

# NEW PHASE TRANSITIONS IN CHERN-SIMONS MATTER THEORY

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**ABSTRACT.** Applying the machinery of random matrix theory and Toeplitz determinants we study the level  $k$ ,  $U(N)$  Chern-Simons theory coupled with fundamental matter on  $S^2 \times S^1$  at finite temperature  $T$ . This theory admits a discrete matrix integral representation, i.e. a unitary discrete matrix model of two-dimensional Yang-Mills theory. In this study, the partition function and phase structure of the Chern-Simons matter theory in an special case with Gross-Witten-Wadia potential are investigated. We obtain an exact expression for the partition function of the Chern-Simons matter theory as a function of  $k, N, T$ , for finite values and in the asymptotic regime. In the Gross-Witten-Wadia case, we show that ratio of the Chern-Simons matter partition function and the continuous two-dimensional Yang-Mills partition function, in the asymptotic regime, is the Tracy-Widom distribution. Consequently, using the explicit results for free energy of the theory, new second order and third-order phase transitions are observed. Depending on the phase, in the asymptotic regime, Chern-Simons matter theory is represented either by a continuous or discrete two-dimensional Yang-Mills theory, separated by a third-order domain wall.

## 1. INTRODUCTION

Study of the phase structure of  $(p + 1)$ -dimensional Yang-Mills (YM) theory coupled to matter on  $S^p$  sphere and especially the confinement/deconfinement transition in large  $N$  gauge theories on sphere sheds light on various phenomena in gravitational phase transition such as black hole nucleation, via the AdS/CFT correspondence. In this direction, the phase structure of Chern-Simons (CS) theory coupled to matter and its gravitational dual, Vasiliev higher spin gravity, [Va], have been studied recently, [Ch-Mi] and [Gi]. In a recent work [Ta], it has been argued that level  $k$ ,  $U(N)$  CS theory coupled to fundamental matter on  $S^2 \times S^1$  at finite temperature  $T$ , can be explained by a discrete unitary matrix model, and its phase structure, the eigenvalue density in different phases, has been studied in the limit of large  $k, N, T$ , with fixed parameters  $\lambda = N/k, \zeta = T/N$ , by using steepest descent method. This study is based on the reduction of the Chern-Simons matter (CSM) theory first to the CS theory with an effective potential and second to a discrete version of two-dimensional YM theory and its representation, one matrix model. In an special case with Gross-Witten-Wadia (GWW) potential, [Gr-Wi] and [Wa], in addition to well-known lower-gap phase, an upper-gap phase is observed, [Ta], in which the upper bound of the eigenvalue density is saturated.

In another discipline of research, discrete Toeplitz determinants are studied in [Ba-Li], and by using the continuous orthogonal polynomials their precise forms are obtained in terms of the continuous Toeplitz determinants and the Fredholm determinant. A key fact to understand the application of the Toeplitz determinant in the CSM theory is the Heine-Sezgo identity which relates the matrix model integral and the Toeplitz determinant, see appendix A.

Using the relation between Toeplitz determinants and matrix integrals, and the above results for discrete matrix model representation of the CSM theory, in this paper, the techniques and methods from discrete Toeplitz determinant is employed to study the CSM theory. The matrix integral representations of the YM theory and CSM theory provide the application of well-developed techniques of random matrix theory and specially the Toeplitz determinant.

In our setting, the partition functions of YM theory and CSM theory with an arbitrary potential  $V(T)$  can be written as Toeplitz determinant and discrete Toeplitz determinant with respect to the probabilistic weight  $f(T) = e^{-V(T)}$ , respectively,

$$(1.1) \quad \mathcal{Z}_{YM} = D_N(f), \quad \mathcal{Z}_{CSM} = D_N^k(f).$$

In fact, by using recent results, [Ba-Li], about the discrete and continuous Toeplitz determinants, an explicit expression for the partition function of CSM theory in terms of the partition function of YM theory, as a function of  $k, N, T$  at the finite values and in the asymptotic regime, is obtained. Then, the special case of GWW potential is studied in detail and an analytic form of the partition function and free energy of CSM theory are obtained. Furthermore, the sub-leading corrections in the asymptotic results is computed and the result is interpreted as the energy of the upper bound. These results lead to a phase transition in the asymptotic regime of the discrete matrix model, related to the phase transition between upper-bound phase and no-upper bound phase, previously studied in [Ta]. In brief, the phase structure of the model in the asymptotic limit of large  $N, T, k$  is studied via the obtained analytic expression for the free energy and as a result, new second-order and third-order phase transitions in this model are determined. Moreover, relation between the obtained result and recent results, in [Ta], is explored.

The phases of CSM theory, can be discussed from the viewpoint of the existence of the continuum limit. In fact, the continuum limit does not exist in whole moduli space of the parameters and in fact,  $N, T, k$  should satisfy certain relations in order to determine the continuum limit.

This paper is organized as follows: In section two, relevant results from [Ta], to our study are reviewed. The CSM theory and the discrete matrix integral representation of the partition function are explained in brief. Furthermore, the phase structure of the CSM theory with GWW potential is described. In section three, the main results of this study are expressed. First, the partition function and free energy of the CSM theory with GWW potential are explicitly obtained and subsequently the phase structure of the model is studied. In section four, possible interpretations of the obtained results are discussed and directions for further studies are introduced. Finally, there are two appendices, where necessary materials from the random matrix theory such as orthogonal polynomials, Toeplitz determinant and Riemann-Hilbert problem (RHP) and their inter-relations are introduced and well-known results in two-dimensional YM theory with GWW potential in terms of Toeplitz determinant and RHP problem are summarized. Finally, some necessary definitions and results about Tracy-Widom distribution are collected.

## 2. CHERN-SIMONS MATTER THEORY AND ITS PHASE STRUCTURE

In this section, the level  $k$ ,  $U(N)$  CSM theory coupled to fundamental matter at temperature  $T$  on  $S^2 \times S^1$  is described. In the first part, we review the following observations in [Ta]; I) the action of pure CS theory coupled with fundamental matter in the 't Hooft limit and at high temperature, up to leading order in  $N$ , can be expressed as the sum of the action of pure CS theory and an effective action with an effective potential which is a local function of holonomy matrix  $U(x)$ . II) the partition function of the CSM theory with effective potential can be written as a discrete version of the matrix integral representation of the  $2d$  YM theory with the effective potential.

In the second part, we collect the results in [Ta], about the phase structure of a toy CSM theory, i.e. the CSM theory with the GWW potential. These results will be compared to ours in the next section.

**2.1. CSM theory and discrete YM theory.** CS theory is a three-dimensional topological field theory of the Schwarz type [Sc], with no metric dependence in the action. The partition function and correlation functions of the theory define global topological invariants of the manifold as well as other topological invariants associated to the manifold such as knot invariants, Jones polynomials, etc. [Wi]. Moreover, CS theory have been applied successfully in condensed matter systems such as quantum Hall effect, [Su]. The partition function of CS theory on Seifert manifolds such as  $S^3$  admits a matrix model representation and thus analytic closed expressions for the partition function have been obtained in [Ma].

Consider a three-dimensional manifold  $M$  with a principal  $G$ -bundle with a connection  $A$ , then the CS action is defined by

$$(2.1) \quad \mathcal{S}_{CS}(A) = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

where  $k$  is called level and it is an integer in the quantized theory. The above action functional defines a three-dimensional topological field theory of the Schwarz type [Sc], i.e. a field theory with observables, explicitly independent of the metric on  $M$ . In this study, we consider  $G = U(N)$ .

