; rectangle \; \setminus \{z'\} \\ 2\pi \mathrm{i} \, \mathrm{Res}_{z=z'} f^{\uparrow}_{z'}(z) = -1, \; 2\pi \mathrm{i} \, \mathrm{Res}_{z=z'} f^{\downarrow}_{z'}(z) = 1 \\ For \; z \in \partial_{rectangle}, \; f^{\uparrow}_{z'}(z) \parallel \frac{1}{\sqrt{-\nu_z}}, \; f^{\downarrow}_{z'}(z) \parallel \frac{1}{\sqrt{-\nu_z}}, \end{cases} \tag{14}$$

where $\partial_{rectangle}$ is the boundary of the rectangle and ν_z is the counter clock-wise tangent vector at point z on the boundary of rectangle. These conditions determine a unique function on the rectangle which transform conformally covariant under the conformal transformations between the rectangle and any other domains. We start with the scaling limit of the correlation functions of Ising fermions on the upper-half plane $\mathbb{H} := \{z \in \mathbb{C} | \Im z > 0\}$. The holomorphic functions on the half-plane which satisfy eq. (14) can be obtained as

$$f^{\uparrow;\mathbb{H}}_{z'}(z) = \frac{\mathrm{i}}{2\pi} \left(\frac{1}{z - z'} + \frac{1}{z - \bar{z}'} \right), \quad f^{\downarrow;\mathbb{H}}_{z'}(z) = \frac{\mathrm{i}}{2\pi} \left(\frac{-1}{z - z'} + \frac{1}{z - \bar{z}'} \right). \tag{15}$$

As we mentioned, the relations between fermionic correlation functions and parafermionic observables are obtained in theorem (22) in [HKZ12]. In slightly different notation they are as follow:

$$
\begin{aligned}
<++|\psi(z)\psi(z')|++> &= 2A_\psi^2 Z(F^{\uparrow}_{z'}(z) - F^{\downarrow}_{z'}(z)),\\
<++|\psi(z)\bar{\psi}(\bar{z'})|++> &= 2\mathrm{i}A_\psi A_{\bar{\psi}} Z(F^{\uparrow}_{z'}(z) + F^{\downarrow}_{z'}(z)),\\
<++|\bar{\psi}(\bar{z})\bar{\psi}(\bar{z'})|++> &= 2A_{\bar{\psi}}^2 Z(\overline{F^{\uparrow}_{z'}(z)} - \overline{F^{\downarrow}_{z'}(z)}).
\end{aligned} \tag{16}
$$

By using the above relations between correlation functions and parafermionic observables, we can deduce also the convergence of the scaling limit of the fermion correlation functions $\lim_{\delta\to 0}\frac{1}{\delta} <++|\psi(z)\psi(z')|++>$. In this spirit, the correlation functions on the half-plane, such as $<\psi(z)\psi(z')>_{\mathbb{H}} = 2A_\psi^2 Z'(f^{\uparrow;\mathbb{H}}_{z'}(z) - f^{\downarrow;\mathbb{H}}_{z'}(z))$ etc. with the choice of parameters $A_\psi = \frac{1}{\sqrt{2}}(-\mathrm{i}-1)$, $A_{\bar{\psi}} = \overline{A_\psi}$ and $Z' = -\frac{\pi}{2}$ can be obtained as

$$
<\psi(z)\psi(z')>_{\mathbb{H}} = \left(\frac{1}{z-z'}\right),\ <\psi(z)\bar{\psi}(\bar{z'})>_{\mathbb{H}} = \left(\frac{1}{z-\bar{z'}}\right),\ <\bar{\psi}(\bar{z})\bar{\psi}(\bar{z'})>_{\mathbb{H}} = \left(\frac{1}{\bar{z}-\bar{z'}}\right). \tag{17}
$$

Moreover, we can obtain the correlation functions in an arbitrary domain D by using the fermion transformation rule $\psi(z) = g'(z)^{\frac{1}{2}}\psi(g(z))$, as a defining rule for fermions, under a conformal map $g: D \to \mathbb{H}$, as

$$
<\psi(z)\psi(z')>_D = \frac{g'(z)^{\frac{1}{2}} g'(z')^{\frac{1}{2}}}{g(z)-g(z')},\ <\psi(z)\bar{\psi}(\bar{z'})>_D = \frac{g'(z)^{\frac{1}{2}} \overline{g'(z')}^{\frac{1}{2}}}{g(z)-\overline{g(z')}},\ <\bar{\psi}(\bar{z})\bar{\psi}(\bar{z'})>_D = \frac{\overline{g'(z)}^{\frac{1}{2}} \overline{g'(z')}^{\frac{1}{2}}}{\overline{g(z)}-\overline{g(z')}}, \tag{18}
$$

where $g'(z)$ is the derivative of $g(z)$ with respect to z.

In the scaling limit on the upper-half plane, by using the two-point correlation functions of fermions $\psi(z)$, any $2n$-point correlation function can be written in terms of two-point functions, via the Wick's theorem,

$$
<\psi(z_1)...\psi(z_{2n})>_{\mathbb{H}} = Pf\left(\left[\frac{1}{z_i - z_j}\right]_{i,j=1}^{2n}\right), \tag{19}
$$

where definition of the Pfaffian of an anti-symmetric matrix $A \in \mathbb{C}^{n\times n}$ is

$$
Pf(A) = \begin{cases} \frac{1}{2^k k!}\sum_P Sgn(P)\prod_{i=1}^k A_{P(2i-1),P(2i)} & \text{for } n = 2k \\ 0 & \text{for } n = 2k-1 \end{cases}.
$$

where P is any permutation of $\{1, 2, ..., 2n\}$ and $Sgn(P)$ is the sign of the permutation. This is called the Pfaffian formula. This result can be proved by using the Pfaffian formula for the lattice fermion correlation function (section (4.4) in [HKZ12]) and then taking the scaling limit. Furthermore, by using the above Pfaffian formula and eq. (18) we have the following equation for the multi-point fermion correlation functions on the domain D

$$
<\prod_{i=1}^{2n}\psi(z_i)>_D = Pf\left(\left[\frac{\sqrt{g'(z_i)}\sqrt{g'(z_j)}}{g(z_i)-g(z_j)}\right]_{i,j=1}^{2n}\right). \tag{20}
$$

There are similar formulas for correlation functions of anti-fermions.

3. Fermionic vertex operator algebra and CFT

In this section, the vertex operator algebra and conformal field theory of the Ising free fermions are studied. The Fock space of fermionic states and fields are constructed in explicit forms and furthermore, the relation between them has been investigated.

3.1. **Vertex operator algebra and fermionic Fock space of states.** In this part, the basic definitions of vertex operator algebra as a rigorous algebraic approach to CFT is reviewed. The *vertex operator algebra* (VOA) was introduced by R. Borcherds in order to provide a rigorous mathematical definition of the chiral algebra, the symmetry of the two-dimensional CFT and its ingredients such as operator product expansion [Bo86]. We start with the definition of the axioms of a general vertex algebra and a restriction of the vertex algebra into a conformal vertex operator algebra [Ka98], [Ga06] and [Scht08]. The discussion will be continued by an explicit example of VOA which has Clifford algebra symmetry, the VOA of free fermions (FVOA or Clifford VOA). The Clifford VOA leads to the Fock space of fermionic states.

The Fock space of states is defined as a graded vector space $V = \bigoplus_{n=-\infty}^{\infty} V_n$ consisting of vacuum state $\mathbf{1} \in V$ and other states which are generated from the vacuum state and we denote them by small letter such as $a, b, ... \in V$. A field operator $a(z)$ is a formal power series $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ where $a_{(n)} \in End(V)$ and for each $v \in V$ we have

$$a_{(n)} v = 0, \tag{21}$$

for $n \gg 0$. Moreover, we can define the vertex operator subalgebras by a $\mathbb{Z}_2$-grading of $V = V_0 + V_1$ into even $(p = 1)$ and odd $(p = 0)$ parity subspaces. Moreover, we assume that the fields have a definite parity. Then we can define a vertex algebra.

Vertex operator algebra. A quadruple $(V, Y, \partial, \mathbf{1})$ is called vertex algebra if for all $a \in V$ there exists a mapping $Y : V \to End(V)\left[[z, z^{-1}]\right]$, $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ satisfying the following axioms:

- *Vacuum*: $Y(\mathbf{1}, z) = I_V$ is the identity;
- *State-field correspondence*:

$$Y(a, z)\mathbf{1}|_{z=0} = a, \Rightarrow \ a_{(n)}\mathbf{1} = 0 \ for\ n \geq 0 \ and\ a_{(-1)}\mathbf{1} = a; \tag{22}$$

- *Translation*:

$$[T, Y(a, z)] = \partial_z Y(a, z), \quad [T, a_{(n)}] = -n a_{(n-1)}; \tag{23}$$

where $T \in End(V)$ is defined by $T(a) = a_{(-2)}\mathbf{1}$.

- *Locality*: $(z-w)^N [Y(a, z), Y(b, w)] = 0$ for some large N; where $[Y(a, z), Y(b, w)] = Y(a, z)Y(b, w) + (-1)^{p(a)p(b)} Y(b, w)Y(a, z)$ and $p(a)$ is parity of field $a(z)$.
- *Regularity*: There is an M such that $a_{(n)} b = 0$, for all $n \geq M$;

The mapping $Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ as a formal power series with operator modes $a_{(n)}$ is called vertex operator.

In order the state the main theorem of this part we need some definitions and properties in VOA. For a moment, let us briefly introduce the Virasoro operator $L(z)$ and the operator product expansion of operators in VOA. We will explain these topics in the following sections, more carefully. The Virasoro field $L(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}$ as an even formal power series is defined such that it satisfies the following OPE

$$L(z)L(w) \sim \frac{1/2C}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)}, \tag{24}$$

where $C = cI$ and c is a central charge. Furthermore, We will define the conformal vertex operator algebra in the following sections.

Furthermore, the $j-$th OPE product $a(z) *_j b(w)$ of the OPE between two fields $a(z)$ and $b(w)$ can be obtained form the following formula

$$a(z)b(w) \sim \sum_{j=0}^{N} \frac{a(z) *_j b(w)}{(z-w)^{j+1}}. \tag{25}$$

We will explain further the OPE product later in the next section.

3.2. **Fermionic correlation functions and VOA.** In previous section, the Fock space of states, V is constructed in the context of VOA. We will define the fermionic Fock space of states later in this section. In order to make contact of the space V to the Fock space of fields $\mathcal{F}$ and the correlation functions of fields we introduce a map χ in the following theorem.

($\mathcal{F} \leftrightsquigarrow V$) *theorem.* There exist a unique collection of maps indexed by domain and the number of points from a tensor powers of fermionic Fock space of states to the correlation functions of fermions in any domain D; $\chi^{(D)}_{(z_1,...z_n)}(v_1 \otimes ... \otimes v_n) : V^{\otimes n} \to \mathbb{C}$, for $z_1, ..., z_n \in D$, such that it satisfies the following properties:

1)

$$\chi^{(D)}_{(z_1,...,z_n)}(\psi \otimes ... \otimes \psi) = Pf\left(\left[\frac{\sqrt{g'(z_i)}\sqrt{g'(z_j)}}{g(z_i) - g(z_j)}\right]^n_{i,j=1}\right),$$

2)

$$\chi^{(D)}_{(z_1,...,z_n)}(v_1 \otimes ... \otimes L_{-1}v_i \otimes ... \otimes v_n) = \frac{\partial}{\partial z_i}\chi^{(D)}_{(z_1,...,z_n)}(v_1 \otimes ... \otimes v_n),$$

3)

$$\chi^{(D)}_{(z_1,...,z_n)}(v_1 \otimes ... \otimes v_m \otimes v_{m+1} \otimes ... \otimes v_n) \sim$$

$$\sum_{j=0}^{N-1} \frac{1}{(z_m - z_{m+1})^{j+1}} \chi^{(D)}_{(z_1,...,\hat{z}_m,z_{m+1},...,z_n)}(v_1 \otimes ... \otimes v_m *_j v_{m+1} \otimes ... \otimes v_n), \tag{26}$$

where the Pf is the Pfaffian, $g : D \to \mathbb{H}$ is a conformal map, L_{-1} is the mode $m = -1$ of the Virasoro field $L(z)$, $\hat{z}_m$ is removed and $v_m *_j v_{m+1}$ is the $j-$th OPE of the vectors v_m and v_{m+1}.

This theorem provides a mathematically rigorous approach to the Fock space of the fermionic fields and their correlation functions from the VOA. We will explicitly construct the Fock space of the fermionic conformal fields in section (3.3). Moreover, we will see in section (4) that the theorem provides us with a rigorous realization of fermionic CFT/SLE_3 correspondence at the level of correlation functions and SLE martingale observables.

In order to prove the theorem we need basically two results, namely the Wick's and reconstruction theorems in VOA and in CFT on domain D. In the following we review the results in VOA without proofs.

Normal order product in VOA. [Ka98] (Theorem 2.3) Let us introduce the following notations,

$$a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-n-1}, \quad a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}. \tag{27}$$

Then, the normal order product of two fields is defined by

$$: a(z)b(w) := a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w)a(z)_- = a(z)b(w) - [a(z)_-, b(w)]. \tag{28}$$

The normal order product of more than two fields is defined inductively from right to left as follow : $a^1(z)a^2(z)...a^N(z) :=: a^1(z)... : a^{N-1}(z)a^N(z) : ... :$.

Furthermore, as we have seen in the axioms of the VOA, two fields $a(z)$ and $b(z)$ are called mutually local if they satisfy $(z - w)^N[a(z), b(w)] = 0$ for $N \gg 0$.

OPE theorem in VOA. [Ka98] (Theorem 2.3) It has been shown that the operator product expansion (OPE) of two mutually local fields $a(z)$ and $b(w)$ in VOA is given by

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{c_j(w)}{(z - w)^{j+1}} + : a(z)b(w) :, \tag{29}$$

where $c_j(w) \in End(V)\left[[w, w^{-1}]\right]$. In fact, it has been proved that above OPE product is equivalent to the locality axiom for the $a(z)$ and $b(w)$ fields; $(z-w)^N[a(z), b(w)] = 0$ for $N \gg 0$. Moreover, the singular part of the OPE is often written as

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}}.$$

And, the $j-$th product $a(w) *_j b(w)$ of OPE $a(z)b(w)$ is defined as follow

(30) $$a(z)b(w) = \sum_{j\in\mathbb{Z}} \frac{a(w) *_j b(w)}{(z-w)^{j+1}} = \sum_{j=0}^{N-1} \frac{a(w) *_j b(w)}{(z-w)^{j+1}} + :a(z)b(w): .$$

The Wick's theorem in VOA. [Ka98] (Theorem 3.3) Let $a^1(z), ..., a^n(z)$ and $b^1(z), ..., b^m(z)$ be two collections of fields such that the following properties hold:

1) $[[a^i(z)_-, b^j(w)], c^k(z)] = 0$ for all i, j, k, and $c = a$ or b.

2) $[a^i(z)_\pm, b^j(w)_\pm] = 0$ for all i, j.

let $< a^i b^j > := [a^i(z)_-, b^j(w)]$ denotes the contraction of $a^i(z)$ and $b^j(w)$. Then the following equality holds in the domain $|z| > |w|$:

$$:a^1(z)...a^n(z)::b^1(w)...b^m(w):=$$

(31)
$$\sum_{s=0}^{min(n,m)} \sum_{i_1<...<i_s, j_1\neq...\neq j_s} (\pm < a^{i_1}b^{j_1} > ... < a^{i_s}b^{j_s} >: a^1(z)...a^n(z)b^1(w)...b^m(w) :_{(i_1,...,i_s;j_1,...,j_s)}),$$

where the sign $\pm$ is obtained by the rule that each permutation of the adjacent odd fields changes the sign and subscript $(i_1, ..., i_s; j_1, ..., j_s)$ means that the fields $a^{i_1}(z), ..., a^{i_s}(z)$ and $b^{j_1}(w), ..., b^{j_s}(w)$ are removed.

Clifford vertex algebra for Ising free fermions. The goal of this part is to construct the fermionic Fock space for states in VOA and their corresponding fields of the Ising model.

The set of generators $\{\psi_n\}$ for $n \in \mathbb{Z} + \frac{1}{2}$ with the following algebra,

(32) $$\{\psi_n, \psi_m\} = \delta_{n+m,0},$$

are the generators of the fermionic Fock space and by acting them on the vacuum state one can generate the basis elements of a fermionic Fock space which is a vector space V.

The Clifford vertex operator algebra is a vector space V consisting of fermionic states including the vacuum state $|0>$, and fermionic vertex operator,

(33) $$Y(\psi_{-\frac{1}{2}}|0>, z) = \psi(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \psi_n z^{-n-\frac{1}{2}},$$

which is an odd formal power series with $\psi_n = \frac{1}{2\pi \mathrm{i}} \oint_{(0)} \zeta^{n-\frac{1}{2}} \psi(z) d\zeta$. This Clifford VOA satisfies the axioms of the VOA. The generator ψ_n acts on V as a linear operator such that for any $v \in V$, $\psi_n v = 0$ for $n \gg 0$. An example of field/state correspondence in the case of fermions is $|\psi> = \psi_{-\frac{1}{2}}|0> = \psi(0)|0>$.

Furthermore, fermion fields satisfy the conditions of the Wick's theorem and they are mutually local fields with the following OPE

(34) $$\psi(z)\psi(w) \sim \frac{1}{z-w}.$$

Conformal vertex algebra for Ising free fermions. [Ka98] (Theorem 4.10) A conformal vector $\nu \in V$ is an even vector such that the corresponding vertex operator $Y(\nu, z) = L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is a Virasoro field of the central charge c with the following properties

$$L_{-1} = T, \ \ \textit{and } L_0 \textit{ is diagonalizable on } V. \tag{35}$$

The vertex algebra $(V, Y, \partial, \mathbf{1})$ is called conformal vertex algebra if it has a conformal vector $\nu \in V_2$. According to the theorem (4.10) in [Ka98], $Y(\nu, z)$ is a Virasoro field with central charge c if

$$L_{-1} = T, \ \ L_2\nu = \frac{c}{2}|0>, \ \ L_n\nu = 0 \, \textit{for} \, n > 2, \ \ L_0\nu = 2\nu, \tag{36}$$

Moreover, $\nu \in V_2$ is conformal vector if it satisfies the above properties and the property that L_0 is diagonalizable on V.