The partition function of the theory is given by a path integral,

$$(2.2) \quad \mathcal{Z}_{CS}(A) = \int [\mathcal{D}A] e^{i\mathcal{S}_{CS}},$$

and the observables of the theory are defined via the correlation functions of the Wilson loops,  $\mathcal{W}_{C_i, R_i}(A) = \text{Tr} \left[ \mathcal{P} e^{i\oint_{C_i} A} \right]$  in  $R_i$  representation with closed loops  $C_i$ , as,

$$(2.3) \quad \left\langle \prod_{i=1}^n \mathcal{W}_{C_i, R_i}(A) \right\rangle = \int [\mathcal{D}A] e^{i\mathcal{S}_{CS}} \left( \prod_{i=1}^n \mathcal{W}_{C_i, R_i}(A) \right).$$

The CS theory coupled with the matter in fundamental representation on  $S^2 \times S^1$ , in  $T, N, k \rightarrow \infty$  limit with the fixed 't Hooft parameter,  $\lambda = N/k$ , and  $\zeta = T/N$ , has been studied in [Ta]. The goal of this part is to compute the partition function of the theory,

$$(2.4) \quad \mathcal{Z}_{CSM}(A, \mu) = \int [\mathcal{D}A][\mathcal{D}\mu] e^{i\mathcal{S}_{CS} - \mathcal{S}_\mu},$$

where  $\mu$  represents the matter fields and  $\mathcal{S}_\mu$  is the corresponding action. In brief, after integration over the massive modes, a local effective action,  $\mathcal{S}_{eff}(U)$ , for the zero modes,  $2d$  unitary matrix valued holonomy  $U(x)$  around  $S^1$  with  $x \in S^2$  is obtained in [Ah]. Then, the partition function is obtained by summing up all the separate contributions from the vacuum graphs including the holonomy fields and matter fields,

$$(2.5) \quad \mathcal{Z}_{CSM}(A) = \int [\mathcal{D}A] e^{i\mathcal{S}_{CS} - \mathcal{S}_{eff}(U)}.$$

At large  $N$  with  $T \sim N$ , up to leading order in  $N$ , the entire effect of matter loops in vacuum graphs on the dynamics of the CS theory is represented by the high temperature effective action of the following form,

$$(2.6) \quad \mathcal{S}_{eff} = T \int d^2x \sqrt{g} v(U(x)),$$

where  $v(U(x))$  is an effective potential which depends on the matter content and CS couplings. Therefore, the thermal partition function of the CSM theory is an  $x$ -independent observable of the theory, i.e. an expected value of linear combination of the Wilson loops in pure CS theory,

$$(2.7) \quad \mathcal{Z}_{CSM} = \int [\mathcal{D}A] e^{i\frac{k}{4\pi} \int_M \text{Tr}(AdA + \frac{2}{3}A^3) - T \int d^2x \sqrt{g} v(U(x))} = \langle e^{-TV_2 v(U)} \rangle_{N,k},$$

where  $V_2$  is the  $2d$  volume of  $S^2$  and the integral of the effective potential over  $S^2$  is approximated. For more convenience assume  $V_2 = 1$ . For the computation of the  $v(U)$  for any given vector matter CS theory, see [Ta], [Ah] and references therein.

An explicit form of the CSM partition function can be obtained by using the path integral techniques, [Bl-Th], for pure CS and its modification to include the effect of potential, [Ta]. Taking the integral of the CS action over  $S^2$ , the partition function over the space of gauge fields reduces to a matrix integral over the holonomy matrices. There are contributions from the  $U(1)^N$  flux sector which lead to the discretization of the holonomy eigenvalues. It has been shown in [Ta], that the partition function of CSM theory, Eq. (2.7), is

given by a discrete unitary matrix integral,

$$\begin{aligned}
\mathcal{Z}_{CSM} &= \int [\mathcal{D}A] e^{(iS_{CS} - T \int d^2x \sqrt{g} v(U(x)))} \\
&= \prod_{j=1}^N \int_{-\infty}^{\infty} d\alpha_j \left[ \prod_{l < p} \left( 2 \sin \left( \frac{\alpha_l(\vec{n}) - \alpha_p(\vec{n})}{2} \right) \right)^2 e^{-N\zeta v(U)} \right] \sum_{m_j=-\infty}^{\infty} e^{ikm_j \alpha_j} \\
&= \prod_{j=1}^N \int_{-\infty}^{\infty} d\alpha_j \left[ \prod_{l < p} \left( 2 \sin \left( \frac{\alpha_l(\vec{n}) - \alpha_p(\vec{n})}{2} \right) \right)^2 e^{-N\zeta v(U)} \right] \sum_{n \in \mathbb{Z}} \delta(k\alpha_j - 2\pi n) \\
(2.8) \quad &= \prod_{i=1}^N \sum_{n_i=-\infty}^{\infty} \left[ \prod_{l < i} \left( 2 \sin \left( \frac{\alpha_l(\vec{n}) - \alpha_i(\vec{n})}{2} \right) \right)^2 e^{-N\zeta v(U)} \right],
\end{aligned}$$

where  $m_j$ 's are constant units of flux in  $U(1)_j$ 's factors, from the second line to the third line we used  $\sum_{m_j=-\infty}^{\infty} e^{ikm_j \alpha_j} = \sum_{n \in \mathbb{Z}} \delta(k\alpha_j - 2\pi n)$  and the delta function puts restriction on eigenvalues,  $\alpha_j(\vec{n}) = \frac{2\pi n_j}{k}$ , such that no two  $n_i$  are allowed to be equal, which in return transform the integral over the eigenvalues to a discrete sum in the last line.

This matrix integral representation of  $\mathcal{Z}_{CSM}$  is a discrete version of the matrix integral representation of the two-dimensional YM theory partition function, Eq. (A.1) with  $V_{YM} = N\zeta v(U)$ .

A toy model of the CSM theory, Eq. (2.8), with interesting features is the CS theory with the GWW potential,  $v(U) = -\frac{1}{2} \text{Tr}(U + U^\dagger)$ , or  $v(\alpha) = -\sum_{i=1} \cos \alpha_i$ . This model shares some features with the real CS theory such as the minimum of  $v(U)$  is always at  $U = I$  and the depth of the potential increases as a function of  $\zeta$ .

**2.2. Phase structure of CSM theory with GWW potential.** By definition, the eigenvalue density is positive and restricted from below by zero lower bound. This leads to lower gaps defined by  $\{\alpha | \alpha \sim \alpha + 2\pi, \rho(\alpha) = 0\}$ . As it was told, summation over flux sector leads to discretization of the eigenvalues and this leads to an upper bound for the eigenvalue density,

$$(2.9) \quad 0 \leq \rho(\alpha) \leq \frac{k}{2\pi} \times \frac{1}{N} = \frac{1}{2\pi\lambda},$$

where  $2\pi/k$  can be seen as the distance between two consecutive eigenvalues. The eigenvalue density is normalized by  $\int_{-\pi}^{\pi} \rho(\alpha) = 1$ . The upper bound on the eigenvalue density leads to upper gaps defined by  $\{\alpha | \alpha \sim \alpha + 2\pi, \rho(\alpha) = \frac{1}{2\pi\lambda}\}$ . The phase structure of the CSM theory defined by Eq. (2.8), can be understood via the comparison of the competing forces in the theory. There are two competing factors in this problem. I) the attractive potential  $V(U)$  which tends to clump the eigenvalues  $\alpha_i$ 's and II) the repulsive force from the measure  $\mathcal{D}U$  or equivalently from the Vandermonde determinant which repels the eigenvalues because it vanishes when two eigenvalues coincide.

As a result, in the CSM theory with the GWW potential, there are four phases, namely *no-gap* phase, *lower-gap* phase, *upper-gap* phase and *two-gap* phase. Depending on the parameters,  $\lambda$  and  $\zeta$ , the phases of the CSM theory are classified in [Ta]. In this part, the qualitative and quantitative results for the phase structure of CSM theory with GWW potential are reviewed. The explicit results are obtained from the saddle point analysis of the discrete GWW model, [Ta]. For alternative derivations of the eigenvalue measures for models of this type, see [Jo-Ja-Ke].