In the fermionic case, $\nu = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0>$ is a conformal vector and the Virasoro vertex operator is given by

$$Y(\frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0>, z) = L(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}, \tag{37}$$

where the modes are given by $L_m = \frac{1}{2\pi i}\oint_{(0)} \zeta^{m+1} L(z) d\zeta$. The conformal vector $\nu = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0>$ and $Y(\frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0>, z)$ satisfy all the conditions for the conformal vertex algebra with $c = 1/2$. The explicit form of the Virasoro operator L_m for Ising fermions that satisfies the VOA axioms can be obtained in the Sugawara construction of the FVOA as

$$L_m = -\frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} (k + \frac{m}{2}) : \psi_{m+k}\psi_{-k} : + \frac{1}{16}\delta_m, \tag{38}$$

where the normal order means that

$$: \psi_n\psi_m := \begin{cases} \psi_n\psi_m & \text{for } n \le m \\ -\psi_m\psi_n & \text{for } n > m \end{cases}. \tag{39}$$

The Wick's theorem and the Taylor expansion can be used to obtain the OPE between $L(z)$ and $\psi(w)$ as $L(z)\psi(w) \sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial_w\psi(w)}{z-w}$ which is equivalent to the following commutation relations

$$[L_m, \psi_n] = -(\frac{1}{2}m + n)\psi_{m+n}, \ \ [L_{-1}, \psi(z)] = \partial_z\psi(z). \tag{40}$$

Notice that the fermionic Virasoro operator (38) satisfies the above commutation relation.

Furthermore, the OPE of Virasoro fields eq. (24) implies the following commutation relation,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}. \tag{41}$$

One can check that the explicit form of the fermionic Virasoro operator (38) satisfies the above commutation relation with the central charge $c = \frac{1}{2}$.

Moreover, a fermionic singular state at level two is defined as an state $|\chi>$ which satisfies

$$L_0|\chi>= \frac{5}{2}|\chi>, \ \ L_n|\chi>= 0, \tag{42}$$

for $n > 0$. Using the commutation relation (41), a fermionic singular state at level two can be constructed from the state $\psi_{-\frac{1}{2}}|0>$ as follow

$$|\chi>= (L_{-2} + \frac{3}{4}L_{-1}^2)\psi_{-\frac{1}{2}}|0> . \tag{43}$$

By using the fermionic representation of L_m in eq. (38) and the left hand commutation relation in eq. (40), one can check that $|\chi>= 0$.

Reconstruction theorem for fermion fields. [Ka98] (Theorem 4.5) As we mentioned, the fermionic Fock space is the vector space V including an even vector, the vacuum $|0>\in V$, and it satisfies the following properties: 1) for the fermion fields $\psi(z)$ we have $[T,\psi(z)]=\partial\psi(z)$, 2) $T|0>=0$ and 3) vectors of the following form span V,

$$\psi_{-k_n-\frac{1}{2}}\psi_{-k_{n-1}-\frac{1}{2}}...\psi_{-k_2-\frac{1}{2}}\psi_{-k_1-\frac{1}{2}}|0>, \tag{44}$$

for $k_n>...>k_2>k_1>0$. Then by reconstruction theorem, for any state in V, there is a corresponding field or a vertex operator of the following form

$$Y(\psi_{-k_n-\frac{1}{2}}\psi_{-k_{n-1}-\frac{1}{2}}...\psi_{-k_2-\frac{1}{2}}\psi_{-k_1-\frac{1}{2}}|0>,z)=:(\partial^{k_n}\psi(z))(\partial^{k_{n-1}}\psi(z))...(\partial^{k_2}\psi(z))(\partial^{k_1}\psi(z)):, \tag{45}$$

where $\partial^{k_i}=\frac{1}{(k_i+1)!}\frac{\partial^{k_i}}{\partial z^{k_i}}$. It defines a unique structure of a vertex algebra on V such that $|0>$ is the vacuum vector, T is the infinitesimal translation operator and $Y(\psi_{-\frac{1}{2}}|0>,z)=\psi(z)$.

In the next section we will explain the necessary construction of fermionic Fock space of fields and its conformal structure which will be used in the proof of the theorem and in relation between CFT and Schramm Loewner evolution in the Ising model.

3.3. **Fermionic Fock space of fields $\mathcal{F}$.** In this section we study the analytic and algebraic aspects of a field theory which describes the scaling limit of free fermions of Ising model on domains with boundaries, a two-dimensional boundary conformal field theory. The goal of this chapter is to construct a fermionic Fock space of fields and to study their properties which will be used in the CFT/SLE correspondence for the Ising model. We study the holomorphic part of the theory but the anti-holomorphic part can be studied similarly.

We define the scaling limit of the Ising fermions, the free fermion fields $\psi(z)$, and other fermionic fields which are called descendant fields, as Fock space fields in domain $D\subset\mathbb{C}$. All the descendant fields are constructed by normally ordered product of derivatives of free fermion field, for example $:\partial^2\psi(z)\partial\psi(z)\psi(z):$.

In general, a finite Fock space field $X_k(z)=:\partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z):\in\mathcal{F}$ for $z\in D$ is defined by

$$:\partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z):=\partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z)-\lim_{z_{i_m}\to z_{j_m}\to z}$$
$$\sum_{s=1}^{n}\sum_{i_1<...<i_s,j_1\neq...\neq j_s}\Big(\pm\partial^{k_{i_1}}\partial^{k_{j_1}}[\frac{1}{z_{i_1}-z_{j_1}}]...\partial^{k_{i_s}}\partial^{k_{j_s}}[\frac{1}{z_{i_s}-z_{j_s}}]$$
$$:\partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z):_{(i_1,...,i_s;j_1,...,j_s)}\Big), \tag{46}$$

for $m=1,...,s$.

As a special example of the general definition of Fock space fields we define a Fock space field of special interest, the fermionic Virasoro field $T(z)$,

$$T(z)=-\frac{1}{2}:\psi(z)\partial_z\psi(z):=\lim_{w\to z}[-\frac{1}{2}(\psi(z)\partial_w\psi(w)-\partial_w(\frac{1}{z-w}))]. \tag{47}$$

We will explain the role of Virasoro field in the Virasoro representation of the CFT.

Conformal transformations. In field theories which are relevant for studies about statistical mechanics, domain conformal transformation $h:D\to D'$ is useful to study. Roughly speaking, the values of the fields on domain D, $\psi(z)$ for $z\in D$ conformally transform to values of fields on domain D', $\psi(h(z))$ for $h(z)\in D'$. By definition, the fermion field $\psi(z)$ is a conformal primary field of dimension $1/2$ that satisfies

$$\psi(z)=h'(z)^{\frac{1}{2}}\psi(h(z)), \tag{48}$$

where $h'(z)$ is derivative of $h(z)$ with respect to z. Furthermore, the Virasoro field is called conformal quasi-primary field and it satisfies

$$T(z)=h'(z)^2T(h(z))+\frac{1}{24}S_h(z), \tag{49}$$

where $S_h(z) = \frac{h'''(z)}{h'(z)} - \frac{3}{2}(\frac{h''(z)}{h'(z)})^2$ is the Schwarzian derivative of function $h(z)$. Moreover, $\psi(z)$ is called a differential form of order $1/2$ and $T(z)$ is called a Schwarzian form of order $1/24$, [KaMa11].

However, the transformation rules for general conformal Fock space fields are much more complicated than for free fermion field and Virasoro field,

$$X(z) = h'(z)^{\lambda_X} X(h(z)) + ..., \tag{50}$$

where λ_X is the conformal dimension of X and ... represents complicated function of higher order derivatives of $h(z)$.

3.4. **Correlation functions and operator product expansion.**

Correlation functions. A CFT correlation function of the general Fock space fields is denoted by $< X_1(z_1)...X_n(z_n) >_D$, where $z_1, ..., z_n \in D$. Using the discrete holomorphicity results, the correlation functions of free fermion fields $\psi(z)$ are obtained rigorously in section (2.3). The $n-$point correlation functions of free fermion fields $\psi(z)$ in $z_i \in D$ as the special case of the general Fock space fields are obtained as

$$(z_1, ..., z_n) \mapsto < \psi(z_1)...\psi(z_n) >_D = Pf((< \psi(z_i)\psi(z_j) >_D)_{i,j=1}^n), \tag{51}$$

where $< \psi(z_i)\psi(z_j) >_D = \frac{\sqrt{g'(z_i)}\sqrt{g'(z_j)}}{g(z_i)-g(z_j)}$. The correlation function of derivatives of fermion fields is simply given by

$$< (\frac{\partial^{m_1}}{\partial z_1^{m_1}})\psi(z_1)...(\frac{\partial^{m_n}}{\partial z_n^{m_n}})\psi(z_n) >_D = \frac{\partial^{m_1+...+m_n}}{\partial z_1^{m_1}...\partial z_n^{m_n}} Pf\left(\left[\frac{\sqrt{g'(z_i)}\sqrt{g'(z_j)}}{g(z_i)-g(z_j)}\right]_{i,j=1}^n\right). \tag{52}$$

All the other correlation functions of fermionic Fock space fields $X(z)$ can be obtained from the correlation functions of free fermion fields $\psi(z)$ by using the Wick's theorem and taking the derivatives of the two-point fermion correlation functions.

In this part, we find asymptotic results for the correlation functions of fermions on domain D by using the Laurent expansion of the function $g(z) : D \to \mathbb{H}$ and its derivative up to some fixed order. Up to a fixed order, one can check that

$$[g(z) - g(w)]^{-1} = \frac{1}{\epsilon g'(w)}(1 - \frac{\epsilon g''(w)}{2g'(w)} - \frac{\epsilon^2 g'''(w)}{6g'(w)} + \frac{\epsilon^2 g''^2(w)}{4g'^2(w)}),$$

$$g'(z)^{\frac{1}{2}} g'(w)^{\frac{1}{2}} = g'(w)(1 + \frac{\epsilon g''(w)}{2g'(w)} + \frac{\epsilon^2 g'''(w)}{4g'(w)} - \frac{\epsilon^2 g''^2(w)}{8g'^2(w)}), \tag{53}$$

where $\epsilon = z - w$. These expansions lead to an asymptotic formula for the two-point function of fermions in the domain D,

$$< \psi(z)\psi(w) >_D = \frac{\sqrt{g'(z)}\sqrt{g'(w)}}{g(z) - g(w)} = \frac{1}{z - w} + (z - w)\left(\frac{1}{12}\frac{g'''(w)}{g'(w)} - \frac{1}{8}(\frac{g''(w)}{g'(w)})^2\right) + ...$$

$$= < \psi(z)\psi(w) >_{\mathbb{H}} + \frac{(z - w)}{12} S_g(w) + ..., \tag{54}$$

where $S_g(w) = \frac{g'''(w)}{g'(w)} - \frac{3}{2}(\frac{g''(w)}{g'(w)})^2$ is the Schwarzian derivative of function g.

About the higher point correlation functions, by using the Pfaffian formula we can obtain that

$$< \psi(z_1)...\psi(z_n) >_D \sim < \psi(z_1)...\psi(z_n) >_{\mathbb{H}}, \tag{55}$$

just hold for $n \leq 4$ and it does not hold in general and $\sim$ means that the two sides have the same divergent terms in the limit $z \to w$.

The operator product expansion. The OPE between two Fock space fields is an expansion of the Wick's formula on domain D, (see lectures (1) and (2) in [KaMa11]), when the positions of two fields become close. Notice that, in general the OPE is domain dependent. Thus, OPE is an asymptotic expansion of $X(z)Y(w)$ on domain D as $z \to w$,

$$X(z)Y(w) = \sum_{n\in\mathbb{Z}} C_n(w)(z-w)^n;\ as\ z \to w, \tag{56}$$

where the OPE coefficients C_n (usually denoted by $X *_n Y$) are also Fock space fields, for further description see lecture (2) in [KaMa11]. We define a OPE product as $X * Y$ where $*_0 = *$. Moreover the singular part of the OPE is defined by

$$X(z)Y(w) \sim \sum_{n<0} C_n(w)(z-w)^n. \tag{57}$$

By using the definition of OPE product one can check that the Virasoro field can be written as $T(z) = -\frac{1}{2}\psi(z) * \partial\psi(z)$.

In the case of fermionic CFT, the formal OPE between free fermions is

$$\psi(z)\psi(w) = \sum_{n\in\mathbb{Z}} c_n(w)(z-w)^n. \tag{58}$$

This can be explicitly written by means of Wick's formula on domain D,

$$\psi(z)\psi(w) =< \psi(z)\psi(w) >_D + \psi(z)\odot\psi(w) = \frac{1}{z-w} + \psi(z)\odot\psi(w) + reg(D), \tag{59}$$

where $\psi(z)\odot\psi(w)$ is called normal order product in domain D and $reg(D)$ denotes the terms which do not diverge in the limit $z \to w$. So the singular part of the OPE is given by the first term since the other terms vanish as $z \to w$ and we have $\psi(z)\psi(w) \sim \frac{1}{z-w}$.

By simple calculations using the definition of the Virasoro field, Wick's formula on domain D and Taylor expansion, the OPE of fermion fields and Virasoro fields for $z, w \in D$ can be obtained as

$$\psi(z)\psi(w) = \frac{1}{z-w} + ..., \tag{60}$$

$$T(z)\psi(w) = \frac{1/2}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial_w\psi(w) + \frac{3}{4}\partial_w^2\psi(w) + ..., \tag{61}$$

$$T(z)T(w) = \frac{1/4}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) + ..., \tag{62}$$

where ... denote non-singular and domain dependent terms in the limit $z \to w$. Therefore, by comparing the above results and the known CFT results about the OPE on the half plane we observe that the OPE singular parts of fermion and Virasoro fields are domain independent,

$$\psi(z)\psi(z')|_D \sim \psi(z)\psi(z')|_{\mathbb{H}},\ \ T(z)\psi(z')|_D \sim T(z)\psi(z')|_{\mathbb{H}},\ \ T(z)T(z')|_D \sim T(z)T(z')|_{\mathbb{H}}. \tag{63}$$

Using the OPE results eqs. (60)-(62), the singular parts of the correlation functions of an arbitrary operator $\mathcal{O}$, fermion fields and Virasoro fields on domain D are given by

$$< \psi(z)\psi(w)\mathcal{O} >_D = \frac{< \mathcal{O} >_D}{(z-w)} + reg(D), \tag{64}$$

$$< T(z)\psi(w)\mathcal{O} >_D = \frac{1/2 < \psi(w)\mathcal{O} >_D}{(z-w)^2} + \frac{< \partial_w\psi(w)\mathcal{O} >_D}{z-w} + \frac{3}{4} < \partial_w^2\psi(w)\mathcal{O} >_D + reg(D), \tag{65}$$

$$< T(z)T(w)\mathcal{O} >_D = \frac{1/4}{(z-w)^4} + \frac{2 < T(w)\mathcal{O} >_D}{(z-w)^2} + \frac{< \partial_w T(w)\mathcal{O} >_D}{(z-w)} + reg(D). \tag{66}$$

The multiplication of the two general Fock space fields can be obtained by using the definition of the Fock space fields in eq. (46) and the general Wick's theorem in field theory [DMS96], as follow

$$: \partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z) :: \partial^{l_m}\psi(w)\partial^{l_{m-1}}\psi(w)...\partial^{l_2}\psi(w)\partial^{l_1}\psi(w) :=$$

$$\sum_{s=0}^{min(n,m)} \sum_{i_1<...<i_s, j_1\neq...\neq j_s} (\pm \prec \partial_z^{k_{i_1}}\psi(z)\partial_w^{l_{j_1}}\psi(w) \succ ... \prec \partial_z^{k_{i_s}}\psi(z)\partial_w^{l_{j_s}}\psi(w) \succ$$

$$: \partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z)\partial^{l_m}\psi(w)\partial^{l_{m-1}}\psi(w)...\partial^{l_2}\psi(w)\partial^{l_1}\psi(w) :_{(i_1,...,i_s;j_1,...,j_s)}), \tag{67}$$

where we define the contractions by $\prec \partial_z^{k_{i_s}}\psi(z)\partial_w^{l_{j_s}}\psi(w) \succ = \partial_z^{k_{i_s}}\partial_w^{l_{j_s}}[\frac{1}{z-w}]$ for $z, w \in D$. Notice that this result is similar to the Wick's theorem in VOA in section (3.2). We will use this similarity later in the proof of the theorem.

3.5. **Virasoro algebra representation of the CFT.** The underlying algebraic structure of the CFT is Virasoro algebra. In fact, fermionic conformal field theory as a field theory is a representation of the Virasoro algebra and Clifford algebra. In this section we review the Virasoro algebra and its representation for the Virasoro generators and Virasoro fields, for further descriptions see lecture (5) in [KaMa11].

So far we have defined the Virasoro field $T(z)$ as a Schwarzian form. The Virasoro field in a CFT with central charge c, satisfies the Virasoro operator product expansion,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w), \tag{68}$$

and we observe that the central charge can be defined by $c = 2\lim_{z\to w}(z-w)^4 < T(z)T(w) >_D$. Next, for each point $p \in D$ and each local chart ϕ we define the Virasoro operator at point $\phi(p) = z$ as

$$L_n(z) = \frac{1}{2\pi \mathrm{i}}\oint_{(z)} (\zeta - z)^{1+n} T(\zeta) d\zeta. \tag{69}$$

Virasoro operator represents the Virasoro algebra, and thus one can check that the above definition satisfies the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{70}$$

where $c = 12\mu$ and μ is the order of T as a Schwarzian form.

In order to obtain a Virasoro algebra representation in the space of all Fock space fields in domain D, the action of Virasoro operator L_n on Fock space fields such as X is defined as

$$L_n X = T *_{(-n-2)} X. \tag{71}$$

From the explicit form of the OPE of Virasoro field T and a primary field X, one can check that for $n \geq -1$,

$$(L_n X)(z) = (v_n \partial_z + \lambda v_n')X(z), \tag{72}$$

where $v_n(z) = (\zeta - z)^{1+n}$, v_n' is the derivative of v_n with respect to z and λ is the conformal dimension of X. And for $n \leq -2$ we have

$$L_n = \frac{\partial^{-n-2}T}{(-n-2)!} * . \tag{73}$$

A Fock space field X is called a primary field with the conformal dimension λ if it satisfies

$$L_n X = 0, \quad L_0 X = \lambda X, \quad L_{-1} X = \partial X, \tag{74}$$

for $n \geq 1$. For example, the free fermion field ψ is a primary field of conformal dimension $1/2$ in CFT with central charge $c = 1/2$. Furthermore, a Schwarzian form Y of order μ in CFT with $c = 12\mu$ is defined by

$$L_m Y = 0, \quad L_2 Y = 6\mu I, \quad L_1 Y = 0, \quad L_0 Y = 2Y, \quad L_{-1} Y = \partial Y, \tag{75}$$

for $m \geq 3$. For example, Virasoro field T is a Schwarzian form of order $1/24$ in CFT with central charge $c = 1/2$ and it is called quasi-primary field.

In a conformal field theory with central charge $c = \frac{2\lambda}{2\lambda+1}(5-8\lambda)$ and a primary field X with conformal dimension λ, the Fock space field $X_s = (L_{-2} - \frac{3}{2(2\lambda+1)}L_{-1}^2)X$ is also a primary field and is called singular vector at level two of the Virasoro algebra representations. The field X is called a primary field degenerate (non-degenerate) at level two if $X_s = 0$ ($X_s \neq 0$). Moreover, $L_0 X_s = (\lambda+2)X_s$.