At low temperatures, in the regions ( $\lambda < 1/2$ ,  $\zeta < 1$ ) and ( $\lambda > 1/2$ ,  $\zeta < 1/\lambda - 1$ ), the factor (II) is stronger and eigenvalue density has support everywhere on the unit circle. The eigenvalue density is given by  $\rho(\alpha) = (1 + \zeta \cos \alpha)/2\pi$  and the system is in no-gap phase. As the temperature increases, factor (I) plays more effective role and depending on  $\lambda$ , the lower-gap or the upper-gap forms first. For small  $\lambda$ ,  $\lambda < 1/2$ , in the region  $1 < \zeta < 1/4\lambda^2$ , the upper bound is large and first the lower-gap phase forms with eigenvalue density,  $\rho(\alpha) = \zeta \cos(\alpha/2)/\pi \sqrt{1/\zeta - \sin^2(\alpha/2)}$  supported on  $[-\alpha_c, \alpha_c]$  and  $\alpha_c$  is given by the condition  $\sin^2(\alpha_c/2) = 1/\zeta$ . Then, in the region  $\zeta > 1/4\lambda^2$  the two-gap phase with one lower-gap and one upper-gap

forms. The lower-gap and the upper-gap are located on the arcs  $[e^{ib}, e^{-ib}]$  and  $[e^{ia}, e^{-ia}]$  on the unit circle, respectively, where  $a$  and  $b$  are determined by

$$(2.10) \quad \begin{aligned} \frac{1}{4\pi\lambda} \int_{-a}^a d\alpha \frac{1}{\sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\alpha}{2}} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}} &= \zeta, \\ \frac{1}{4\pi\lambda} \int_{-a}^a d\alpha \frac{\cos \alpha}{\sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\alpha}{2}} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2}}} &= 1 + \frac{\zeta}{2}(\cos a + \cos b). \end{aligned}$$

Between the arcs, in the complement of two gaps, the eigenvalue density is determined by

$$(2.11) \quad \begin{aligned} \rho(\alpha) &= \frac{|\sin \alpha|}{4\pi^2\lambda} \sqrt{(\sin^2 \frac{a}{2} - \sin^2 \frac{\alpha}{2})(\sin^2 \frac{b}{2} - \sin^2 \frac{\alpha}{2})} \\ &\times \int_{-a}^a d\theta \frac{1}{(\cos \theta - \cos \alpha) \sqrt{\sin^2 \frac{a}{2} - \sin^2 \frac{\theta}{2}} \sqrt{\sin^2 \frac{b}{2} - \sin^2 \frac{\theta}{2}}}. \end{aligned}$$

For large  $\lambda$ ,  $\lambda > 1/2$ , in the region  $1/\lambda - 1 < \zeta < 1/(4\lambda(1 - \lambda))$ , the upper bound is small and upper-gap phase forms first, with eigenvalue density,

$$(2.12) \quad \rho(\alpha) = \begin{cases} \frac{1}{2\pi\lambda} - \frac{\zeta}{\pi} |\sin \frac{\alpha}{2}| \sqrt{\frac{1-\lambda}{\lambda\zeta} - \cos^2 \frac{\alpha}{2}} & \cos^2 \frac{\alpha}{2} < \frac{1-\lambda}{\lambda\zeta}, \\ \frac{1}{2\pi\lambda} & \cos^2 \frac{\alpha}{2} > \frac{1-\lambda}{\lambda\zeta}, \end{cases}$$

then, in the region  $\zeta > 1/(4\lambda(1 - \lambda))$ , the two-gap phase forms.

In summary, the phase structure of the CSM theory with GWW potential is

$$(2.13) \quad \text{for } \lambda < \lambda_c : \begin{cases} \text{no-gap phase} & \zeta < 1 \\ \text{lower-gap phase} & 1 < \zeta < \frac{1}{4\lambda^2} \\ \text{two-gap phase} & \zeta > \frac{1}{4\lambda^2} \end{cases}, \quad \text{for } \lambda > \lambda_c : \begin{cases} \text{no-gap phase} & \zeta < \frac{1}{\lambda} - 1 \\ \text{upper-gap phase} & \frac{1}{\lambda} - 1 < \zeta < \frac{1}{4\lambda(1-\lambda)} \\ \text{two-gap phase} & \zeta > \frac{1}{4\lambda(1-\lambda)} \end{cases},$$

where  $\lambda_c = \frac{1}{2}$ .

In the following section, explicit formulas for the free energy of the CSM theory in each of the phases are obtained. Furthermore, the order of the phase transition between upper-gap phase and no-upper-gap phase is computed.

### 3. NEW RESULTS IN CHERN-SIMONS MATTER THEORY

In this section, new phase transitions in the CSM theory with GWW potential are observed. In order to obtain the final result, in the first part, the partition function of the CSM theory with an arbitrary potential is obtained via a recent result in the context of the discrete Toeplitz determinants, [Ba-Li]. In the second part, the CSM theory with GWW potential is studied and explicit asymptotic results, in the limit of large parameters, for the partition function is obtained. In the third part, a careful study of the obtained results leads to an explicit expression for the free energy of the CSM theory with GWW potential which consequently reveals new phase transitions in this case. In the fourth part, some consistency checks of the obtained results as well as some explicit computations for the free energy in different limits are performed.

**3.1. CSM theory partition function.** It has been shown in [Ba-Li], that the analysis of discrete Toeplitz determinants can be done by means of continuous orthogonal polynomials associated to continuous weight function. The main result in [Ba-Li] which is used in this study is the explicit relation between the discrete and continuous versions of the Toeplitz determinant for a given weight function in terms of a Fredholm determinant. Then, this relation is used to connect the partition functions of CSM theory (discrete YM theory) and continuous YM theory, via the relation between the Toeplitz determinants and partition functions, see appendix A. Adopting the Theorem (1.1) in [Ba-Li] for the problem in this study, the following explicit

formula is obtained. Let  $d$  be a finite discrete subset of a unit circle  $\Sigma$  and let  $\Omega$  be a neighborhood of  $\Sigma$  and  $f(z) = e^{-V(z)}$  be an analytic positive weight function with continuous measure,  $f(z) \frac{dz}{2\pi iz}$ , then,

$$(3.1) \quad \frac{\mathcal{Z}_{CSM}}{\mathcal{Z}_{YM}} = \frac{D_N^k(f, d)}{D_N(f)} = \det(1 + K(z, w)),$$

where  $K$  is the kernel of the Fredholm determinant. And the kernel is given by

$$(3.2) \quad K(z, w) = K_{cont}(z, w) \sqrt{v(z)v(w)f(z)f(w)}, \quad K_{cont}(z, w) = z^{-N} \frac{p_N(z)p_N^*(w) - p_N^*(z)p_N(w)}{1 - z^{-1}w},$$

and  $p_N(z)$  is a orthogonal polynomial with respect to weight function  $f(z)$ , and  $p_N^*(z) := z^N \overline{p_N(\bar{z}^{-1})}$ , and  $v(z)$  is a function representing discreteness. This function is defined by

$$(3.3) \quad v(z) := \begin{cases} -\frac{z\gamma'(z)}{|d|\gamma(z)} & z \in \Sigma_{in} \\ \frac{z\gamma'(z)}{|d|\gamma(z)} - 1 & z \in \Sigma_{out} \end{cases},$$

where  $\gamma(z)$  is an analytic function on  $\Omega$  such that it vanishes exactly on  $d$  and all zeros are simple, and  $\Sigma_{in}$  and  $\Sigma_{out}$  are positively-oriented circles of radius  $1 - \epsilon$  and  $1 + \epsilon$ .

In the limit of large parameters the ratio of the partition functions/Toeplitz determinants can be studied via the asymptotic analysis of the Fredholm determinant. As the simplest example, for a positive constant weight function,  $f_c$ , as  $k - N \rightarrow \infty$  and  $N \rightarrow \infty$ , the following result is obtained in [Ba-Li],

$$(3.4) \quad \frac{D_N^k(f_c, d_k)}{D_N(f_c)} = 1 + \mathcal{O}(e^{-c(k+N)}),$$

where  $d_k = \{z \in \mathbb{C} | z^k = 1\}$ , and  $c$  is a positive constant. In this special case, one can take  $\gamma(z) = z^k - 1$ , and by using,

$$(3.5) \quad v(z) = \begin{cases} -\frac{z^k}{1-z^k} & z \in \Sigma_{in} \\ \frac{z^{-k}}{1-z^{-k}} & z \in \Sigma_{out} \end{cases},$$

the above result for the ratio can be proved. However, Eq. (3.4) is expected, since in the leading order, the continuum Toeplitz determinant is the continuum limit of discrete Toeplitz determinant, when the discrete structure vanishes. But as it will be studied in this paper, the asymptotic limit of the ratio depends on the weight function in a nontrivial way and the continuum limit is only obtained in a special region of the moduli space of the parameters.

**3.2. Partition function of CSM theory with GWW potential.** In this part, the partition function of CSM theory with GWW potential is obtained explicitly in terms of the partition function of the GWW model, (see Appendix A.1.2), and by careful analysis of the asymptotic regime, new phase transitions are traced.

In the finite regime, the following equation, obtained from Eq. (3.1), determines the CSM partition function with GWW potential,

$$(3.6) \quad \mathcal{Z}_{CSM}^{(GWW)} = \mathcal{Z}_{YM}^{(GWW)} \det(1 + K_{GWW}(z, w)),$$

where  $K_{GWW}(z, w)$  can be obtained from Eq. (3.2) by putting  $f_{GWW}(z) = e^{\frac{T}{2}(z+z^{-1})}$ .

In the limit  $N, T, k \rightarrow \infty$ , asymptotic results of the ratio in Eq. (3.1) for the case of GWW potential can be obtained by adopting the results in [Ba-Li], (Proposition 4.1. and Theorem 4.1).

In domain  $d_s = \{z \in \mathbb{C} | z^k = s\}$ , using the results in [Ba-Li], one can obtain,

$$(3.7) \quad \lim_{N, T, k \rightarrow \infty} \oint_{|s|=1} \frac{D_N^k(f_{GWW}, d_s)}{D_N(f_{GWW})} \frac{ds}{2\pi i s} = F\left(\frac{k - \mu}{\sigma}\right),$$

where  $F$  is the Tracy-Widom distribution (see appendix B) and functions  $\mu$  and  $\sigma$  are defined by

$$(3.8) \quad \mu := \begin{cases} N + T & N \geq T \\ 2\sqrt{NT} & N < T \end{cases}, \quad \sigma := \begin{cases} 2^{-\frac{1}{3}}T^{\frac{1}{3}} & N \geq T \\ 2^{-\frac{2}{3}}T^{\frac{1}{3}}(\sqrt{\frac{N}{T}} + \sqrt{\frac{T}{N}})^{\frac{1}{3}} & N < T \end{cases}.$$

For  $s = 1$ , the above result, Eq. (3.7), implies

$$(3.9) \quad \lim_{N,T,k \rightarrow \infty} \frac{\mathcal{Z}_{CSM}^{(GWW)}}{\mathcal{Z}_{YM}^{(GWW)}} = \lim_{N,T,k \rightarrow \infty} \frac{D_N^k(f_{GWW}, d)}{D_N(f_{GWW})} = F\left(\frac{k - \mu}{\sigma}\right).$$

Using an expression for the Tracy-Widom distribution in terms of the kernel of the Airy function,  $F = \det(1 + K_{Air})$ , (see Appendix B), Eqs. (3.6) and (3.9), imply that  $\lim_{N,T,k \rightarrow \infty} K_{GWW}(z, w) = K_{Ai}(z, w)$ .

In the next step, an asymptotic formula for the ratio of the partition functions, Eq. (3.9) is obtained via the asymptotic formula for the Tracy-Widom distribution, Eq. (B.4) in Appendix B,

$$(3.10) \quad F(x) = \begin{cases} 1 - \mathcal{O}(e^{-x^{3/2}}) & x \rightarrow \infty \\ \mathcal{O}(e^{-|x|^3}) & x \rightarrow -\infty \end{cases}.$$

Let us denote the ratio by  $\mathcal{R}(N, T, k) = \lim_{N,T,k \rightarrow \infty} \frac{\mathcal{Z}_{CSM}^{(GWW)}}{\mathcal{Z}_{YM}^{(GWW)}}$ , then by using the above result with  $x = \frac{k - \mu}{\sigma}$ , the ratio of the partition functions is given by,

$$(3.11) \quad \mathcal{R}(N, T, k) = \begin{cases} 1 - \mathcal{O}(e^{-N}) & k > (1 + \epsilon)\mu(N, T) \\ 0 + \mathcal{O}(e^{-N^2}) & k < (1 - \epsilon)\mu(N, T) \end{cases},$$

where  $\epsilon$  is a positive infinitesimal parameter. Equivalently, the above result can be written separately for two phases,

$$(3.12) \quad \text{for } N \geq T : \mathcal{R}(k, N, T) \approx \begin{cases} 1 & k > N + T \\ 0 & k < N + T \end{cases}, \quad \text{for } N < T : \mathcal{R}(k, N, T) \approx \begin{cases} 1 & k > 2\sqrt{NT} \\ 0 & k < 2\sqrt{NT} \end{cases}.$$

Since the ratio of the partition functions jumps across the line  $k = \mu$ , one would expect a phase transition of order zero, however, a careful analysis is needed to determine the order of the phase transition via the explicit form of the free energy. As it is observed, CSM theory has a very complicated moduli space of three parameters  $N, T, k$  at asymptotic regime.

A possible interpretation of the above result can be expressed in terms of the continuum limit and upper-gap phase. Notice that the difference between the CSM theory and the YM theory is the presence of the discreteness and upper bound in eigenvalue density for the CSM theory. Therefore, the continuum limit of the CSM is expected in the regime or the phase that upper bound is not formed. On the other hand, in the continuum limit the partition function of the CSM theory becomes the partition function of the YM theory and the ratio becomes one. Therefore, in terms of  $\lambda$  and  $\zeta$  the ratio with the correct interpretation can be written as

$$(3.13) \quad \text{for } \zeta \leq 1 : \mathcal{R}(\lambda, \zeta) \approx \begin{cases} 1 & \text{no upper-bound, } \lambda^{-1} > \zeta + 1 \\ 0 & \text{upper-bound, } \lambda^{-1} < \zeta + 1 \end{cases},$$

$$(3.14) \quad \text{for } \zeta > 1 : \mathcal{R}(\lambda, \zeta) \approx \begin{cases} 1 & \text{no upper-bound, } \lambda^{-1} > 2\sqrt{\zeta} \\ 0 & \text{upper-bound, } \lambda^{-1} < 2\sqrt{\zeta} \end{cases}.$$

In the next part, explicit calculations for free energy in each region of the moduli space determine the order of this new phase transition and other features of the CSM theory.

Above interpretation is consistent with the results in [Ta]. This consistency can be seen by comparing our above result with the rearranged form of (2.13),

$$(3.15) \quad \text{for } \lambda < \frac{1}{2} : \begin{cases} \text{no-gap or lower-gap,} & \zeta < \frac{1}{4\lambda^2}, \\ \text{two-gap,} & \zeta > \frac{1}{4\lambda^2}, \end{cases} \text{ for } \lambda > \frac{1}{2} : \begin{cases} \text{no-gap,} & \zeta < \frac{1}{\lambda} - 1, \\ \text{upper-gap or two-gap,} & \zeta > \frac{1}{\lambda} - 1. \end{cases}$$

However, as it will be clarified, our result contains more information about the phase structure such as the explicit formulas for the free energy and the order of phase transition between continuous and discrete YM theories. The actual properties and features of this new phase transition can only be described by careful analysis of the Tracy-Widom distribution which will be presented in the next part.

**3.3. Free energy of CSM theory and new phase transitions.** In this part, an explicit formula for the free energy of the CSM theory in different phases is obtained. The free energy determines the order of the new phase transition introduced in the previous part. The obtained results will be checked in nontrivial calculations. Furthermore, some limits of the free energy in interesting points of the moduli space are computed.