Free fermionic field theory of fermionic Fock space fields as a conformal field theory is a representation of the Virasoro algebra and Clifford algebra. In fermionic CFT with $c = 1/2$, the fermion field $\psi(z)$ with conformal dimension $h = \frac{1}{2}$ is a primary field degenerate at level two since

$$(L_{-2} - \frac{3}{4}L_{-1}^2)\psi(z) = 0. \tag{76}$$

Ward identity and null field differential equation. In this section, we review the standard results in CFT, such as Ward identity and null field differential equation. The Ward identity in $\mathbb{H}$ for the correlation functions of Fock space fields with insertion of the stress tensor field (which on the half plane with identity chart is the same as Virasoro field), is given by

$$< T(z)\psi(w_1)...\psi(w_n) >_{\mathbb{H}} = \sum_{i=1}^{N}\left[\frac{1/2}{(z-w_i)^2} + \frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right] < \psi(w_1)...\psi(w_n) >_{\mathbb{H}} . \tag{77}$$

For derivation of the Ward identity for Gaussian free fields see lecture (4) in [KaMa11]. In the case of fermionic Fock space fields, one can explicitly check the Ward identity by inserting the Pfaffian form of the correlation functions of fermions in both sides of the above equation.

Using equation (76), one can insert the relation $L_{-2}\psi(z) = \frac{3}{4}L_{-1}^2\psi(z) = T * \psi(z)$ in the Ward identity (77) to obtain the null field differential equation on the half-plane $\mathbb{H}$,

$$\left[\frac{3}{4}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{N}\left(\frac{1/2}{(z-w_i)^2} + \frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right)\right] < \psi(z)\psi(w_1)...\psi(w_n) >_{\mathbb{H}} = 0. \tag{78}$$

Having introduced enough background from fermionic Fock space of conformal fields and their properties and also the VOA structure of fermionic Fock space of states, we will give a proof of the theorem in the next section.

Proof of the $(\mathcal{F} \leftrightsquigarrow V)$ theorem. In order to set up the stage for the proof, we need to associate to each vector $v_i \in V$, a Fock space field $Y_i(z_i) \in \mathcal{F}$, then we define the mapping $\chi^{(D)}_{(z_1,...,z_n)}(v_1 \otimes ... \otimes v_m \otimes v_{m+1} \otimes ... \otimes v_n)$ for an even n as the correlation function of Fock space fields

$$\chi^{(D)}_{(z_1,...,z_n)}(v_1 \otimes ... \otimes v_n) = < Y_1(z_1)...Y_n(z_n) >_D . \tag{79}$$

Then, the first axiom follows immediately

$$\chi^{(D)}_{(z_1,...,z_n)}(\psi \otimes ... \otimes \psi) = < \psi(z_1)...\psi(z_n) >_D = Pf\left(\left[\frac{\sqrt{g'(z_i)}\sqrt{g'(z_j)}}{g(z_i)-g(z_j)}\right]^{n}_{i,j=1}\right). \tag{80}$$

By our definition, the basis vectors v_k of Fock space of states V,

$$v_k = \psi_{-k_i-\frac{1}{2}}\psi_{-k_{i-1}-\frac{1}{2}}...\psi_{-k_2-\frac{1}{2}}\psi_{-k_1-\frac{1}{2}}|0> \tag{81}$$

are associated to Fock space fields at point $z_k \in D$, $Y_k(z_k) \in \mathcal{F}$ of the following form,

$$Y_k(z_k) =: \partial^{k_i}\psi(z_k)\partial^{k_{i-1}}\psi(z_k)...\partial^{k_2}\psi(z_k)\partial^{k_1}\psi(z_k) : . \tag{82}$$

The second axiom can be checked by using the commutation relation $[L_m, \psi_n] = -(\frac{1}{2}m+n)\psi_{m+n}$, which leads to

$$L_{-1}v_k \leftrightarrow \partial_{z_k} : \partial^{k_i}\psi(z_k)\partial^{k_{i-1}}\psi(z_k)...\partial^{k_2}\psi(z_k)\partial^{k_1}\psi(z_k) :, \tag{83}$$

and so we obtain

$$\begin{aligned}\chi^{(D)}_{(z_1,...,z_n)}(v_1\otimes ...\otimes L_{-1}v_k\otimes ...\otimes v_n) &= < Y_1(z_1)...\partial_{z_k}Y_k(z_k)...Y_n(z_n) >_D\\ &= \partial_{z_k} < Y_1(z_1)...Y_n(z_n) >_D\\ &= \frac{\partial}{\partial z_k}\chi^{(D)}_{(z_1,...,z_n)}(v_1\otimes ...\otimes v_n).\end{aligned} \tag{84}$$

To prove the third axiom, let us first write the third axiom formally in terms of Fock space fields. The OPE between two general Fock space fields can be written formally as

$$Y_i(z_i)Y_j(z_j) = \sum_{n=-\infty}^{N}\frac{\{Y_iY_j\}_n(z_j)}{(z_i-z_j)^n} = \overbrace{Y_i(z_i)Y_j(z_j)} + : Y_i(z_i)Y_j(z_j) :, \tag{85}$$

in which normal order is denoted by $: Y_i(z_i)Y_j(z_j) := \sum_{k\geq 0}\frac{(z_i-z_j)^k}{k!} : \partial^k Y_iY_j : (z_j)$, $: Y_iY_j : (z_j) = \{Y_iY_j\}_0(z_j)$ and contraction is denoted by $\overbrace{Y_i(z_i)Y_j(z_j)} = \sum_{n=1}^{N}\frac{\{Y_iY_j\}_n(z_j)}{(z_i-z_j)^n}$. Replacing the OPE in the third axiom and taking the limit $z_m\to z_{m+1}$ lead to

$$\begin{aligned}&\chi^{(D)}_{(z_1,...,z_n)}(v_1\otimes ...\otimes v_m\otimes v_{m+1}\otimes ...\otimes v_n) =\\ &< Y_1(z_1)...Y_m(z_m)Y_{m+1}(z_{m+1})...Y_n(z_n) >_D \sim\\ &\sum_{n=1}^{N}\frac{1}{(z_m-z_{m+1})^n} < Y_1(z_1)...\{Y_mY_{m+1}\}_n(z_{m+1})...Y_n(z_n) >_D =\\ &\sum_{n=1}^{N}\frac{1}{(z_m-z_{m+1})^n}\chi^{(D)}_{(z_1,...,\hat{z}_m,z_{m+1},...,z_n)}(v_1\otimes ...\otimes v_m *_{n-1} v_{m+1}\otimes ...\otimes v_n).\end{aligned} \tag{86}$$

In order to prove the third axiom explicitly, we have to obtain the explicit form of the OPE between two general Fock space fields $Y_1(z_1)$ and $Y_2(z_2)$ by using the Wick's theorem for Fock space fields, and the OPE between two vertex operator in VOA by using the Wick's theorem in VOA and the reconstruction theorem. If the OPEs in VOA and CFT match, then the third axiom holds.

The OPE between any two general Fock space fields can be obtained by taking the limit of the formula (67) and keeping the singular terms when the points become close,

$$\begin{aligned}&: \partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z) :: \partial^{l_m}\psi(w)\partial^{l_{m-1}}\psi(w)...\partial^{l_2}\psi(w)\partial^{l_1}\psi(w) :\sim\\ &\lim_{z\to w}\sum_{s=0}^{min(n,m)}\sum_{i_1<...<i_s,j_1\neq ...\neq j_s}(\pm\partial_z^{k_{i_1}}\partial_w^{l_{j_1}}[\frac{1}{z-w}]...\partial_z^{k_{i_s}}\partial_w^{l_{j_s}}[\frac{1}{z-w}]\\ &: \partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z)\partial^{l_m}\psi(w)\partial^{l_{m-1}}\psi(w)...\partial^{l_2}\psi(w)\partial^{l_1}\psi(w) :_{(i_1,...,i_s;j_1,...,j_s)}),\end{aligned} \tag{87}$$

where in the above expression the singular terms in the limit $z\to w$ are kept.

Since the explicit form of Fock space fields in known, the OPE between them can be explicitly calculated. As we explained, in order to find the OPE we first use the Wick's formula and then write the Taylor expansion of the fields around the second argument. The claim is that the eq. (87) can be formally written in terms of the vertex operators in VOA as the following

$$Y(v_1,z_1)Y(v_2,z_2) = \sum_{p=0}^{N}\frac{Y(v_1 *_p v_2, z_2)}{(z_1-z_2)^{p+1}}. \tag{88}$$

In order to show that, the next step is to write down the OPE for vertex operators in VOA. As we have seen, using the state/operator correspondence, any state in V is associated to a vertex operator in $End(V)[[z,z^{-1}]]$.

Hence, to any basis vector v_i there exists a vertex operator $Y(v_i, z_i)$. Then, by reconstruction theorem we have the assignment

(89) $$Y(\psi_{-k_n-\frac{1}{2}}\psi_{-k_{n-1}-\frac{1}{2}}...\psi_{-k_2-\frac{1}{2}}\psi_{-k_1-\frac{1}{2}}|0>, z) =: \partial^{k_n}\psi(z)\partial^{k_{n-1}}\psi(z)...\partial^{k_2}\psi(z)\partial^{k_1}\psi(z) : .$$

Then, we need to obtain the OPE of two vertex operator of the above form. To obtain the OPE we need the Wick's formula for the product of two vertex operator of the above form and the Taylor expansion. This can be done by assuming that the fields in Wick's formula of VOA in eq. (31) are of the form $a^i(z) = \partial^{k_i}\psi(z)$ and $b^j(z) = \partial^{l_j}\psi(z)$ and the OPE of fermion operators in eq. (34) which leads to $< a^i(z)b^j(w) >= \partial_z^{k_i}\partial_w^{l_j}[\frac{1}{z-w}]$. Then, the OPE in VOA is given by the same equation as for the Fock space fields eq. (87) but with the correct interpretation of objects in VOA.