In order to study the phase structure of the theory, the first step is to compute the sub-leading corrections to ratio formula, Eq. (3.11), by using the expansion of the Tracy-Widom distribution. In the asymptotic regime, the free energy of the YM and CSM theories is defined via their partition functions,  $\mathcal{F}_{YM/CSM} = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log Z_{YM/CSM}$ . In the case of GWW potential, by using Eq. (3.9), and the integral representation of the Tracy-Widom distribution (see Appendix B), the free energy of the CSM theory is given by

$$(3.16) \quad \mathcal{F}_{CSM} = \mathcal{F}_{YM} + \frac{1}{N^2} \log F(x) = \mathcal{F}_{YM} - \frac{1}{N^2} \left[ \int_x^\infty (s-x) q^2(s) ds \right],$$

where  $x = \frac{k-\mu}{\sigma}$ . In different phases, by using YM free energy, Eq. (A.7), the free energy of the CSM theory is given by

$$(3.17) \quad \mathcal{F}_{CSM} = \mathcal{F}_{YM} + \frac{1}{N^2} \log F(N^{\frac{2}{3}}j) = \begin{cases} \frac{\zeta^2}{4} - \frac{1}{N^2} \left[ \int_{N^{\frac{2}{3}}j}^\infty (s - N^{\frac{2}{3}}j) q^2(s) ds \right] & \zeta \leq 1 \\ \zeta - \frac{3}{4} - \frac{\log \zeta}{2} - \frac{1}{N^2} \left[ \int_{N^{\frac{2}{3}}j}^\infty (s - N^{\frac{2}{3}}j) q^2(s) ds \right] & \zeta > 1 \end{cases},$$

where

$$(3.18) \quad j := \begin{cases} \frac{k-(N+T)}{2^{-\frac{1}{3}}T^{\frac{1}{3}}} N^{-\frac{2}{3}} = \frac{\lambda^{-1}-(\zeta+1)}{2^{-\frac{1}{3}}\zeta^{\frac{1}{3}}} & \zeta \leq 1 \\ \frac{k-(2\sqrt{NT})}{2^{-\frac{2}{3}}T^{\frac{1}{3}}(\sqrt{\frac{N}{T}}+\sqrt{\frac{T}{N}})^{\frac{1}{3}}} N^{-\frac{2}{3}} = \frac{\lambda^{-1}-2\zeta^{\frac{1}{2}}}{2^{-\frac{2}{3}}\zeta^{\frac{1}{3}}(\zeta^{\frac{1}{2}}+\zeta^{-\frac{1}{2}})^{\frac{1}{3}}} & \zeta > 1 \end{cases}.$$

Following the interpretation of the ratio result in the previous section, Eq. (3.16) can be written as

$$(3.19) \quad \mathcal{F}_{CSM} = \mathcal{F}_{YM} + \mathcal{F}_{ub},$$

where  $\mathcal{F}_{ub} = \frac{1}{N^2} \log F(N^{\frac{2}{3}}j) = -\frac{1}{N^2} \int_{N^{\frac{2}{3}}j}^\infty (s - N^{\frac{2}{3}}j) q^2(s) ds$  is the upper bound energy. For  $j < 0$ , in the large  $N$  limit, by using the expansion of  $q(s)$  in the limit  $s \rightarrow \pm\infty$ , (see Appendix B), the following



approximate result up to leading order is obtained,

$$\begin{aligned}
\mathcal{F}_{ub} &= -\frac{1}{N^2} \int_{N^{\frac{2}{3}}j}^{\infty} (s - N^{\frac{2}{3}}j) q^2(s) ds \\
&\approx -\frac{1}{N^2} \int (s - N^{\frac{2}{3}}j) \left(\frac{-s}{2}\right) ds \Big|_{s=N^{\frac{2}{3}}j \rightarrow -\infty} \\
&\quad -\frac{1}{N^2} \int (s - N^{\frac{2}{3}}j) q^2(s) ds \Big|_{s=finite} \\
&\quad -\frac{1}{N^2} \int (s - N^{\frac{2}{3}}j) \left(-\frac{e^{-\frac{2}{3}s^{\frac{3}{2}}}}{2\sqrt{\pi}s^{\frac{1}{4}}}\right)^2 ds \Big|_{s \rightarrow \infty} \\
(3.20) \quad &\approx \frac{j^3}{12},
\end{aligned}$$

where the third and fourth lines vanish in the large  $N$  limit and only the second line gives a finite contribution. Above approximate calculation produce the correct result as it will be shown in the following. In the next-to-leading order, the upper bound energy and total free energy of CSM theory in each phase can be computed, either by using the above result, Eq. (3.20), or by direct computation from the expansion of the Tracy-Widom distribution, Eq. (B.4), as follows,

$$(3.21) \quad \text{for } \zeta \leq 1 : \mathcal{F}_{CSM} = \begin{cases} \frac{\zeta^2}{4} + \frac{1}{N^2} \log\left(1 - \frac{e^{-c_1 j^{3/2} N}}{32\pi j^{3/2} N}\right) & \lambda^{-1} > \zeta + 1 \\ \frac{\zeta^2}{4} + \frac{1}{N^2} \log\left(c_3 \frac{e^{-c_2 |j|^3 N^2}}{|j|^{1/8} N^{1/12}}\right) & \lambda^{-1} < \zeta + 1 \end{cases},$$

where  $c_1 = 4/3$ ,  $c_2 = 1/12$ ,  $c_3 = 2^{\frac{1}{42}} e^{(\zeta^*)^{(-1)}}$  and  $j = \frac{\lambda^{-1} - (\zeta + 1)}{2^{-\frac{1}{3}} \zeta^{\frac{1}{3}}}$ , and in the other phase,

$$(3.22) \quad \text{for } \zeta > 1 : \mathcal{F}_{CSM} = \begin{cases} \zeta - \frac{3}{4} - \frac{\log \zeta}{2} + \frac{1}{N^2} \log\left(1 - \frac{e^{-c_1 j^{3/2} N}}{32\pi j^{3/2} N}\right) & \lambda^{-1} > 2\zeta^{\frac{1}{2}} \\ \zeta - \frac{3}{4} - \frac{\log \zeta}{2} + \frac{1}{N^2} \log\left(c_3 \frac{e^{-c_2 |j|^3 N^2}}{|j|^{1/8} N^{1/12}}\right) & \lambda^{-1} < 2\zeta^{\frac{1}{2}} \end{cases},$$

where  $j = \frac{\lambda^{-1} - 2\zeta^{\frac{1}{2}}}{2^{-\frac{2}{3}} \zeta^{\frac{1}{3}} (\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}})^{\frac{1}{3}}}$ .

In the large  $N$  limit, by expanding the logarithm, the CSM free energy can be computed from above equations,

$$(3.23) \quad \text{for } \zeta \leq 1 : \mathcal{F}_{CSM} = \begin{cases} \frac{\zeta^2}{4} & \lambda^{-1} > \zeta + 1 \\ \frac{\zeta^2}{4} - \frac{c_2}{2^{-1}\zeta} (\lambda^{-1} - (\zeta + 1))^3 & \lambda^{-1} < \zeta + 1 \end{cases},$$

and

$$(3.24) \quad \text{for } \zeta > 1 : \mathcal{F}_{CSM} = \begin{cases} \zeta - \frac{3}{4} - \frac{\log \zeta}{2} & \lambda^{-1} > 2\zeta^{\frac{1}{2}} \\ \zeta - \frac{3}{4} - \frac{\log \zeta}{2} - \frac{c_2}{2^{-2}\zeta(\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}})} (\lambda^{-1} - 2\zeta^{\frac{1}{2}})^3 & \lambda^{-1} < 2\zeta^{\frac{1}{2}} \end{cases}.$$

Thus, in different regions of the moduli space, the upper bound energy is given by

$$(3.25) \quad \mathcal{F}_{ub} = \begin{cases} -\frac{c_2}{2^{-1}\zeta} (\lambda^{-1} - (\zeta + 1))^3 & \zeta \leq 1, \lambda^{-1} < \zeta + 1 \\ -\frac{c_2}{2^{-2}\zeta(\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}})} (\lambda^{-1} - 2\zeta^{\frac{1}{2}})^3 & \zeta > 1, \lambda^{-1} < 2\zeta^{\frac{1}{2}} \end{cases}.$$

Depending on the phase, the upper bound forms at a critical coupling  $\lambda_c$ ,

$$(3.26) \quad \lambda_c^{-1} = \begin{cases} 2(1 - \epsilon)\zeta^{\frac{1}{2}} & \zeta > 1 \\ (1 - \epsilon)(1 + \zeta) & \zeta \leq 1 \end{cases},$$

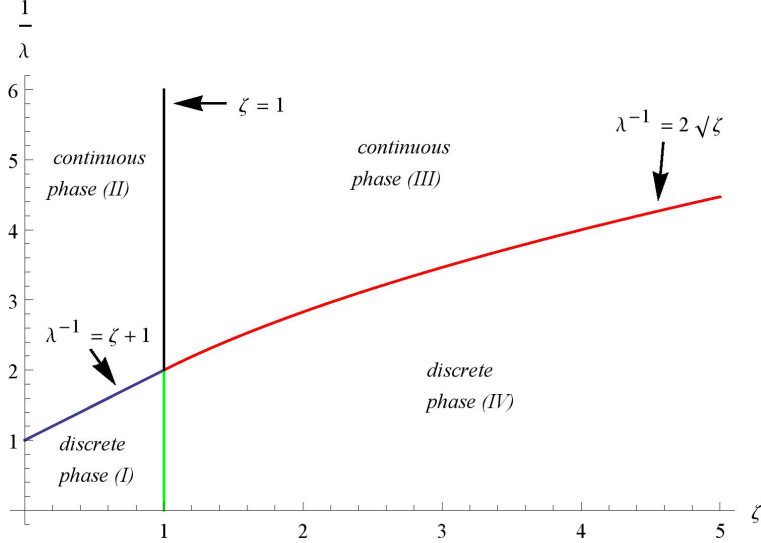


FIGURE 1. Phase structure of Chern-Simons matter theory. Black and blue lines, and red curve are third-order domain walls and green line is a second-order domain wall.

and upper bound critical energy,  $\mathcal{F}_{ub}^{crit}$ , infinitesimally close to the domain wall, at  $\lambda_c$ , is

$$(3.27) \quad \mathcal{F}_{ub}^{crit} = \begin{cases} 32c_2\epsilon^3\zeta^{\frac{3}{2}}/(\zeta^{\frac{3}{2}} + \zeta^{\frac{1}{2}}) & \zeta > 1 \\ 2c_2\epsilon^3(1 + \zeta)^3/\zeta & \zeta \leq 1 \end{cases}.$$

The free energy of CSM theory, Eqs. (3.23) and (3.24), completely determines the phase structure of the CSM theory. In fact, the explicit formula for the free energy of the upper bound, determines the order of the phase transition between the continuous theory (YM theory) and discrete theory (CSM theory). From the upper bound energy, Eq. (3.25), it is easy to find that the phase transition at  $\lambda^{-1} = \zeta + 1$ , between region (I) and (II), and the phase transition at  $\lambda^{-1} = 2\zeta^{\frac{1}{2}}$ , between region (III) and (IV) are third order, i.e. the third derivative of the free energy, with respect to  $\lambda$  or  $\zeta$ , jumps at the critical line. By direct computation one can observe that a second-order phase transition between region (I) and (IV) happens at domain wall  $\zeta = 1$ . One needs to check the continuity of derivative of free energy, with respect to  $\zeta$ , in regions (I) and (IV) at  $\zeta = 1$  up to second derivative. However, this domain wall separates also region (II) and (III) and in those regions, this is a third-order domain wall by simple computation from YM free energy, Eq. (A.7). Thus, upper bound changes the third-order domain wall to a second-order domain wall, see Figure 1.

In summary, the above results in CSM theory with GWW potential determine the phase structure of the theory. In this theory, there are four different domain walls separating four different phases. First, at the GWW domain wall, the black line,  $\zeta = 1$ , where a third-order phase transition occurs and second, at the green line,  $\zeta = 1$ , separating two discrete theories, a second-order phase transition occurs, and third, two third-order domain walls separating continuous theories and discrete theories; the blue line,  $\lambda^{-1} = \zeta + 1$  in  $\zeta \leq 1$  phase and the red curve,  $\lambda^{-1} = 2\zeta^{\frac{1}{2}}$  in  $\zeta > 1$  phase.

**3.4. Consistency checks and remarks.** A consistency check can be observed in the following. The YM eigenvalue density,  $\rho(\alpha)$ , Eq. (A.10), is maximum at  $\alpha = 0$ . Hence the upper bound condition,  $\rho(\alpha) < \frac{1}{2\pi\lambda}$ , becomes

$$(3.28) \quad \rho(\alpha) = \begin{cases} \frac{1}{2\pi}(1 + \zeta \cos \alpha) \Big|_{\alpha=0} = \frac{1}{2\pi}(1 + \zeta) < \frac{1}{2\pi\lambda} & \zeta \leq 1 \\ \frac{\zeta}{\pi} \cos(\alpha/2) \sqrt{\frac{1}{\zeta} - \sin^2(\alpha/2)} \Big|_{\alpha=0} = \frac{\sqrt{\zeta}}{\pi} < \frac{1}{2\pi\lambda} & \zeta > 1 \end{cases}.$$

These conditions are precisely the conditions in  $\mathcal{F}_{ub} = 0$  or  $\mathcal{R}(\lambda, \zeta) = 1$ , in region (II) and (III). In this case, the CSM theory is described by the continuum YM theory and thus the eigenvalue density of the CSM theory, which in the phase  $\mathcal{R}(\lambda, \zeta) = 1$  is the eigenvalue density of YM theory, should satisfy the upper bound.

In the following, some explicit values of the upper bound energy in strong and weak coupling regime are computed. Taking the strong coupling limit,  $\lambda \rightarrow \infty$ , is only possible in region (I) and (IV) and the values of the upper bound energy at strong coupling are given by

$$(3.29) \quad \text{in region (I)} : \mathcal{F}_{ub} = \begin{cases} 16c_2 & \zeta \rightarrow 1 \\ \infty & \zeta \rightarrow 0 \end{cases},$$

and

$$(3.30) \quad \text{in region (IV)} : \mathcal{F}_{ub} = \begin{cases} 16c_2 & \zeta \rightarrow 1 \\ 32c_2 & \zeta \rightarrow \infty \end{cases}.$$

More generally, in region (IV), for any fixed  $\lambda$ ,  $\mathcal{F}_{ub} = 32c_2$  as  $\zeta \rightarrow \infty$ . The zero coupling limit,  $\lambda \rightarrow 0$ , is only possible and meaningful in region (IV) and in this region it implies  $\zeta \rightarrow \infty$  which leads to  $\mathcal{F}_{ub} \rightarrow 0$ , as  $\lambda \rightarrow 0$  and  $\zeta \rightarrow \infty$ , because  $\lambda^{-1} - 2\zeta^{\frac{1}{2}} \rightarrow 0$ , as  $\epsilon \rightarrow 0$ .

On the line  $\lambda = \frac{1}{2}$ , in region (II),  $\mathcal{F}_{ub} = 0$  and in region (IV),  $\mathcal{F}_{ub} = \frac{-32c_2}{\zeta(\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}})}(1 - \zeta^{\frac{1}{2}})^3$ . At the point  $\zeta = 1, \lambda = \frac{1}{2}$ ;  $\mathcal{F}_{ub} = 0$  and at  $\zeta \rightarrow \infty, \lambda = \frac{1}{2}$ ;  $\mathcal{F}_{ub} = 32c_2$ .

#### 4. INTERPRETATIONS AND DISCUSSIONS

There are some possible explanations as well as interpretations of our results. A possible explanation is via the understanding of the physical picture of the phase transition. The level  $k$  is a formal source of discreteness in the system and the real source of discreteness is the existence of the upper bound. In fact, since all the parameters are going to infinity, taking the continuum limit strictly depends on the relations between the parameters. The plausible continuum limit exists under certain conditions. In the naive continuum limit, we expect to remove the upper bound,  $\lambda \rightarrow 0$ , on the eigenvalue density, but at the same time, for larger and larger  $T$  the eigenvalue density is sharper and pick of the eigenvalue density is higher, so,  $\lambda \rightarrow 0$  limit should be taken before the limit  $T \rightarrow \infty$ . However, for finite  $\lambda$  in some phases the continuum limit of the CSM theory exists, in the sense that the eigenvalue density is not bounded. Therefore, we can explain our results as a sufficient and necessary conditions for the continuum limit. In summary, in order to get the expected continuum limit,  $k$  should go to infinity faster than  $N$  (and  $T$ ), i.e. the relations  $k > N + T$  and  $k > 2\sqrt{NT}$  in two phases.

In the upper-gap phases, the partition function and free energy of the CSM theory, in addition to the YM part, contain an extra term originated from the upper bound. The upper gap is a consequence of discretization of the eigenvalues of the holonomy matrix and the discreteness is itself a direct consequence of the summation over flux sector in the partition function. Discrete fluxes are dual to the monopole operators and discrete flux configurations are effectively the same as the configurations generated by monopoles at the centre of  $S^2$ . However, in our case, there is no physical monopole or any matter inside  $S^2$ . Having said that, we might interpret the formation of the upper bound configuration and its energy  $\mathcal{F}_{ub}$  as the formation of the monopole in the CSM theory with the same energy. Possible physical implications of our results in the context of monopoles, e.g. monopole condensation and confinement is highly interesting and remain for future studies.