Without performing the combinatorics for the OPE in VOA and in CFT and just by using the fact that the both sides (OPEs in VOA and CFT) follow the same combinatorics because the OPE of $\psi(z)\psi(z')$ are the same in both sides and contractions of fermion fields in VOA and contractions of fermionic Fock space fields in $\mathcal{F}$ have the same form, thus we conclude the proof of the third axiom.

This theorem leads to an explicit constructions of fermionic correlation functions and differential equations that they satisfy from the fermionic Fock space of states in FVOA. Moreover, in the next section the ($\mathcal{F} \leftrightsquigarrow V$) theorem will be used to provide an explicit form for SLE_3 martingale observables from the correlation functions of fermionic Fock space fields which are rigorously written by using the theorem.

4. Fermionic CFT/SLE$_3$ correspondence

In this section we introduce a rigorous approach to CFT/SLE correspondence, namely the VOA/SLE correspondence. We demonstrate this correspondence in an explicit example, the FVOA/SLE_3. We describe the relation between SLE and the scaling limit of statistical lattice model, namely in a concrete example of the Ising model. We provide an explicit realization of the CFT/SLE correspondence in the Ising model, first by using the explicit Fock space of fermionic states and its relation to the martingale generators in the case of chordal SLE_3. And second, by using the correlation functions of fermions on the upper-half plane and their differential equations, Ward identity and null field differential equation, we show that the correlation functions of fermionic Fock space fields on domain D produce chordal $n - SLE_3$ martingale observables.

Fermionic realization of CFT/SLE in Ising model. The relation between CFT and SLE have been studied from different perspectives, for good reviews see [BaBe06], [Car05], [Gr06] and [Kon03]. In one perspective, the relation between the interfaces in the scaling limit of critical lattice models and the field theory describing that limit is a natural question to study. The interfaces are classified by chordal SLE curves which are characterized by a parameter κ. Moreover, the scaling limit of critical lattice models are usually described by conformal field theory which is characterized by a central charge c. This classifying number, determines the universality class of the scaling limit of the different lattice models such as Ising model.

Although, there is a common belief that a CFT with $c = \frac{1}{2}$ such as $m = 3$ minimal model describes the scaling limit of the critical Ising model [DMS96] (chapter 12), but there were no systematic approach proposed to CFT based on probability theory until recently. We study towards an algebraic construction of a conformal quantum field theory of free fermions for Ising model based on probability theory, SLE. In fact, in the Ising model example, we apply SLE_3 and FVOA to construct and study fermionic CFT of the scaling limit of the Ising model, rigorously.

On the one hand, we study the rigorous scaling limit of Ising model and its fermionic correlation functions which lead to a fermionic conformal field theory. As we have seen in previous chapter, this is composed of algebraic Fock space of states, local Fock space fields, their correlation functions and differential equations that they satisfy. On the other hand, SLE_3 curves and observables appear in the scaling limit of the Ising model at criticality. We study the relation between these two distinct scaling limit of Ising model in different aspects.

First, we study the algebraic operator formalism of fermionic Fock space of fields and states in CFT and VOA and its relation to martingale generators of SLE_3. Second, we explain how a certain fermionic observable of $2d$ critical Ising model, which is obtained from the scaling limit of Ising fermion correlation functions, is related to a martingale observable of SLE_3.

In general, a CFT corresponds to a chordal SLE if the parameters of CFT, c and h, and the parameter of chordal SLE, κ satisfy

(90) $$c_\kappa = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}, \qquad h = \frac{6-\kappa}{2\kappa}.$$

In our example the values of parameters, $c = \frac{1}{2}, h = \frac{1}{2}$ and $\kappa = 3$, satisfy the above equations. The first explicit realization of $FCFT/SLE_3$ in the case of Ising model is the explicit form of the B.C.C. operator, a Ising fermion field $\psi(z)$ that is obtained from the scaling limit of the Ising lattice fermion operator. This is an operator that changes the boundary conditions $\pm/\mp$ on the lattice row configurations. As we have shown in previous section, the state $\psi_{-\frac{1}{2}}|0>$ is a primary state and degenerate at level two. Therefore, this state can be considered as an explicit B.C.C. state for the chordal SLE_3 curve. In fact, the field $\psi(z)$ is a boundary operator, inserted in a boundary point z, the starting point of the SLE_3 curve.

4.1. SLE_3 **and interfaces in Ising model.** Schramm Loewner evolution is a conformally invariant stochastic process introduced in 1999 by O. Schramm [Sch00], for further details see [La08]. It has been used widely in the study of lattice models such as Ising model, percolation etc. The scaling limit of lattice model interfaces at criticality is described by the SLE curves.

Let us start with an intuitive picture of how SLE curves emerge in the Ising model. A chordal SLE_3 curve appears as the scaling limit of discrete interface or boundary between two boundary points where the boundary conditions change. The interface separates two clusters of plus and minus spins in the Ising model. The claim is that in the scaling limit, the interface converges to the chordal SLE_3 curve. The convergence of the interfaces in the Ising model to the SLE_3 curves has been proved recently by Smirnov etal. [CDHKS12].

The probability measure of the spin configurations in the Ising model induces a probability measure on the interfaces, in the following sense. Consider a domain D in complex plane with two boundary points a, b. Approximate the domain and boundary points by square lattice domain $D_n = \frac{1}{n}\mathbb{Z}^2 \cap D$ and a_n, b_n and define a probability measure on interfaces in this domain, $P_n(D_n, a_n, b_n)$. In the limit $n \to \infty$, P_n converges to $\mu(D, a, b)$ which is the law of chordal SLE_3 in the domain D from a to b. This procedure can be easily generalized for the the case of several interfaces.

Let us briefly describe the simplest case of chordal SLE curves in domain (D, a, b). The chordal SLE_κ, with $\kappa \geq 0$ is conformally invariant random curve processes in domain D from a to b. It is described by the Loewner equation with a one-dimensional Brownian motion B_t as a driving force. Let $g_t(z)$ for $z \in D$ and $t < \tau_z \in (0, \infty]$ be the solution of the equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}, \qquad B_0 = 0,$$

where $g_0 : (D, a, b) \to (\mathbb{H}, 0, \infty)$ is a conformal map. Then, the function $h_t(z) := g_t(z) - \sqrt{\kappa}B_t$ for all t is a conformal map, $h_t(z) : D_t \to \mathbb{H}$ from domain $D_t := \{z \in D : t < \tau_z\}$ onto $\mathbb{H}$, where $h_t(z)$ at $t = \tau_z$ maps the tip of the curve to zero; $\lim_{t\to\tau_z} h_t(z) = 0$. The chordal SLE_κ curve γ is defined by $\gamma(t) := \lim_{z\to 0} g_t^{-1}(z + \sqrt{\kappa}B_t)$.

Let us consider the general case of chordal $n - SLE_3$, where there are $2n$ source points and arbitrary number of bulk points. The general picture of the chordal $n - SLE_3$ consists of several interface curves, growing by Loewner chain, a collection of g_t, with random driving force, which connect the boundary points in the critical Ising model. Moreover, in order to define a chordal $n - SLE_3$ we have to write down the Loewner equation in $\mathbb{H}$ with explicit conditions. By using a hydrodynamically normalized conformal map g_t, we can uniformize the complement of the several SLE_3 curves, labeled by integer i with the starting point X_i. This conformal map satisfies the Loewner equation

$$\begin{aligned} dg_t(z) &= \sum_{i=1}^{2n} \frac{2v_t^i dt}{g_t(z) - X_t^i}, \\ g_0(z) &= z, \end{aligned} \tag{91}$$

where v_t^i is the speed of growth of $i-$th curve which can be set to $v_t^i = 1$ and X_t^i are the images of the tips of the curves under the map g_t.

In order to define a martingale observable from the fermionic correlation functions for chordal $n - SLE_3$ we claim that X_t^i should satisfy

$$dX_t^i = \sqrt{3} dB_t^i + 3(\partial_{x_i} \log Z^{\mathbb{H}}_{n-SLE_3}) dt + \sum_{l \neq i} \frac{2dt}{X_t^i - X_t^l}, \tag{92}$$

where dB_t^i are $2n$ independent Brownian motions, $Z^{\mathbb{H}}_{n-SLE_3}$ is called the SLE_3 partition function which will be defined in the following section and initial conditions are $X_0^i = X_i$, and they are ordered as $X_1 < X_2 < ... < X_{2n}$. The claim will be proved in section (4.3). Moreover, the chordal $n - SLE_3$, by definition, satisfies the domain Markov property and domain conformal invariance.