As it is defined in [Ta], the level-rank duality in CSM theory leads to conservation of the partition function under the transformation  $N \rightarrow k - N$  and  $k \rightarrow k$  or equivalently,  $\lambda \rightarrow 1 - \lambda$  and  $\zeta \rightarrow \zeta(\lambda/(1 - \lambda))$ . At  $\lambda = \lambda_c = 1/2$ , the level-rank duality is valid. But in the general case, the level-rank duality does not seem to be valid. At  $\lambda = \lambda_c$ , the CSM free energy in each phase satisfies the level-rank duality but further studies

are needed to find other solutions,  $\lambda^*$ , satisfying,

$$(4.1) \quad \mathcal{F}_{CSM}(\lambda, \zeta) \Big|_{\lambda^*} = \mathcal{F}_{CSM}\left(1 - \lambda, \frac{\zeta\lambda}{1 - \lambda}\right) \Big|_{\lambda^*}.$$

A similar third-order phase transition, between discrete and continuous two-dimensional gauge theories, occurs in the Douglas-Kazakov model, [Do-Ka]. It would be interesting to explore possible relations between the two works.

In this paper, the CSM theory is considered with the GWW potential, however, the general results in section (3.1) allow to study the asymptotic limit of the ratio of the partition functions of CSM and YM theories for any arbitrary potential. One would expect to get similar phase structure for an arbitrary nontrivial potential. However, the actual computations remain for future studies.

Possible generalizations of the obtained results in this paper, e.g. the third-order phase transitions, to CSM theory with other gauge groups  $G = O(N)$  and  $Sp(N)$ , remain for future studies, [Ch-Jo].

Another direction for further study is possible applications and implications of our results, especially the new phase structure of the model, in AdS/CFT correspondence with Vasiliev higher spin gravity.

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## A. RANDOM MATRIX THEORY, TOEPLITZ DETERMINANT AND RIEMANN-HILBERT PROBLEM IN GAUGE THEORY

In this appendix, two-dimensional Yang-Mills theory and its relation to matrix model and Toeplitz determinant are reviewed. The example of YM theory with the Gross-Witten-Wadia potential is considered and explicit formulas for the partition function and free energy are reviewed. Furthermore, the phase structure of the model and the eigenvalue distribution are described. Finally, the GWW model as the solution of the Riemann-Hilbert problem is briefly stated.

**A.1. Two-dimensional YM theory, matrix models and Toeplitz determinant.** The interesting and rich phase structure of  $2d$  YM theory and its matrix model representation with some special potentials has been studied in both mathematics and physics literature, [Gr-Wi], [Wa] and [Ba-De-Jo]. The two-dimensional YM theory is defined via the following matrix integral representation of the partition function

$$(A.1) \quad \begin{aligned} \mathcal{Z}_{YM} &= \int [DU] e^{-V_{YM}} = \prod_{j=1}^N \int_{-\infty}^{\infty} d\alpha_j \prod_{l < p} \left( 2 \sin\left(\frac{\alpha_l - \alpha_p}{2}\right) \right)^2 e^{-V_{YM}} \\ &= \prod_{j=1}^N \int_{-\infty}^{\infty} d\alpha_j \prod_{l < p} |e^{i\alpha_l} - e^{i\alpha_p}|^2 e^{-V_{YM}}, \end{aligned}$$

where  $V_{YM}$  is the potential of the YM theory. This is a partition function of coulomb gas of unit charges on the unit circle with an attractive potential  $V_{YM}$  and a logarithmic repulsive potential,  $\log |e^{i\alpha_l} - e^{i\alpha_p}|$ , from the Vandermonde determinant. The partition function is also equal to the  $N \times N$  Toeplitz determinant,  $D_N(f)$ ,

$$(A.2) \quad \mathcal{Z}_{YM} = D_N(f),$$

where  $f = e^{-V_{YM}}$  is the generating function.

**A.1.1. Toeplitz determinant and matrix model.** The relation between matrix integral and Toeplitz determinant is briefly introduced in the introduction. In this part, in order to elaborate more on Eq. (A.2), the continuous and discrete Toeplitz determinants are defined and their relations to continuous and discrete matrix models are stated.

A continuous Toeplitz determinant with a weight function  $f$  is defined by

$$(A.3) \quad D_N(f) = \det \left[ \int_{|z|=1} z^{-j+l} f(z) \frac{dz}{2\pi i z} \right]_{j,l=0}^{N-1},$$

where  $z = e^{i\alpha}$ . Similarly, the discrete Toeplitz determinant is defined in a finite domain  $d$  with the size  $|d| = k$  as follows

$$(A.4) \quad D_N^k(f, d) = \det \left[ \frac{1}{k} \sum_{z \in d} z^{-j+l} f(z) \right]_{j,l=0}^{N-1}.$$

The discrete and continuous Toeplitz determinants are related to the discrete and continuous matrix integrals via a relation called, Heine-Szegö identity,

$$(A.5) \quad \begin{aligned} D_N(f) &= \prod_{j=1}^N \int_0^{2\pi} \frac{d\alpha_j}{2\pi} f(e^{i\alpha_j}) \prod_{l < j} |e^{i\alpha_l} - e^{i\alpha_j}|^2, \\ D_N^k(f, d) &= \frac{1}{N! k^N} \sum_{(z_1, \dots, z_N) \in d^N} \prod_{j=1}^N f(z_j) \prod_{1 < j < l \leq N} |z_j - z_l|^2. \end{aligned}$$

A.1.2. *Gross-Witten-Wadia model.* Consider a specific potential  $V_{YM} = -(T/2)Tr(U + U^\dagger)$  where  $U$  is a  $N \times N$  holonomy matrix with eigenvalues  $\alpha_i$ 's. This defines an interesting two-dimensional YM theory, called Gross-Witten-Wadia model, which is a unitary random one-matrix model. This model possesses a third-order phase transition. The partition function reads,

$$(A.6) \quad \begin{aligned} \mathcal{Z}_{YM}^{(GWW)} &= \int [DU] e^{\frac{T}{2}Tr(U+U^\dagger)} \\ &= \prod_{j=1}^N \int_{-\infty}^{\infty} d\alpha_j \prod_{l < j} \left( 2 \sin\left(\frac{\alpha_l - \alpha_j}{2}\right) \right)^2 \prod_{j=1}^N e^{\frac{T}{2}(z_j + z_j^{-1})} \\ &= \prod_{j=1}^N \int_{-\infty}^{\infty} d\alpha_j \prod_{l < j} |e^{i\alpha_l} - e^{i\alpha_j}|^2 e^{T \sum_{j=1}^N \cos \alpha_j}. \end{aligned}$$

The above partition function is equivalently given by the Toeplitz determinant,  $D_N(f)$  with  $f = e^{T \cos \alpha}$ .

Free energy of the GWW model is defined by  $\mathcal{F}_{YM}^{(GWW)}(\zeta) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \mathcal{Z}_{YM}^{(GWW)}(e^{\zeta N \sum_j \cos \alpha_j})$ , and with a change of variable  $T = \zeta N$ , it has been computed in [Gr-Wi],

$$(A.7) \quad \mathcal{F}_{YM}^{(GWW)}(\zeta) = \begin{cases} \frac{\zeta^2}{4} & 0 < \zeta < 1 \\ \zeta - \frac{3}{4} - \frac{\log \zeta}{2} & \zeta > 1 \end{cases}.$$

A third-order phase transition, i.e. discontinuity in  $d^3 \mathcal{F} / d^3 \zeta$ , occurs at  $\zeta = 1$ .

A.1.3. *Eigenvalue distribution function.* In the YM theory, in order to find the equilibrium measure one has to look for the measure on the circle which is non-negative and satisfies an energy minimization problem. Roughly speaking, competition of the attractive force from the potential and a repulsive force from the Vandermonde determinant in the matrix integral, results to a phase transition. In fact, there are two phases; in one phase the attraction force is weaker than the repulsive force and the eigenvalues has support on the whole circle. In another phase the repulsive force is weaker and this leads to the existence of a gap in the eigenvalue density and therefore the eigenvalues have only support on finite arcs on the circle.