SLE_3 partition function. In the simplest case, a partition function of the chordal SLE_3 on the domain D with two boundary points $a, b \in \partial D$ can be written in terms of the two-point fermionic correlation function,

$$Z^D_{SLE_3} = \chi^D_{a,b}(\psi \otimes \psi) =< \psi(a)\psi(b) >_D \sim \frac{1}{a-b}. \tag{93}$$

In the case of chordal $n - SLE_3$ the partition function is given by the correlation functions of fermions on domain D and it has the Pfafffian structure,

$$Z^D_{n-SLE_3} = \chi^D_{x_1,...,x_{2n}}(\psi \otimes ... \otimes \psi) =< \prod_{i=1}^{2n} \psi(x_i) >_D = Pf\left(\left[\frac{\sqrt{\varphi'(x_i)}\sqrt{\varphi'(x_j)}}{\varphi(x_i) - \varphi(x_j)}\right]^{2n}_{i,j=1}\right), \tag{94}$$

where operators $\psi's$ are inserted at source points, $x_1, ..., x_{2n} \in D$ at B.C.C. points $\pm/\mp$ and $\varphi : D \to \mathbb{H}$ is a conformal map. It can be checked that the chordal $n - SLE_3$ partition function on the half-plane, $Z^{\mathbb{H}}_{n-SLE_3}$ as a CFT correlation function of fermion fields with Pfaffian form satisfies the homogeneity and scaling equations,

$$\sum_i \frac{\partial}{\partial x_i} Z^{\mathbb{H}}_{n-SLE_3} = 0, \quad \sum_i (x_i \frac{\partial}{\partial x_i} + \frac{1}{2}) Z^{\mathbb{H}}_{n-SLE_3} = 0, \quad \sum_i (x_i^2 \frac{\partial}{\partial x_i} + x_i) Z^{\mathbb{H}}_{n-SLE_3} = 0, \tag{95}$$

and null field differential equation,

$$\left[\frac{3}{4}\frac{\partial^2}{\partial x_i^2} + \sum_{l \neq i}\left[\frac{1}{x_l - x_i}\frac{\partial}{\partial x_l} - \frac{1/2}{(x_l - x_i)^2}\right]\right] Z^{\mathbb{H}}_{n-SLE_3} = 0, \tag{96}$$

for $i = 1, ..., 2n$. As it will be explained in the following section, this partition function can be used to define the local martingales in chordal $n - SLE_3$.

4.2. SLE_3 **Martingale generators and fermionic Fock space.** In this section we study the fermionic Fock space of states of the chordal SLE_3 curves. By using the Clifford VOA, we want to construct explicitly the Fock space of states of chordal SLE_3 curves, or the chordal SLE_3 martingale generators, in domain $H_t := \mathbb{H} \setminus \gamma[0, t]$.

Let us review construction of the operator formalism in SLE. It is known that, to any formal power series in particular function $f \in z + \mathbb{C}[[z^{-1}]]$ of the form $f(z) = z + \sum_{m \leq -1} f_m z^{1+m}$, one can associate an operator $G_f \in \overline{\mathcal{U}(\mathfrak{vir}_-)} = \prod_{d=0}^{\infty} \mathcal{U}(\mathfrak{vir}_-)_d$. The algebra $\overline{\mathcal{U}(\mathfrak{vir}_-)}$ is a completion of the universal enveloping algebra $\mathcal{U}(\mathfrak{vir}_-) = \bigoplus_{d=0}^{\infty} \mathcal{U}(\mathfrak{vir}_-)_d$ of the Virasoro subalgebra $\mathfrak{vir}_-$ generated by $L_n (n < 0)$, [BaBe04].

The fermionic Fock space of states V consists of basis vectors of the form $\psi_{-k_n-\frac{1}{2}}\psi_{-k_{n-1}-\frac{1}{2}}...\psi_{-k_2-\frac{1}{2}}\psi_{-k_1-\frac{1}{2}}|0>$. The fermionic descendant states in V and descendant Fock space fields in $\mathcal{F}$ can be constructed by the action of Virasoro generators on the state $|\psi>=\psi_{-\frac{1}{2}}|0>$ and its corresponding field $\psi(z)$, as follow

(97)
$$\mathcal{M}=\mathcal{U}(\mathfrak{vir}_-)|\psi>\subset V, \quad \mathcal{N}=\mathcal{U}(\mathfrak{vir}_-)\psi(z)\subset\mathcal{F}.$$

Since we have the explicit fermionic Fock space of states V from the Clifford VOA, then by using the intertwining operator G_{h_t}, between domains $\mathbb{H}$ and H_t, associated to the conformal map $h_t=g_t-\sqrt{3}B_t$ where $g_t:H_t\to\mathbb{H}$, we can construct the SLE_3 martingale generators and their Fock space in the domain H_t.

The Fock space of states in H_t (the states of SLE_3 curves) and the Fock space of fields in H_t are obtained from the corresponding Fock spaces in $\mathbb{H}$ by means of the action of the transformation operator G_{h_t} as follow

(98)
$$G_{h_t}|\psi>, \quad G_{h_t}^{-1}\psi(z)G_{h_t},$$

and therefore,

(99)
$$\mathcal{M}_t=G_{h_t}\mathcal{U}(\mathfrak{vir})|\psi>, \quad \mathcal{N}_t=G_{h_t}^{-1}(\mathcal{U}(\mathfrak{vir})\psi(z))G_{h_t}.$$

The elements of $\mathcal{M}_t$ are of the form $G_{h_t}\psi_{-k_n-\frac{1}{2}}\psi_{-k_{n-1}-\frac{1}{2}}...\psi_{-k_2-\frac{1}{2}}\psi_{-k_1-\frac{1}{2}}|0>$.

Using the grading of $V=\bigoplus_{h\in\frac{1}{2}\mathbb{N}}V_h$, for $\forall h$ we can write explicitly the vector valued graded martingale generators M_h of the chordal SLE_3 as

(100)
$$\frac{1}{Z}G_{h_t}|\psi>=\sum_{h\in\frac{1}{2}\mathbb{N}}M_h\in\bar{V},$$

where Z is the chordal SLE_3 partition function on the half plane and in fact $Z=<\psi(\infty)\psi(0)>_{\mathbb{H}}=1$, and $\bar{V}=\prod_{h\in\frac{1}{2}\mathbb{N}}V_h$ is the completion of V.

As we observed, the state $|\psi>\in V$ is a primary state degenerate at level two with singular descendant state at level two, $(L_{-2}+\frac{3}{4}L_{-1}^2)|\psi>=0$. Using that, it can be shown that the state $G_{h_t}|\psi>$ is a local martingale of chordal SLE_3, which roughly means that $<v|G_{h_t}|\psi>$ is conserved in mean for any $<v|$,

(101)
$$\mathbb{E}[<v|G_{h_t}|\psi>|\{G_{h_u}\}_{u\leq s}]=<v|G_{h_s}|\psi>,$$

for $t\geq s$.

In order to prove the local martingale property for $G_{h_t}|\psi>$ we need to show that the drift term in Itô derivative of this state vanishes. Since G_{h_t} is an intertwining operator corresponding to the conformal map h_t, it satisfies the Itô differential equation, see section (5.3.3) in [BaBe06],

(102)
$$G_{h_t}^{-1}dG_{h_t}=dt(-2L_{-2}+\frac{3}{2}L_{-1}^2)-d\xi_tL_{-1}.$$

Then, if we apply both sides of above equation to the state $|\psi>$ and use the level two singular vector equation then we obtain

(103)
$$dG_{h_t}|\psi>=G_{h_t}L_{-1}|\psi>d\xi_t,$$

which makes perfect sense in $\bar{V}$ and shows that the drift term in the Itô derivative vanishes and thus $G_{h_t}|\psi>$ is a local martingale of chordal SLE_3.

Since that is a local martingale, all the fermionic correlation functions of CFT in the domain H_t which are SLE_3 observable on the domain H_t, and are constructed from the vector $G_{h_t}|\psi>$ will be local martingale observables of chordal SLE_3. Therefore, $G_{h_t}|\psi>$ is called a generating function of local martingales of chordal SLE_3.

In general, the scaling limit of the interfaces in Ising model are related to the scaling limit of correlation functions of local operators and fields and physical observables in the Ising model. We will elaborate on this point in the following section.

4.3. SLE_3 **Martingale observables and fermion correlation functions.** Let us first describe the martingale observables. It is easy to show that an observable is a local martingale if the drift term in its Itô derivative vanishes. The general claim of this part is that a large collection of SLE_3 martingale observables can be constructed systematically by using the rigorously constructed correlation functions of fermionic Fock space fields in $(\mathcal{F} \leftrightsquigarrow V)$ theorem. It is known that all the correlation functions of descendant Fock space fields can be reduced to correlation functions of primary Fock space fields through the action of differential operators in Ward-like identities, [DMS96],

$$< \psi^{(-k_n,-k_{n-1}...-k_2,-k_1)}(z) \prod_{i=1}^{N} \psi(w_i) >= \mathcal{L}_{-k_n}...\mathcal{L}_{-k_1} < \psi(z) \prod_{i=1}^{N} \psi(w_i) >, \tag{104}$$

where $\psi^{(-k_n,-k_{n-1}...-k_2,-k_1)}(z) = L_{-k_n}L_{-k_{n-1}}...L_{-k_1}\psi(z) = \frac{1}{2\pi i}\oint dw \frac{1}{(w-z)^{n-1}} T(w)(L_{-k_{n-1}}...L_{-k_1}\psi(z))$ and $\mathcal{L}_n$ is defined by

$$\mathcal{L}_{-n} = \sum_{i=1}^{N} \left\{ \frac{1}{2}\frac{(n-1)}{(w_i-z)^n} - \frac{1}{(w_i-z)^{n-1}}\partial_{w_i} \right\}. \tag{105}$$

The correlation functions including more than one descendant fields can be explicitly calculated similarly. This means that the correlation functions of descendant fields can be reduced to the correlation functions of primary fields. We will use this observation later.

In this section we present an explicit form of some chordal $n - SLE_3$ martingale observables as the correlation functions of free fermion fields and descendant fermionic Fock space fields. Furthermore, we will see that basic SLE_3 martingale observables have an explicit Pfaffian structure. We study the relations between fermionic null field differential equation (78) and vanishing of the drift term in the Itô formula for the SLE_3 martingale observables.

Let us construct an example of chordal $n - SLE_3$ martingale observable, see [BBK05]. Consider an operator of the form $\mathcal{O} = \prod_{k=1}^{m} \psi(W_k)$. A martingale observable of this operator in the domain H_t, where the $2n$ SLE_3 curves are removed from $\mathbb{H}$ and the tips of the SLE_3 curves are $\gamma_1, ..., \gamma_{2n}$, is claimed to be

$$< \mathcal{O} >_{H_t} = \frac{1}{Z^{H_t}_{n-SLE_3}} \chi^{H_t}_{\gamma_1,...,\gamma_{2n};W_1,...,W_m}((\psi \otimes ... \otimes \psi) \otimes (\psi \otimes ... \otimes \psi)), \tag{106}$$

where $Z^{H_t}_{n-SLE_3} = \chi^{H_t}_{\gamma_1,...,\gamma_{2n}}(\psi \otimes ... \otimes \psi)$ is a partition function of chordal $n - SLE_3$ in domain H_t. In this case, the explicit form of the correlation function $\chi^{H_t}_{\gamma_1,...,\gamma_{2n};W_1,...,W_m}((\psi \otimes ... \otimes \psi) \otimes (\psi \otimes ... \otimes \psi))$ and the partition function are known as Pfaffian formulas. In the following, we prove the claim that eq. (106) gives a local martingale observable of the chordal $n - SLE_3$.

The observable $< \mathcal{O} >_{H_t}$ can be transformed to $\mathbb{H}$ by using the mapping $g_t : H_t \to \mathbb{H}$ as follow

$$< \mathcal{O} >_{H_t} = \frac{1}{Z^{\mathbb{H}}_{n-SLE_3}} J_t \chi^{\mathbb{H}}_{x_1,...,x_{2n};w_1,...,w_m}((\psi \otimes ... \otimes \psi) \otimes (\psi \otimes ... \otimes \psi)), \tag{107}$$

where $Z^{\mathbb{H}}_{n-SLE_3} = \chi^{\mathbb{H}}_{x_1,...,x_{2n}}(\psi \otimes ... \otimes \psi)$, $x_i = g_t(\gamma_i)$, $w_k = g_t(W_k)$ and the Jacobian is $J_t = \prod_{k=1}^{m} g_t'^{\frac{1}{2}}(W_k)$. Above chordal $n - SLE_3$ observable is a local martingale if the drift term in the Itô derivative of the observable vanishes. To show that the drift term vanishes, we need Itô formula for the $\psi(X^i_t)$, which is $d\psi(X^i_t) = \psi'(X^i_t)dX^i_t + \frac{3}{2}\psi''(X^i_t)dt$, and Loewner equation for $g_t(W_k)$ and its derivative with respect to W_k which lead to,

$$d(\psi(w_k)w_k'^{\frac{1}{2}}) = w_k'^{\frac{1}{2}} \sum_i 2dt \left(\frac{\psi'(w_k)}{w_k - X^i_t} - \frac{\frac{1}{2}\psi(w_k)}{(w_k - X^i_t)^2} \right). \tag{108}$$

In addition, we need the null field differential equation for the chordal $n - SLE_3$ partition function on $\mathbb{H}$, eq. (96), and for the correlation function of fermions on $\mathbb{H}$,

$$D\chi^{\mathbb{H}}_{x_1,...,x_{2n};w_1,...,w_m}((\psi \otimes ... \otimes \psi) \otimes (\psi \otimes ... \otimes \psi)) = 0, \tag{109}$$

where operator D is

$$D = (\frac{3}{4}\frac{\partial^2}{\partial x_i^2} \quad + \quad \sum_{l\neq i}\left[\frac{1}{x_l - x_i}\frac{\partial}{\partial x_l} - \frac{1/2}{(x_l - x_i)^2}\right]$$

$$+ \quad \sum_{k=1}^{m}\left[\frac{1}{w_k - x_i}\frac{\partial}{\partial w_k} - \frac{1/2}{(w_k - x_i)^2}\right]). \tag{110}$$

Then, let us write the Itô derivative of the numerator of eq. (107)

$$d(J_t\chi^{\mathbb{H}}_{x_1,\dots,x_{2n};w_1,\dots,w_m}((\psi\otimes\dots\otimes\psi)\otimes(\psi\otimes\dots\otimes\psi))) =$$

$$J_t\left[\sum_i dX_t^i\partial_{x_i} + \sum_i dt\left(\frac{3}{4}\frac{\partial^2}{\partial x_i^2} + \sum_{k=1}^{m}\left[\frac{1}{w_k - x_i}\frac{\partial}{\partial w_k} - \frac{1/2}{(w_k - x_i)^2}\right]\right)\right]$$

$$\chi^{\mathbb{H}}_{x_1,\dots,x_{2n};w_1,\dots,w_m}((\psi\otimes\dots\otimes\psi)\otimes(\psi\otimes\dots\otimes\psi)). \tag{111}$$

Then, by using the null differential eq. (109), the Itô derivative becomes

$$d(J_t\chi^{\mathbb{H}}_{x_1,\dots,x_{2n};w_1,\dots,w_m}((\psi\otimes\dots\otimes\psi)\otimes(\psi\otimes\dots\otimes\psi))) =$$

$$J_t\left[\sum_i dX_t^i\partial_{x_i} - \sum_i 2dt\left(\sum_{l\neq i}\left[\frac{1}{x_l - x_i}\frac{\partial}{\partial x_l} - \frac{1/2}{(x_l - x_i)^2}\right]\right)\right]$$

$$\chi^{\mathbb{H}}_{x_1,\dots,x_{2n};w_1,\dots,w_m}((\psi\otimes\dots\otimes\psi)\otimes(\psi\otimes\dots\otimes\psi)). \tag{112}$$

As an special case, the above equation implies that

$$dZ^{\mathbb{H}}_{n-SLE_3} = \left[\sum_i dX_t^i\partial_{x_i} - \sum_i 2dt\left(\sum_{l\neq i}\left[\frac{1}{x_l - x_i}\frac{\partial}{\partial x_l} - \frac{1/2}{(x_l - x_i)^2}\right]\right)\right] Z^{\mathbb{H}}_{n-SLE_3}. \tag{113}$$

Finally, by using the above equations the Itô derivative of $< \mathcal{O} >_{H_t}$ becomes

$$d < \mathcal{O} >_{H_t} = J_t\sum_i\left[dX_t^i - (3\partial_{x_i}\log Z^{\mathbb{H}}_{n-SLE_3} + 2\sum_{l\neq i}\frac{1}{x_i - x_l})dt\right]\partial_{x_i}$$

$$(\frac{1}{Z^{\mathbb{H}}_{n-SLE_3}}\chi^{\mathbb{H}}_{x_1,\dots,x_{2n};w_1,\dots,w_m}((\psi\otimes\dots\otimes\psi)\otimes(\psi\otimes\dots\otimes\psi))). \tag{114}$$

Thus, we observe that if the process X_t^i satisfies

$$dX_t^i = \sqrt{3}dB_t^i + (3\partial_{x_i}\log Z^{\mathbb{H}}_{n-SLE_3} + 2\sum_{l\neq i}\frac{1}{x_i - x_l})dt, \tag{115}$$

then the drift term in Itô formula for $< \mathcal{O} >_{H_t}$ vanishes and we have

$$d < \mathcal{O} >_{H_t} = \sqrt{3}J_t\sum_i dB_t^i\partial_{x_i}[\frac{1}{Z^{\mathbb{H}}_{n-SLE_3}}\chi^{\mathbb{H}}_{x_1,\dots,x_{2n};w_1,\dots,w_m}((\psi\otimes\dots\otimes\psi)\otimes(\psi\otimes\dots\otimes\psi))]. \tag{116}$$

To summarize, we observe that $\frac{1}{Z^{H_t}_{n-SLE_3}}\chi^{H_t}_{\gamma_1,\dots,\gamma_{2n};W_1,\dots,W_m}((\psi\otimes\dots\otimes\psi)\otimes(\psi\otimes\dots\otimes\psi))$ is a martingale observable of chordal $n - SLE_3$ with the condition (115).

Without explicit calculations, but because of the fact that $< v|G_{h_t}|\psi >$ (for any arbitrary $< v|$ which can be obtained from the action of descendant fields on the $< \psi|$) are local martingales, we claim that the correlation functions of arbitrary Fock space fields can be used to write the following expression as a chordal $n-SLE_3$ martingale observable

$$\frac{1}{Z^{H_t}_{n-SLE_3}}\chi^{H_t}_{\gamma_1,...,\gamma_{2n};W_1,...,W_m}((\psi\otimes...\otimes\psi)\otimes(v_1\otimes...\otimes v_m)), \tag{117}$$

where v_i's are arbitrary vectors in Fock space. Moreover, we claim that all the SLE_3 observables obtained from the fermionic correlation functions are reducible to the basic ones by using the generalization of eq. (104). By basic SLE_3 observables we mean those that are obtained from the correlation functions of free fermion fields.

5. CONCLUSIONS

In this article we have studied a concrete and explicit realization of the CFT/SLE correspondence in the case of Ising model. We obtained the correlation functions of free fermionic fields on domain D by taking the scaling limit of the lattice correlation functions of the Ising free fermions. These results are obtained by using the rigorous methods of discrete holomorphicity and Riemann boundary value problem introduced in [Hon10a]. We investigated the algebraic and analytic fermionic conformal field theory by studying the correlation functions of fermion fields and differential equations that they satisfy such as Ward identity and singular vector differential equations. Moreover, we developed the algebraic aspects of the fermionic Fock space and fermionic vertex operator algebra and especially we found a mapping between the Fock space of states and the correlation functions of the fermionic fields which respects the conformal structure of the theory. The relation between these results and the probability theory of martingale generators and observables in chordal SLE_3 are studied and we have worked out all these relations explicitly in a concrete example of the Ising model.

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