The distribution of the eigenvalues on the unit circle is determined by solving the following variational problem, [Ba-De-Jo]. For a given measure  $d\mu_V(z)$ , the equilibrium measure  $\rho_V(\alpha)$  is obtained as a unique minimizer of the following variational problem,

$$(A.8) \quad E^V = \inf\{I^V((\rho)) : \rho \text{ is a probability measure on the unit circle } \Sigma\},$$

where

$$(A.9) \quad I^V(\rho) = \int \int_{\Sigma \times \Sigma} \log |z - w|^{-1} d\rho(z) d\rho(w) + \int_{\Sigma} V(z) d\rho(z).$$

By solving above problem for the GWW model,  $V(z) = \frac{T}{2}(z + z^{-1})$ , the eigenvalue density is obtained,

$$(A.10) \quad \rho(\alpha) = \begin{cases} \frac{1}{2\pi}(1 + \zeta \cos \alpha) & \zeta \leq 1 \\ \frac{\zeta}{\pi} \cos(\alpha/2) \sqrt{\frac{1}{\zeta} - \sin^2(\alpha/2)} & \zeta > 1 \end{cases}.$$

As it is mentioned before, phase transition happens at  $\zeta = 1$ , and the eigenvalue density  $\rho(\alpha)$  has different features in different phases. In  $\zeta \leq 1$  phase,  $\rho(\alpha)$  is supported on whole circle and this phase is called *no-gap* phase. In  $\zeta > 1$  phase,  $\rho(\alpha)$  is supported on a subset of circle,  $[-\alpha_c, \alpha_c]$  where  $\alpha_c$  is given by the condition  $\sin^2(\alpha_c/2) = 1/\zeta$  and this phase is called *lower-gap* phase.

**A.2. Riemann-Hilbert problem in Yang-Mills theory.** In this part, the YM theory as a continuous matrix model is formulated via a well-defined mathematical problem, namely the Riemann-Hilbert problem (RHP), [Ba-De-Jo]. In the following, orthogonal polynomials and RHP associated to the GWW model are reviewed. Lets  $z \in \mathbb{C}$ , and  $T \in \mathbb{R}$  is a parameter of the model, then the following polynomial,

$$(A.11) \quad p_N(z, T) = a_N(T)z^N + \dots$$

is called an orthogonal polynomial with respect to weight function,  $f(e^{i\alpha}) = \exp((T/2)(e^{i\alpha} + e^{-i\alpha}))$ , on the unit circle if it satisfies

$$(A.12) \quad \int_{-\pi}^{\pi} p_N(e^{i\alpha}) \overline{p_M(e^{i\alpha})} d\mu_f = \delta_{N,M},$$

where  $N, M > 0$  and measure is  $d\mu_f = f(e^{i\alpha}) \frac{d\alpha}{2\pi}$ . The well-known connection between the orthogonal polynomial and Toeplitz determinant is established in [Sz]. As a result, the leading coefficient in orthogonal polynomial,  $a_N(T)$ , is expressed in terms of the Toeplitz determinant,

$$(A.13) \quad a_N(T) = \frac{D_{N-1}(T)}{D_N(T)},$$

where  $D_N(T) = D_N(e^{T \cos \alpha})$ .

It has been observed earlier that the partition function of the YM theory with the GWW potential which is given by a unitary matrix integral is expressed in terms of the Toeplitz determinant. Thus using the above connection to the orthogonal polynomials and the Szegő strong limit,  $\lim_{N \rightarrow 1} D_N(T) = 1$ , one can obtain,

$$(A.14) \quad \log \mathcal{Z}_{YM}^{(GWW)}(T) = \log D_{N-1}(e^{T \cos \alpha}) = \sum_{l=N}^{\infty} \log a_l^2(T).$$

To control the behavior of the partition function in the limit of large  $N, T$ , one needs to control the behavior of orthogonal polynomial  $a_l^2(T)$  in the large  $N, T$  limit for all  $l \geq N$ . To this aim, the steepest descent methods has been used to compute the asymptotic behavior of the following RHP, [De-Zh].

The following RHP uniquely determines  $a_l^2(T)$  and partition function of the YM theory with the GWW potential. Let  $\Sigma$  be a unit circle with the counter clock-wise orientation and  $Y(z, l+1, T)$  is a  $2 \times 2$  matrix-valued function satisfying

$$(A.15) \quad \begin{cases} Y(z, l+1, T) \text{ is analytic in } \mathbb{C} - \Sigma, \\ Y_+(z, l+1, T) = Y_-(z, l+1, T) \begin{pmatrix} 1 & \frac{1}{z^{l+1}} e^{\frac{T}{2}(z+z^{-1})} \\ 0 & 1 \end{pmatrix}, \\ Y(z, l+1, T) z^{-(l+1)\sigma_3} = I + \mathcal{O}(\frac{1}{z}) \text{ as } z \rightarrow \infty \end{cases},$$

where  $Y_{\pm}$  denotes  $Y$  inside/outside of the unit circle,  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $z^{-(l+1)\sigma_3} = \begin{pmatrix} z^{-(l+1)} & 0 \\ 0 & z^{l+1} \end{pmatrix}$ .

It has been proved that the largest coefficient in the orthogonal polynomial is given by 21-entry of matrix  $Y$ ,

$$(A.16) \quad a_l^2(T) = -Y_{21}(0; l+1, T).$$

Asymptotic results for  $a_N^2(T)$  and  $Z_{YM}^{(GW)}(T)$  for  $N, T \rightarrow \infty$  is obtained in [Ba-De-Jo].

## B. TRACY-WIDOM DISTRIBUTION

Tracy-Widom distribution  $F(x)$  is the probability distribution of the largest eigenvalue of a Hermitian random matrix and it is defined by

$$(B.1) \quad F(x) = e^{-\int_x^{\infty} (s-x)q^2(s)ds},$$

where  $q(s)$  is the solution of the Painlevé II equation,  $q''(s) = 2q^3(s) + sq(s)$ , with boundary condition  $q(s) \sim -Ai(s)$  as  $s \rightarrow \infty$ , where  $Ai(s)$  is the Airy function. The Tracy-Widom distribution is also given by  $F(x) = \det(1 + K_{Ai})$  where the kernel of Airy function is given by

$$(B.2) \quad K_{Ai}(z, w) = \frac{Ai(z)Ai'(w) - Ai(w)Ai'(z)}{z - w}.$$

In the asymptotic regime,  $s \rightarrow \pm\infty$ , the following result is obtained in [Ba-Bu-Di],

$$(B.3) \quad q(s) = \begin{cases} -Ai(s) + \mathcal{O}(\frac{e^{-\frac{4}{3}s^{\frac{3}{2}}}}{s^{\frac{1}{4}}}) & s \rightarrow +\infty \\ \sqrt{\frac{-s}{2}}(1 + \frac{1}{8s^3} - \frac{73}{128s^6} + \frac{1020}{1024s^9} + \mathcal{O}(|s|^{-12})) & s \rightarrow -\infty \end{cases},$$

where in the limit  $s \rightarrow +\infty$ ,  $Ai''(s) = sAi(s)$  and  $Ai(s) \approx \frac{e^{-\frac{2}{3}s^{\frac{3}{2}}}}{2\sqrt{\pi}s^{\frac{1}{4}}}$ .

Moreover, the asymptotic of the Tracy-Widom distribution is studied in [Ba-Bu-Di], and the following result is obtained,

$$(B.4) \quad F(x) = \begin{cases} 1 - \frac{e^{-\frac{4}{3}x^{\frac{3}{2}}}}{32\pi x^{\frac{3}{2}}}(1 - \frac{35}{24x^{\frac{3}{2}}} + \mathcal{O}(x^{-3})) & \text{as } x \rightarrow \infty \\ 2^{\frac{1}{42}} e^{(\zeta^*)'(-1)} \frac{e^{-\frac{1}{12}|x|^{\frac{3}{2}}}}{|x|^{\frac{1}{8}}}(1 + \frac{3}{26|x|^3} + \mathcal{O}(|x|^{-6})) & \text{as } x \rightarrow -\infty \end{cases},$$

where  $\zeta^*$  is the Riemann zeta function.

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